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On the Statistical Theory of Electromagnetic Waves in a Fluctuating Medium (I)

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The subject of electromagnetic wave scattering by a randomly varying medium is reviewed giving special emphasis to the technical method of approach. The symbolic representation of Maxwell's equations is introduced to make it easier to survey the whole subject and to formulate the equations. The Feynman diagram method is applied to the computation of the correlation of the fields at different points in space to any order of approximation. The differential equation to be satisfied by the latter correlation function is also derived from another point of view. Then the theory is developed on the "renormalization" of the constants, i.e., the effective propagation constant in a fluctuating medium and the effective coupling constant between the field and the medium, etc.; the explicit expression of the former is obtained to the first order of approximation. The dispersion relation is derived as a connected problem. In Part II of this series of papers, a fundamental theory of statistics of the electromagnetic field in a fluctuating medium will be developed In Part III, a few applications to tropospheric scattering will be given.

1. Introduction

The entire subject of electromagnetic propagation through a fluctuating medium has been investigated theoretically and experimentally by many authors. Even though the Born approximation might be valid for the scattered electromagnetic waves in many cases, there still remain several theoretical problems to be solved such as the effect of earth diffraction, the effect, if any, of using very narrow beam antennas, the restrictions for the validity of simple formulas for the scattered power, etc. As to the theory of multiple scattering, there are also many fundamental problems to be solved.

The purposes of this series of papers are first to remark that an almost perfect correspondence exists between the statistical treatments of the electromagnetic field in a fluctuating medium, which could be anisotropic, and the treatments in quantum field theory, and thus that the many powerful methods used in the latter, e.g., Feynman's diagram method, the method of renormalization, the methods used in solving bound-state problems, etc., are available when solving the statistical problems in the former (Part I). Then a fundamental statistical theory of the field in a fluctuating medium will be developed (Part II); the field and the medium are inevitably quantized there, and the reason for keeping a high correspondence between the statistical theory and the quantum field theory will naturally be understood.

The contents of part I can be summarized as follows: In section 2, a symbolic representation of Maxwell's equations is introduced [Furutsu, 1952, 1956; Marcuvitz, 1962]. This representation is very convenient for studying the subject from a broader point of view, and for achieving a unified and brief equation formulation. In section 3, the theory of the power transmitted to the receiver through an arbitrary medium, including anisotropic media, is extended. Here the antenna patterns of the transmitter and receiver are represented in terms of the respective equivalent electromagnetic current distributions on suitable surfaces enclosing the transmitter and the receiver. These distributions could refer directly to pure electric currents of all the antenna elements which may be distributed on a surface of some extension, or they could be derived from the electromagnetic fields themselves, as occurring, e.g., on the surface of the mouth of a parabolic antenna. In the same way, the cross product of the amplitudes of two different receivers at arbitrary points in space is also derived in terms of the antenna patterns of the receiver and transmitter. In section 4, the expression for any

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cross correlation of fields is derived with the aid of the results of section 3. All the equations are first transformed into their Fourier representations, and next Feynman's diagram method [Feynman, 1949a, 1949b; Dyson, 1949a, 1949b] is extensively used to any order of approximation with respect to the correlation function of the refractive index. This diagram method has been one of the most convenient techniques in quantum electrodynamics. Here the same technique is introduced and, in section 5, the discussions are completely devoted to the "renormalization," i.e., the effective medium constants in a fluctuating medium, the effective coupling constant between the field and the fluctuating medium, etc.

In section 6, another approach to the multiple scattering problem is shown. It is used to derive directly the differential equation for the correlation function between the fields at two arbitrary points in space. The equation takes a simple form, and there are many possibilities to solve it.

2. Symbolic Representation of Maxwell's Equations

In order to survey the subject from a broader point of view and also to facilitate the analytic representations in the following sections, it is convenient to represent the six components of the electromagnetic field in a unified form. In this section, Maxwell's equations are expressed in symbolic form, and the Green function is properly defined for later convenience.

Here we employ the following notations: Latin subscripts assume values ranging from 1 to 3, and a repeated index is to be so summed. The coordinate vector in space is denoted by $x_i=(x)$.

Using the conventional notations, Maxwell's equations are given in Gaussian units (time factor $e^{i\omega t}$) by

$$\operatorname{rot} H-i(\omega/c)\varepsilon E=4\pi I/c, \quad \operatorname{rot} E+i(\omega/c)\mu H=0. \tag{2.1}$$

Introducing the matrices

$$s_1=\begin{pmatrix}0&0&0\\0&0&i\\0&-i&0\end{pmatrix},\quad s_2=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},\quad s_3=\begin{pmatrix}0&i&0\\-i&0&0\\0&0&0\end{pmatrix}, \tag{2.2}$$

the operator "rot" as a three-rowed matrix is expressed by

$$\operatorname{rot}=s_i\partial_i=(s\partial), \quad \partial_j=i\frac{\partial}{\partial x_j} \qquad (j=1,2,3). \tag{2.3}$$

Thus, (2.1) can be expressed in the form

$$(s\partial)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}iE\\H\end{pmatrix}-\frac{\omega}{c}\begin{pmatrix}\varepsilon&0\\0&\mu\end{pmatrix}\begin{pmatrix}iE\\H\end{pmatrix}=\begin{pmatrix}\frac{4\pi}{c}I\\0\end{pmatrix}, \tag{2.4}$$

which takes the form

$$[\rho_1(s\partial)-k]\psi=j. \tag{2.5}$$

Here

$$\psi=\begin{pmatrix}iE\\H\end{pmatrix} \quad \text{and} \quad j=\begin{pmatrix}\frac{4\pi}{c}I\\0\end{pmatrix} \tag{2.6a}$$

are one-column matrices of 6 elements, and

$$k=\frac{\omega}{2c}\{(1+\rho_3)\varepsilon+(1-\rho_3)\mu\}, \tag{2.6b}$$

a 6×6 matrix and, in the case of isotropic media, the elements ε and μ of the latter, as well as those of the two-rowed Pauli's matrices

$$\rho_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad \rho_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\qquad \rho_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\tag{2.7}$$

represent 3×3 unit diagonal matrices multiplied by the indicated quantity.

On the other hand, in the case of anisotropic media, $\mathcal{E}$ and μ are generally 3×3 matrices, which can be represented as functions of s_1, s_2, and s_3. Therefore $\mathcal{E}$ and μ in general do not commute with the s_i's (see (2.10)).[1]

Equation (2.5) finally takes the form

$$[\gamma_i\partial_i-k]\psi=j,\qquad \gamma_i=\rho_1 s_i.\tag{2.8}$$

The matrices s_i and ρ_i are all Hermitian and especially

$$\gamma_i^*=\gamma_i^\tau=-\gamma_i,\tag{2.9}$$

where γ_i^* and γ_i^τ are the complex conjugate and transposed matrices of γ_i, respectively. Equation (2.9) is evident from the explicit matrix expressions of (2.2).

The matrices s_i and ρ_i satisfy the following algebraic relations which are often very convenient to use:

$$s_is_js_k+s_ks_js_i=\delta_{ij}s_k+\delta_{jk}s_i,\ s_is_j-s_js_i=i^{-1}\epsilon_{ijk}s_k,$$
$$s_i^2=s_1^2+s_2^2+s_3^2=2,\ \rho_1^2=\rho_2^2=\rho_3^2=1,\ \rho_1\rho_2=-\rho_2\rho_1=i\rho_3,\ \text{etc.}\tag{2.10}$$

Here ϵ_{ijk} is the antisymmetrical tensor corresponding to $\epsilon_{123}=1$; its elements are $+1$ or -1 according as the number of permutations necessary to change the order of subscripts i, j, k, into the order 1, 2, 3 is even or odd. From (2.10), it especially follows, for any two orthogonal vectors m and n in the ordinary three-dimensional space, that

$$(n\gamma)^3=n^2(n\gamma),\ (n\gamma)(m\gamma)^2=\{m^2-(m\gamma)^2\}(n\gamma),\ (n\gamma)(m\gamma)(n\gamma)=0,$$
$$(n\gamma)^2(m\gamma)^2=(m\gamma)^2(n\gamma)^2=n^2m^2-(m\times n\cdot\gamma)^2.\tag{2.11}$$

The traces of these matrices are frequently used in actual computations, and the following formulas hold (appendix 1):

$$\mathrm{Tr}'(s_i)=0,\ \mathrm{Tr}'(s_is_j)=2\delta_{ij},\ \mathrm{Tr}'(s_is_js_k)=i^{-1}\epsilon_{ijk},\ \mathrm{Tr}'(s_is_js_ks_l)=\delta_{ij}\delta_{kl}+\delta_{jk}\delta_{il}.\tag{2.12}$$

Here Tr′ stands for the trace in the subspace of s_i's matrices.

The Green theorem in this representation takes the following form: Let ψ'' and ψ' be arbitrary continuous ψ-vectors represented by a one-row and a one-column matrix of 6 elements, respectively, in the space Σ, then with $(dx)=dx_1dx_2dx_3$

$$\int_\Sigma\{\psi''[(\gamma\partial)-k]\psi'-\psi''[-(\gamma\overleftarrow{\partial})-k]\psi'\}(dx)=-i\int_\sigma\psi''(n\gamma)\psi'd\sigma.\tag{2.13}$$

Here the right side represents a surface integral over the surface σ of Σ, and $n=n(\sigma)$ the inward space unit vector normal to σ, and $\overleftarrow{\partial}$ operates on the coordinates (x) on its left side. The excitation of the electromagnetic field will be performed either directly by electric currents, or indirectly by the use of a device such as a waveguide antenna. In the latter case, it will, in practice, be excited by one predominant waveguide mode. However, these two ways of excitation involve no difference whatever in the representation of the field:

[1] For example, in an ionized gas of plasma frequency $\omega_o/2\pi$ under the influence of static magnetic field H_o or corresponding gyrofrequency $\vec{\omega}_H/2\pi$ with collision frequency ν [Furutsu, 1952],

$$\mathcal{E}=1-\omega_0^2[\omega\{\omega-i\nu+(\vec{\omega}_H s)\}]^{-1}.$$

Let $\psi(x, x')$ be the 6×6 tensorial Green function satisfying given conditions in space and at boundaries. We then get, combining (2.8) with a result derived in appendix 2, two different representations for the differential equation for this function, i.e.,

$$[(\gamma\partial)-k]\psi(x,x')=\psi(x,x')[-(\gamma\overleftarrow{\partial'})-k]=\delta(x-x'). \tag{2.14}$$

Here the right side is to be interpreted as the coefficient in front of a 6×6 diagonal unit matrix. On the other hand, $\psi(x, x')$ constitutes a nondiagonal 6×6 matrix, the jth column of which satisfies Maxwell's equations in which the right side of the jth equation is $\delta(x-x')$ instead of zero. The symbol $\partial'=i\partial/\partial x'$ here refers to differentiation with respect to x'.

Then, the solution $\psi(x)$ of (2.8) is given by the one-column matrix

$$\psi(x)=\int\psi(x,x')j(x')(dx'). \tag{2.15}$$

On the other hand, the field which is excited by a source inside the surface σ can be expressed in terms of the tangential components of the field on σ. Thus, applying (2.13) to the space outside σ, while putting $\psi'=\psi(x)$ (the solution to be obtained) and $\psi''=\psi(x', x)$ (the Green function), we find, taking into account the outward propagating condition for $\psi(x)$ at infinity, and also using (2.14),

$$\psi(x')=i\int_\sigma\psi(x', x)(n\gamma)\psi(x)d\sigma,$$

or

$$\psi(x)=\int\psi(x, x')\xi_\sigma(x')(dx'). \tag{2.16}$$

Here

$$\xi_\sigma(x)=i\delta(x, \sigma)(n\gamma)\psi(x), \tag{2.17}$$

while $\delta(x, \sigma)$ is defined as the one-dimensional δ-function of a variable which may be identified with the distance to σ, that is a function different from zero only on the surface σ.[2] It may be remarked that $\xi_\sigma(x)$ depends only on the components of the electromagnetic field that are tangential to the surface σ.

Equations (2.15) and (2.16) take the same form, and thus $\xi_\sigma(x)$ may be interpreted as the effective electromagnetic current accounting for the field inside σ.

In this symbolic representation, the Poynting vector P_i takes the form

$$P_i=\frac{c}{8\pi}\psi^*\gamma_i\psi=\frac{c}{4\pi}[\mathrm{Re}\,(E^*\times H)]_i. \tag{2.18}$$

Here ψ^* stands for the one-row matrix representing the complex conjugate of ψ. Decomposing k into its Hermitian and anti-Hermitian parts,

$$k=\kappa-i(4\pi/c)\sigma, \tag{2.19}$$

both κ and σ (not to be confused with the same symbol for the surface σ) being Hermitian, we readily find by the use of (2.8) and (2.6) that

$$\mathrm{div}\,P+\psi^*\sigma\psi=-\mathrm{Re}\,(E^*I). \tag{2.20}$$

Thus the anti-Hermitian part of k contributes to the waves influenced by the dissipative properties of the medium.

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[2] For an arbitrary function $f(x)$, it holds that

$$\int f(x)\delta(x, \sigma)(dx)=\int_\sigma f(x)d\sigma.$$

On the other hand, from (2.17), we derive, using the relation $(n\gamma)\gamma_i(n\gamma)=n_i(n\gamma)$,

$$\int \xi^*_{\sigma''}\gamma_j\xi_{\sigma'}(dx)=n_j\delta(\sigma'', \sigma'), \tag{2.21}$$

if ψ is normalized by [3]

$$\int_\sigma \psi^*(n\gamma)\psi d\sigma=1. \tag{2.22}$$

The function $\delta(\sigma'', \sigma')$ is the delta function of the "distance" between two "parallel" surfaces σ'' and σ' (according to the usual definitions of these quantities for neighboring surfaces).

In view of (2.18), the normalization (2.22) implies that the total power radiated outwards from the surface σ equals $c/8\pi$ power units. Here, the surfaces σ' and σ'' in (2.21) are assumed to be two arbitrary elements of a continuous set of surfaces, no element of which intersects another.

For later convenience, we shall deduce the Green function $S(x)$ for the case of a homogeneous medium which could be anisotropic; then, from (2.14),

$$[(\gamma\partial)-k]S(x)=\delta(x), \tag{2.23}$$

with the condition of outward propagation at infinity. Let $S(t)$ be the Fourier transform [4] of $S(x)$ as follows:

$$S(t)=\int_{-\infty}^{\infty} e^{i(tx)}S(x)(dx), \quad (dx)=dx_1dx_2dx_3, \tag{2.24}$$

then, from (2.23),

$$[(\gamma t)-k]S(t)=1, \quad S(t)=[(\gamma t)-k]^{-1}, \quad \text{Im }(k)<0. \tag{2.25}$$

Here, Im $(k)<0$ indicates that all the eigenvalues of k to be used in (2.23) should have an infinitesimal negative imaginary part.

On the other hand, the adjoint Green's function $\bar{S}(x)$ is defined by

$$\bar{S}(x)=S(-x)=\frac{1}{(2\pi)^3}\int e^{+i(tx)}S(t)(dt), \quad (dt)=dt_1dt_2dt_3, \tag{2.26}$$

which satisfies, with the condition of outward propagation at infinity,

$$\bar{S}(x)[-(\gamma\overleftarrow{\partial})-k]=\delta(x). \tag{2.27}$$

In the special case of a homogeneous isotropic medium, k commutes with all the s_i matrices (but still does not with ρ_1), and thus, using the algebraic relation $(\gamma\partial)^3=\partial^2(\gamma\partial)$ (see (2.11)), which involves $(\gamma t)^3=(\gamma t)t^2$, we find

$$[(\gamma t)-k][(\gamma t)+\{(\gamma t)^2-t^2+k'^2\}k^{-1}]=t^2-k'^2,$$

in which the new matrix $\bar{k}$ and the new scalar k' are defined by

$$k'^2=\bar{k}k=(\omega/c)^2\mu\epsilon, \quad \bar{k}=\rho_1k\rho_1=k|_{\rho_3\to-\rho_3}. \tag{2.28}$$

Hence, from (2.25),

$$S(t)=g(t)(t^2-k'^2)^{-1}, \quad g(t)=(\gamma t)+\{(\gamma t)^2-t^2+k'^2\}k^{-1}. \tag{2.29}$$

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[3] This normalization is possible only when the total radiated power away from the surface σ into the direction n is positive.

[4] The notation t will be used exclusively for the Fourier transformation variable, and, in this paper, any function $f(t)$ will stand for the Fourier transform of the space coordinate function $f(x)$.

The function $S(t)$ has poles at $t^2=k'^2$ which can be avoided in further integrations by assuming also an infinitesimal negative imaginary part for k'; the latter corresponds to the condition of outward propagation at infinity. Hence, from (2.24) and (2.29), we finally have

$$S(x)=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{-i(tx)}S(t)(dt)=g(\eth)\,\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}(t^2-k'^2)^{-1}e^{-i(tx)}(dt)=g(\eth)\left\{\frac{1}{4\pi r}\,e^{-ik'r}\right\}. \tag{2.30}$$

Here $r=|x|$ and $\eth_j=i\partial/\partial x_j$.

3. Transmission Power Gains

We consider here a transmitter system which is assumed to be in the space Σ_1 enclosed by the surface σ_1 and, in the same way, a receiver system in the space Σ_2 enclosed by the surface σ_2. The conductivity loss in space and in the systems is first neglected, involving a Hermitian k. Let $\psi_1(x)$ and $\psi_2(x)$ be the solutions satisfying the given boundary conditions in space and at the boundaries, whose fields on the surface σ_1 or σ_2 are respectively given by $\psi_i(\sigma_i)$ and are normalized according to (2.22)[5] by (fig. 1)

$$\int_{\sigma_i}\psi_i^*(n\gamma)\psi_i d\sigma=1 \qquad (i=1,\,2). \tag{3.1}$$

Thus, since $\gamma^{T}{}_i=\gamma_i^*=-\gamma_i$ and $\eth_i^*=-\eth_i$,

$$[(\gamma\eth)-k]\psi_i(x)=\psi_i^*(x)[-(\gamma\overleftarrow{\eth})-k]=0, \tag{3.2}$$

the latter being the complex conjugate of the former. However, the transposed expression of the second representation yields,

$$[(\gamma\eth)-k^T]\psi_i^*(x)=0.$$

Hence, ψ_i^* constitute no solutions of Maxwell's equations unless $k=k^T$. This holds for an isotropic medium; the ψ_i^* then are exactly the solutions satisfying the given boundary conditions except for the propagation directions which are just inverse to those of ψ_i.

Although (3.2) is sufficient in the case of isotropic media, it is necessary to introduce the adjoint equation of (3.2) in order to include the case of anisotropic media, i.e.,

$$\bar{\psi}_i[-(\gamma\overleftarrow{\eth})-k]=0, \quad [(\gamma\eth)-k]\bar{\psi}_i^*=0 \qquad (i=1,\,2), \tag{3.3}$$

with the normalization

$$-\int_{\sigma_i}\bar{\psi}_i(n\gamma)\bar{\psi}_i^*d\sigma=\int_{\sigma_i}\bar{\psi}_i^*(n\gamma)\bar{\psi}_i d\sigma=1. \tag{3.4}$$

Here $\bar{\psi}_i$ have to satisfy the same boundary conditions and propagation condition at infinity as ψ_i. It is to be noticed that $\bar{\psi}_i^*$ are the set of solutions of Maxwell's equations whose propagation directions are just inverse to those of ψ_i. In the special case of $k=k^T$, we have $\bar{\psi}_i=\psi_i$, and the normalization (3.4) agrees with (3.1).

[5] In this paper, (σ_i) will stand for the coordinates of arbitrary point bounded on the surface σ_i.

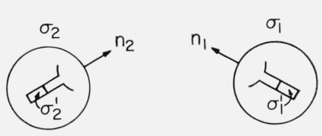

FIGURE 1. *Illustrations of the surfaces σ_1, σ_2, σ_1' and σ_2' for (3.1) and (3.6).*

Now suppose that both the transmitter and receiver use waveguides in each of which only one propagational mode can exist, and that the antenna systems are so well matched that there are no reflected waves at all for these modes. Any cross sections of these waveguides will be denoted by σ_1' and σ_2', respectively, corresponding to the surfaces σ_1 and σ_2 enclosed by them; the predominant modes will be marked $\psi_1(\sigma_1')$ and $\psi_2(\sigma_2')$ inside the respective waveguides, $\psi_1(x)$ and $\psi_2(x)$ being their continuations outside.

The problem is, then, to determine the field strength or the power transmitted into the receiver waveguide by the excitation of the mode wave in the transmitter normalized by (3.1). Putting $\psi'=\psi_1(x)$ and $\psi''=\bar{\psi}_2(x)$ in (2.13), and identifying the space Σ with the whole space outside the surfaces σ_1 and σ_2, we have, on taking into account that both ψ_1 and $\bar{\psi}_2$ are solutions of (3.2) and (3.3) that represent waves propagating outwards at infinity, respectively,

$$\int_{\sigma_1+\sigma_2} \bar{\psi}_2(n\gamma)\psi_1 d\sigma=0. \tag{3.5}$$

Here we could replace the surfaces σ_1 and σ_2 by σ_1' and σ_2', respectively.

On the other hand, deep in the waveguides, say at σ_2' and σ_1', we know $\psi_1(\sigma_2')$ and $\bar{\psi}_2(\sigma_1')$ to be waves entering into the second and the first waveguides, respectively; therefore

$$\psi_1(\sigma_2')=a_{21}\bar{\psi}_2^*(\sigma_2'), \qquad \bar{\psi}_2(\sigma_1')=b_{12}\psi_1^*(\sigma_1'), \text{ etc.} \tag{3.6}$$

Here a_{21} and b_{12} are numerical constants. Multiplying by $\bar{\psi}_2(\sigma_2')(n_2'\gamma)$ on the left side of the first equation of (3.6), and by $(n_1'\gamma)\psi_1(\sigma_1')$ on the right side of the second equation, and integrating both sides over σ_2' and σ_1', respectively, we find, using also the normalization conditions of (3.1) and (3.4),

$$\int_{\sigma_2'}\bar{\psi}_2(n\gamma)\psi_1 d\sigma=-a_{21}, \quad \int_{\sigma_1'}\bar{\psi}_2(n\gamma)\psi_1 d\sigma=b_{12}, \tag{3.7}$$

and, on account of (3.5),

$$\int_{\sigma_2'+\sigma_1'}\bar{\psi}_2(n\gamma)\psi_1 d\sigma=-a_{21}+b_{12}=0, \text{ or } a_{21}=b_{12}. \tag{3.8}$$

This agrees with the theory of reciprocity in the case of isotropic media. However, when referring to (3.6), we should bear in mind that $\bar{\psi}_2(x)$ is not a solution of Maxwell's equations but of the adjoint equation (3.3).

Using the expression (2.18) for the Poynting vector, and also the normalization condition (3.1) (involving a transmitted power of $c/8\pi$ units), the power W_{21} transmitted into the surface σ_2 per unit power of the transmitter takes the form

$$W_{21}=-\int_{\sigma_2}\psi_1^*(n\gamma)\psi_1 d\sigma, \tag{3.9}$$

which, on deforming the surface σ_2 into σ_2', yields after the substitution of (3.6) for $\psi_1(\sigma_2')$

$$W_{21}=-|a_{21}|^2\int_{\sigma_2'}\bar{\psi}_2(n\gamma)\bar{\psi}_2^* d\sigma=|a_{21}|^2, \tag{3.10}$$

or, in view of (3.7),

$$W_{21}=\left|\int_{\sigma_2'}\bar{\psi}_2(n\gamma)\psi_1 d\sigma\right|^2. \tag{3.11}$$

In the same way, in the inverse case of radiation away from the surface σ_2, the power W_{12} transmitted into the surface σ_1 takes the form

$$W_{12}=\left|\int_{\sigma_1'}\bar{\psi}_1(n\gamma)\psi_2 d\sigma\right|^2. \tag{3.12}$$

When the medium is isotropic, we have $\bar{\psi}_i = \psi_i$, and thus, from (3.7) and (3.8), $W_{21} = W_{12}$, as it should.

From (2.16),

$$\psi_1(x) = \int \psi(x, x') \xi_{\sigma_1}(x') (dx').$$

Hence, we have finally, by deforming σ_2' to σ_2,

$$W_{21} = \left| \iint \bar{\xi}_{\sigma_2}(x) \psi(x, x') \xi_{\sigma_1}(x') (dx)(dx') \right|^2. \tag{3.13}$$

Here, in accordance with (2.17) and (2.21),

$$\bar{\xi}_\sigma(x) = -i\delta(x, \sigma) \bar{\psi}(x)(n\gamma) = i\delta(x, \sigma)(n\gamma) \bar{\psi}(x),$$

with

$$\int \bar{\xi}^*_{\sigma''} \gamma_j \bar{\xi}_{\sigma'} (dx) = n_j \delta(\sigma'', \sigma'), \tag{3.14}$$

which is derived from (3.4).

If we would like to have the cross product of the amplitudes of two different receivers labeled by the subscripts 2 and 3, the corresponding product $W_{32;\,11} = a_{21} a^*_{31}$ would be given, instead of (3.13), by

$$W_{32;\,11} = \left[\iint \bar{\xi}_{\sigma_2}(x) \psi(x, x') \xi_{\sigma_1}(x') (dx)(dx') \right] \times \left[\iint \bar{\xi}_{\sigma_3}(x) \psi(x, x') \xi_{\sigma_1}(x') (dx)(dx') \right]^*. \tag{3.15}$$

4. Fourier Transform and Feynman Diagram Method

We assume here that the medium is changing randomly in space and time, but still has some correlation between two points in space which depends on space-independent averages. Also the time variation is assumed to be very slow as compared with the wave frequency. Let $k = k_o + \delta k$, where, for the time being, we put $k_o = \langle k \rangle$, the time average value of k and thus $\langle \delta k \rangle = 0$. In an anisotropic medium, the k_o generally becomes a matrix operating on ψ.

Then, from (2.14) fixing the Green function $\psi(x, x')$, we find the equation

$$[(\gamma\partial) - k_o] \psi(x, x') = \delta k(x) \psi(x, x') + \delta(x - x') \qquad k = k_o + \delta k, \tag{4.1}$$

whose solution is, using the Green function $S(x - x')$ of (2.23) for the homogeneous medium k_o,

$$\psi(x, x') = \int S(x - x'') \delta k(x'') \psi(x'', x') (dx'') + S(x - x') = \psi_0(x, x') + \psi_1(x, x') + \psi_2(x, x') + \ \ldots \tag{4.2}$$

Here

$$\psi_0(x, x') = S(x - x'),\ \psi_1(x, x') = \int (dx'') S(x - x'') \delta k(x'') S(x'' - x'),$$

$$\psi_2(x, x') = \int (dx'') \int (dx''') S(x - x'') \delta k(x'') S(x'' - x''') \delta k(x''') S(x''' - x'), \text{ etc.} \tag{4.3}$$

On the other hand, according to (3.13), the time average $\langle W_{21} \rangle$ of the transmission power gain W_{21} is given by

$$\langle W_{21} \rangle = \iiiint (dx_2)(dx_1)(dx_2')(dx_1') \langle \xi^*_{\sigma_1}(x_1') \psi^\dagger(x_1', x_2') \bar{\xi}^*_{\sigma_2}(x_2') \times \bar{\xi}_{\sigma_2}(x_2) \psi(x_2, x_1) \xi_{\sigma_1}(x_1) \rangle = \sum_{i,j=0}^{\infty} T_{ij}. \tag{4.4}$$

Here $\psi^\dagger$ is the Hermitian conjugate of ψ and defined as $\psi^\dagger(x_1', x_2')$ = complex conjugate of the transposed function $\psi^\tau(x_2', x_1')$. Further, in view of (4.3),

$$T_{ij}=\iiiint(dx_2)(dx_1)(dx_2')(dx_1')\langle\xi^*_{\sigma_1}(x_1')\psi_i^\dagger(x_1',\,x_2')\bar{\xi}^*_{\sigma_2}(x_2')\times\bar{\xi}_{\sigma_2}(x_2)\psi_j(x_2,\,x_1)\xi_{\sigma_1}(x_1)\rangle \tag{4.5}$$

or, in an abbreviated representation,

$$T_{ij}=\langle(\xi^*_{\sigma_1}|\psi_i^\dagger|\bar{\xi}^*_{\sigma_2})(\bar{\xi}_{\sigma_2}|\psi_j|\xi_{\sigma_1})\rangle$$

with

$$(\bar{\xi}_{\sigma_2}|\psi_j|\xi_{\sigma_1})=\iint(dx_2)(dx_1)\bar{\xi}_{\sigma_2}(x_2)\psi_j(x_2,\,x_1)\xi_{\sigma_1}(x_1).$$

Generally, δk will have the form

$$\delta k(x)=\beta\Delta\mathcal{E}(x). \tag{4.6a}$$

Here β is a 6×6 nondiagonal matrix in the case of an anisotropic medium, having the dimension of a propagation constant, and $\Delta\mathcal{E}(x)$ is always a dimensionless scalar such as the fluctuating part of the dielectric constant of the medium. In the case of an isotropic medium, β has the form, in view of (2.6),

$$\beta=\frac{1}{2}(1+\rho_3)(\omega/c). \tag{4.6b}$$

Substituting the expression (4.6a) in (4.5), we see that the average value appears only through the term

$$D(x_1,\,x_2,\,x_3,\,\ldots,\,x_n)=\langle\Delta\mathcal{E}(x_1)\Delta\mathcal{E}(x_2)\;\ldots\;\Delta\mathcal{E}(x_n)\rangle, \tag{4.7}$$

which may be represented with the aid of its Fourier transform $D(t_1,\,t_2,\,\ldots,\,t_n)$, in the form

$$\iint_{-\infty}^{\infty}\ldots\int(dt_1)(dt_2)\ldots(dt_n)e^{-i\{t_1x_1+t_2x_2+\ldots+t_nx_n\}}D(t_1,\,t_2,\,\ldots,\,t_n). \tag{4.8}$$

On the other hand, the function $D(x_1,\,x_2,\,\ldots,\,x_n)$ has two important properties: its invariance with regard to parallel translations and simultaneous rotations of the points x_1, $x_2,\,\ldots,\,x_n$, and also the symmetry with regard to these points. Owing to these invariances, the functional form of $D(t_1,\,t_2,\,\ldots,\,t_n)$ is very restricted: from the translation invariance for $x_i\to x_i+l$, l being an arbitrary vector, we see by the use of (4.8) that the factor $\exp[-i(t_1+t_2+\ldots+t_n)l]$ should effectively be equal to 1 when multiplied by $D(t_1,\,t_2,\,\ldots,\,t_n)$. This implies that, also taking into account the rotational invariance, the latter takes the form

$$D(t_1,\,t_2,\,\ldots,\,t_n)=\delta(t_1+t_2+\;\ldots\;+t_n)D_n(t_i^2,\,t_it_j), \tag{4.9}$$

and thus the vector sum $t_1+t_2+\;\ldots\;+t_n$ should always be equal to zero. Here $D_n(t_i^2,\,t_it_j)$ stands for a function which is symmetric with regard to the indices i and j of the indicated independent variables.

Substituting the expression (4.9) into (4.8), we finally arrive at

$$D(x_1,\,x_2,\,\ldots,\,x_n)=\iint_{-\infty}^{\infty}\ldots\int(dt_1)(dt_2)\ldots(dt_n)\delta(t_1+t_2+\ldots+t_n)$$
$$\times D_n(t_i^2,\,t_it_j)e^{-i(t_1x_1+t_2x_2+\ldots+t_nx_n)}. \tag{4.10}$$

Thus we also infer the invariance for coordinate inversion

$$D(-x_1,-x_2,\,\ldots,-x_n)=D(x_1,\,x_2,\,\ldots,\,x_n).$$

In the same way, we define the Fourier transforms $\xi_\sigma(t)$ of $\xi_\sigma(x)$, and $\bar{\xi}_\sigma(t)$ of $\bar{\xi}_\sigma(x)$ by

$$\xi_\sigma(x)=(2\pi)^{-3/2}\int_{-\infty}^{\infty}\xi_\sigma(t)e^{-i(tx)}(dt),$$

$$\bar{\xi}_\sigma(x)=(2\pi)^{-3/2}\int_{-\infty}^{\infty}\bar{\xi}_\sigma(t)e^{+i(tx)}(dt); \tag{4.11a}$$

hence, if $k^T=k$ (see sec. 3)

$$\bar{\xi}_\sigma(t)=\xi_\sigma(-t),\ k^T=k. \tag{4.11b}$$

The normalization condition (2.21) then takes the form

$$\int_{-\infty}^{\infty}\xi^*_{\sigma''}(t)\gamma_j\xi_{\sigma'}(t)(dt)=n_j\delta(\sigma'',\sigma'). \tag{4.12}$$

Also, from (2.24), the Green function $S(x)$ and its Hermitian conjugate function $S^\dagger(x)$ are found to be

$$S(x)=(2\pi)^{-3}\int_{-\infty}^{\infty}S(t)e^{-i(tx)}(dt),\quad S^\dagger(x)=(2\pi)^{-3}\int_{-\infty}^{\infty}S^\dagger(t)e^{-i(tx)}(dt),\quad S^\dagger(t)=S(t)|_{k\to k^\dagger}, \tag{4.13}$$

where $k^\dagger$ is Hermitian conjugate to k.

It would be instructive to know the Fourier transformed form of T_{11}. From (4.3) and (4.5), we derive, with the aid of (4.6a) and (4.7),

$$T_{11}=\iiiint\!\!\iint(dx_1')(dx_2')(dx')(dx_1)(dx_2)(dx)\xi^*_{\sigma_1}(x_1')S^\dagger(x_1'-x')\beta S^\dagger(x'-x_2')\bar{\xi}^*_{\sigma_2}(x_2')$$
$$\times\bar{\xi}_{\sigma_2}(x_2)S(x_2-x)\beta S(x-x_1)\xi_{\sigma_1}(x_1)\cdot D(x'-x). \tag{4.14a}$$

Here, according to (4.10),

$$D(x'-x)=\langle\Delta\varepsilon(x')\Delta\varepsilon(x)\rangle=\iint_{-\infty}^{\infty}(dt')(dt)\delta(t+t')D(t^2)e^{-i(t'x'+tx)}=\int_{-\infty}^{\infty}D(t^2)e^{-it(x'-x)}(dt). \tag{4.14b}$$

Evaluating (4.14a) with the aid of (4.13) and (4.11a), we arrive at

$$T_{11}=\iiiint\!\!\int(dt_1')(dt_2')(dt_1)(dt_2)(dt)\xi^*_{\sigma_1}(t_1')S^\dagger(t_1')\beta S^\dagger(t_2')\bar{\xi}^*_{\sigma_2}(t_2')\delta(-t_1'+t_2'+t)$$
$$\times\bar{\xi}_{\sigma_2}(t_2)S(t_2)\beta S(t_1)\xi_{\sigma_1}(t_1)\delta(t_1-t_2-t)D(t^2). \tag{4.15}$$

Here, the terms $\delta(t_1-t_2-t)$ and $\delta(-t_1'+t_2'+t)$ result from the integrations over x and x', respectively.

Equation (4.15) finally takes the form

$$T_{11}=\int_{-\infty}^{\infty}D(t^2)f^*(t)f(t)(dt)=\langle D\rangle_f, \tag{4.16a}$$

in which

$$f(t)=\int_{-\infty}^{\infty}\bar{\xi}_{\sigma_2}(t_1-t)S(t_1-t)\beta S(t_1)\xi_{\sigma_1}(t_1)(dt_1) \tag{4.17}$$

constitutes an ordinary scalar function. This integration will be evaluated in Part III for arbitrary functions for ξ_{σ_1} and ξ_{σ_2}.

In the same way, for the cross correlation of the amplitudes at two receivers, we derive, starting from (3.15),

$$W_{32;\,11}=\int_{-\infty}^{\infty}D(t^2)f^*_{31}(t)f_{21}(t)(dt), \tag{4.16b}$$

which is the cross correlation in the first approximation with respect to $D(t^2)$. Here f_{21} is the function given in (4.17) and f_{31} the corresponding function referring to the receiver labeled by the subscript 3 instead of 2. In Part III, we shall see that $f^*_{31}(t)f_{21}(t)$ does oscillate more rapidly with t according as the distance $|x_3-x_2|$ increases, and thus the integral becomes smaller.

The integral (4.15) can be interpreted graphically in the following way (fig. 2a): On a sheet of paper, we first mark four points corresponding to ξ_{σ_1}, $\bar{\xi}_{\sigma_2}$, $\xi^*_{\sigma_1}$ and $\bar{\xi}^*_{\sigma_2}$, respectively, and the two points s and s' corresponding to the two scattering points. Then draw the solid line from the point ξ_{σ_1} to s and from s to $\bar{\xi}_{\sigma_2}$. Here the direction is taken away from the point ξ_{σ_1}; it corresponds to the propagation with the wave-number vector t_1 away from ξ_{σ_1} (connected with the transmitter, for instance) to the scattering point s. On the other hand, from the end point of this vector we draw a line towards the point $\bar{\xi}_{\sigma_2}$; it corresponds to the propagation into $\bar{\xi}_{\sigma_2}$ (connected with the receiver) with the wave-number vector t_2.

In the same way, we draw the solid line from the point $\bar{\xi}^*_{\sigma_2}$ to s', and from s' to $\xi^*_{\sigma_1}$. But the directions of these complex conjugated lines are to be taken just inverse to those of the corresponding original lines. Hence, the new solid line is directed towards the point $\xi^*_{\sigma_1}$, with the corresponding vector t'_1. Finally we connect the points s and s' by a broken line with any prescribed direction of the vector t.

Now there exists a one-to-one correspondence between the lines thus drawn in the diagram, and every factor in the integrand of (4.15). First of all, the factors $\xi_{\sigma_1}(t_1)$ and $\bar{\xi}_{\sigma_2}(t_2)$ in the integrand fix the antenna patterns of the transmitter and receiver in the spaces enclosed by the surfaces σ_1 and σ_2, respectively; they correspond to the points ξ_{σ_1} and $\bar{\xi}_{\sigma_2}$ in the diagram. In the same way, the factors $\xi^*_{\sigma_1}(t'_1)$ and $\bar{\xi}^*_{\sigma_2}(t'_2)$ correspond to the points $\xi^*_{\sigma_1}$ and $\bar{\xi}^*_{\sigma_2}$. Then the vector line t_1 from the point ξ_{σ_1} to s in the diagram corresponds to the factor $S(t_1)$ and, in the same way, the vector lines t_2, t'_1 and t'_2 to the factors $S(t_2)$, $S^{\dagger}(t'_1)$ and $S^{\dagger}(t'_2)$, respectively. The vertices s and s' are situated in accordance with the vector relations $t=t_1-t_2=t'_1-t'_2$, thus expressing geometrically the effect of the momentum functions $\beta\delta(t_1-t_2-t)$ and $\beta\delta(-t'_1+t'_2+t)$, respectively. Finally the broken line represents the last factor $D(t^2)$ of the integrand of (4.15).

Since the mentioned factors are all 6×6 matrices or one-row or one-column vectors of six elements, their orders of succession in the integrand are important: from the right to the left these orders correspond to the directions of the solid lines, thus beginning with the factor ξ_{σ_1} and ending with the factor $\bar{\xi}_{\sigma_2}$ for the solid line considered first, while beginning with the factor $\bar{\xi}^*_{\sigma_2}$ and ending with the factor $\xi^*_{\sigma_1}$ for the complex conjugated second line. On the other hand, broken lines will always represent an ordinary scalar function, that is $D(t^2)$ in the case under consideration.

Though this is just an example of the way for representing the integrand of the multifold integral for T_{11} by a corresponding diagram, its method is quite general and can be applied to all other cases. There is one such diagram for each T_{ij} occurring in (4.4), and any of them comprises two solid lines, the one being the original line starting from $\xi_{\sigma 1}$ and ending at $\bar{\xi}_{\sigma 2}$, and the other the complex conjugate line starting from $\bar{\xi}^*_{\sigma_2}$ and ending at $\xi^*_{\sigma_1}$; in the case of the diagram for T_{ij}, there are j vertices on the original solid line from each of which one broken line is branching off; the corresponding property holds for the i vertices on the conjugate solid line. All these broken lines are connected to the "domain" of D which, including all of them, corresponds to the factor $D(t_1, t_2, \ldots, t_{i+j})$ introduced in (4.9) (fig. 3). The other correspondences of each solid line and each vertex in the diagram with the corresponding factors of the integrand are just the same as explained above for T_{11}.

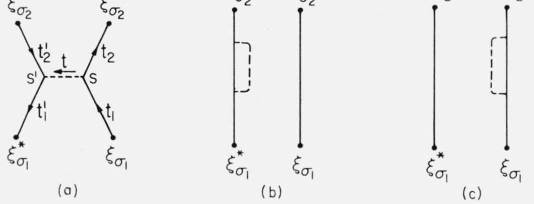

FIGURE 2. *Feynman diagrams for* T_{11}, T_{20}, *and* T_{02}.

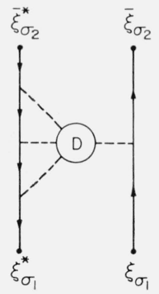

FIGURE 3. *Feynman diagram for* T_{31} *in terms of* $D(t_1, t_2, t_3, t_4)$.

The number of broken lines or bonds of D is equal to the number of factors δk involved in the integrand, and generally there are accordingly $(i+j)$ bonds for the special term T_{ij}. Therefore, for the second-order approximation terms, we have to do with the three terms T_{11}, T_{20}, and T_{02}, the diagrams of which are given by figures 2a, 2b, and 2c, respectively. In the higher approximation of nth order, only the sum of terms T_{ij} with $i+j=n$ would have a clear physical meaning, but in most cases, not the other individual terms.

It remains to discuss the function $D(x_1, x_2, \ldots, x_n)$ or $D_n(t_i^2.\ t_it_j)$ which is of the nth order with respect to $\Delta\mathcal{E}(x)$. Of course $\langle\Delta\mathcal{E}(x)\rangle=0$ and further, assuming a multivariate normal distribution of $\Delta\mathcal{E}(x)$ values at different points, it is well known[6] that

$$\langle\Delta\mathcal{E}(x_1)\Delta\mathcal{E}(x_2)\Delta\mathcal{E}(x_3)\ \ldots\ \Delta\mathcal{E}(x_{2n})\rangle=\Sigma_p\langle\Delta\mathcal{E}(x_1)\Delta\mathcal{E}(x_2)\rangle\langle\Delta\mathcal{E}(x_3)\Delta\mathcal{E}(x_4)\rangle\ \ldots\ \langle\Delta\ x_{2n-1})\Delta\mathcal{E}(x_{2n})\rangle, \quad (4.18)$$

$$\langle\Delta\mathcal{E}(x_1)\Delta\mathcal{E}(x_2)\ \ldots\ \Delta\mathcal{E}(x_{2n+1})\rangle=0.$$

Here Σ_p stands for the sum of all the terms obtained by the various combinations of different pairs composing the set of variables $x_1, x_2, \ldots, x_{2n}$.

The right side of (4.18) is given in terms of the correlation function $D(x)$ between only two points and this fact makes the computation very simple; first of all any diagram having an odd number of vertices yields no contribution to the result, and the function $D(t_1, t_2, \ldots, t_{2n})$ can be divided into $(2n-1)\ (2n-3)\ \ldots\ 1$ independent parts depending only on the function $D(t^2)$.

For instance, the diagram of $D(t_1, t_2, t_3, t_4)$ can be expressed as the sum of the three independent diagrams of figures 4a, 4b, and 4c, and thus the diagram for T_{31} is divided likewise into the three diagrams shown in figures 5a, 5b, and 5c. The contribution from each of these diagrams can be easily formulated in the integral form according to the correspondence rule stated above.

The number of diagrams, however, rapidly increases with the order n, and the diagram method would become impractical. But, just as in the case of quantum electrodynamics, a renormalization method can be applied. The sum of contributions of all the self-energy parts, such as shown in figure 6a, and the sum of contributions of all the vertex parts, such as those of figure 6b, can then formally be taken into account by the use of an effective value of k, and another effective value of the matrix β introduced in (4.6a), respectively. This procedure is in accordance with the method developed by Dyson [1949a, 1949b]; we then only need to consider irreducible graphs, that is, graphs having no self-energy parts and no vertex parts at all. In the next section, the theory of renormalization, especially of k, will extensively be discussed. In particular, the imaginary part of the effective value of k which results from the renormalization, and which represents physically the attenuation of waves due to the scattering by the medium, will be computed to a first approximation, together with the corresponding dispersion relation.

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[6] See, for example, Middleton: Introduction to Statistical Communication Theory, ch. 7- sec. 7.3 (7.28), (McGraw-Hill Book Co., 1960).

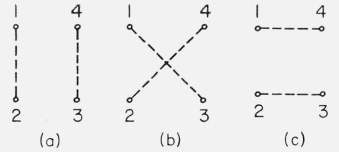


FIGURE 4. *Decomposition of* D(t_1, t_2, t_3, t_4) *into the three independent diagrams in the case of a multivariate Gaussian distribution.*

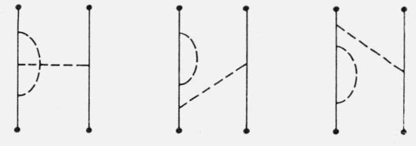
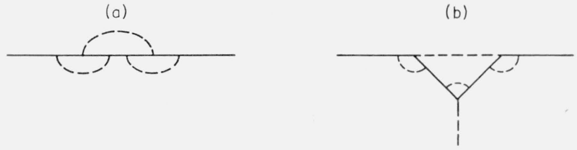


FIGURE 5. *Three independent diagrams for* T_{31}.

FIGURE 6 (a) *Example of self-energy part;*
(b) *example of vertex part.*



5. Effective Medium Constant and Effective Fluctuating Medium-Field Coupling Constant (Theory of Renormalization)

Here, (4.1) for the Green function is replaced by

$$[(\gamma\partial)-k_e]\psi(x, x')=\delta\mathbf{k}(x)\psi(x, x')+\delta(x-x'), \qquad k=k_e+\delta\mathbf{k}. \tag{5.1}$$

Hence, by averaging over the time, we obtain

$$[(\gamma\partial)-k_e]\langle\psi(x, x')\rangle=\langle\delta\mathbf{k}(x)\psi(x, x')\rangle+\delta(x-x'). \tag{5.2}$$

Here, k_e is the time-independent effective value of k, which is different from k_o of (4.1), and which is to be defined by

$$k_e\langle\psi(x, x')\rangle=(\gamma\partial)\langle\psi(x, x')\rangle-\delta(x-x'). \tag{5.3a}$$

Thus k_e could be an operator operating on $\langle\psi\rangle$, but it will be assumed as a space-independent constant for a while. Since $\langle\psi(x'', x')\rangle$ should depend only on the coordinate difference $x''-x'$, its Fourier integral representation will have the form

$$\langle\psi(x'', x')\rangle=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}S_e(t)e^{-it(x''-x')}(dt), \tag{5.4}$$

which is similar to the formula (4.13) for $S(x)$. Here, according to (5.3a), $S_e(t)$ satisfies the equation

$$[(\gamma t)-k_e]S_e(t)=1, \qquad S_e(t)=[(\gamma t)-k_e]^{-1}, \tag{5.5}$$

from which we conclude inversely that $\langle\psi(x, x')\rangle$ satisfies the adjoint equation

$$\langle\psi(x, x')\rangle[-(\gamma\overleftarrow{\partial}')-k_e]=\delta(x-x'). \tag{5.3b}$$

It will be seen later that the k_e in (5.5) is generally not constant but depends on t. This situation corresponds to replacing the left side of (5.3a) by

$$k_e\langle\psi(x, x')\rangle\rightarrow\int k_e(x-x'')\langle\psi(x'', x')\rangle(dx''), \tag{5.6}$$

where

$$k_e(x)=\frac{1}{(2\pi)^3}\int k_e(t)e^{-i(tx)}(dt). \tag{5.7}$$

The Fourier transform of (5.3a) then leads to (5.5).

Now, from (5.2) and (5.3a), we infer

$$\langle\delta\mathbf{k}(x)\psi(x, x')\rangle=0, \quad \langle\psi(x, x')\delta\mathbf{k}(x')\rangle=0, \quad \langle\delta\mathbf{k}(x)\rangle=\delta\kappa\neq0. \tag{5.8}$$

Here we put, as in section 4,

$$\langle k\rangle=k_e+\delta\kappa=k_o. \tag{5.9}$$

By comparing (5.3a) with (2.23) we find, identifying the constant k in the latter equation with the present k_o,

$$\langle\psi(x, x')\rangle|_{k_e\rightarrow k_o}=S(x-x').$$

Equation (5.3a) is then solved for arbitrary k_e by

$$\langle\psi(x, x')\rangle = -\int S(x-x'')\delta\kappa\langle\psi(x'', x')\rangle(dx'') + S(x-x'). \tag{5.10}$$

Here $\delta\kappa\langle\psi\rangle$ should be interpreted as in (5.6).

On the other hand, in view of the wave equations (5.3a) and (5.3b) for $\langle\psi(x, x')\rangle$, the solution of (5.1) can be given in the form

$$\begin{aligned}\psi(x, x') &= \int\langle\psi(x, x'')\rangle\delta\mathbf{k}(x'')\psi(x'', x')(dx'') + \langle\psi(x, x')\rangle \\ &= \int\psi(x, x'')\delta\mathbf{k}(x'')\langle\psi(x'', x')\rangle(dx'') + \langle\psi(x, x')\rangle.\end{aligned} \tag{5.11}$$

Here $\delta\mathbf{k}(x) - \delta\kappa$ should be equal to $\delta k(x)$ defined in (4.6a). Hence,

$$\delta\mathbf{k}(x) - \delta\kappa = \beta\Delta\mathcal{E}(x), \qquad \langle\Delta\mathcal{E}(x)\rangle = 0, \tag{5.12}$$

and thus, substituting (5.12) in (5.8),

$$\delta\kappa\langle\psi(x, x')\rangle + \beta\langle\Delta\mathcal{E}(x)\psi(x, x')\rangle = 0. \tag{5.13}$$

It will be convenient to introduce now the "vertex function" $B(x_1; x_3; x_2)$, to be defined by

$$\langle\psi(x_1, x_2)\Delta\mathcal{E}(x_3)\rangle = \int\langle\psi(x_1, x')\rangle B(x'; x'''; x'')\langle\psi(x'', x_2)\rangle D(x'''-x_3)(dx')(dx'')(dx'''). \tag{5.14}$$

The Fourier transform of B will have the form

$$B(x_1; x_3; x_2) = \frac{1}{(2\pi)^6}\int\delta(t_1+t_2+t_3)B(-t_1, t_2)e^{i\{(t_1x_1)+(t_2x_2)+(t_3x_3)\}}(dt_1)(dt_2)(dt_3). \tag{5.15}$$

Here the δ-function appears because the left side of (5.15) should be invariant for a parallel displacement of the vectors x_1, x_2, and x_3, just as in (4.10). Applying the first expression of (5.11) to $\psi(x_1, x_2)$ on the left side of (5.14) and also using (5.12), we have

$$\langle\psi(x_1, x_2)\Delta\mathcal{E}(x_3)\rangle = \int\langle\psi(x_1, x')\rangle\langle\delta\mathbf{k}(x')\psi(x', x_2)\Delta\mathcal{E}(x_3)\rangle(dx')$$

which gives, applying the second expression of (5.11) to $\psi(x', x_2)$ on the right side,

$$\begin{aligned}&\int\langle\psi(x_1, x')\rangle\beta\langle\Delta\mathcal{E}(x')\Delta\mathcal{E}(x_3)\rangle\langle\psi(x', x_2)\rangle(dx') \\ &\qquad + \int\langle\psi(x, x')\rangle\langle\delta\mathbf{k}(x')\psi(x', x'')\delta\mathbf{k}(x'')\Delta\mathcal{E}(x_3)\rangle\langle\psi(x'', x_2)\rangle(dx')(dx'').\end{aligned}$$

Thus we infer, since the above equation should be identical with the right side of (5.14),

$$\int B(x_1; x; x_2)D(x-x_3)(dx) = \beta\delta(x_1-x_2)D(x_1-x_3) + \langle\delta\mathbf{k}(x_1)\psi(x_1, x_2)\delta\mathbf{k}(x_2)\Delta\mathcal{E}(x_3)\rangle. \tag{5.16}$$

Hence B becomes in the lowest order of approximation with respect to $\Delta\mathcal{E}$

$$B(x_1; x_3; x_2) \sim \beta\delta(x_1-x_3)\delta(x_2-x_3),$$

which agrees with (5.15) if $B(t_1, t_2)$ is given by the corresponding approximation

$$B(t_1, t_2) \sim \beta. \tag{5.17}$$

When the values of $\Delta\mathcal{E}(x)$ follow the Gaussian distribution, it will be proven in Part II of this series of papers that, for any functional $f(\Delta\mathcal{E}[\Sigma])$ of $\Delta\mathcal{E}(x)$ over the space Σ[7],

$$\langle\Delta\mathcal{E}(x)f(\Delta\mathcal{E}[\Sigma])\rangle = \int_\Sigma D(x-x')\left\langle\frac{\delta}{\delta\Delta\mathcal{E}(x')}f(\Delta\mathcal{E}[\Sigma])\right\rangle(dx'). \tag{5.18}$$

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[7] The notation $f(\Delta\mathcal{E}[\Sigma])$ means that the function f involves the variables $\Delta\mathcal{E}(x)$ at all points of the space Σ; for example, any waves are functional of the medium through which they propagate.

Here the functional derivative is defined as follows: When, for arbitrary variation $\delta\Delta\mathcal{E}(x)$, the change δf can be represented in its first order by the form

$$\delta f(\Delta\mathcal{E}[\Sigma])=\int_{\Sigma}\left\{\frac{\delta}{\delta\Delta\mathcal{E}(x)}f(\Delta\mathcal{E}[\Sigma])\right\}\delta\Delta\mathcal{E}(x)\,(dx), \tag{5.19}$$

the factor of $\delta\Delta\mathcal{E}(x)$ in the integrand is defined to be the functional derivative with respect to $\Delta\mathcal{E}(x)$.

Applying the lemma (5.18) to the second term on the right side of (5.16), we find, taking into account (5.12) and also the relations [8]

$$\delta(\delta\mathbf{k}(x))/\delta\Delta\mathcal{E}(x')=\beta\delta(x-x'), \tag{5.20}$$

$$\delta\psi(x_1,\,x_2)=\int\psi(x_1,\,x')\beta\delta\Delta\mathcal{E}(x')\psi(x',\,x_2)\,(dx'),$$

together with (5.8), that

$$\langle\delta\mathbf{k}(x_1)\psi(x_1,\,x_2)\delta\mathbf{k}(x_2)\Delta\mathcal{E}(x_3)\rangle=\int D(x_3-x)\langle\delta\mathbf{k}(x_1)\psi(x_1,\,x)\beta\psi(x,\,x_2)\delta\mathbf{k}(x_2)\rangle\,(dx). \tag{5.21}$$

Hence, it follows from (5.16) that

$$B(x_1;\,x_3;\,x_2)=\beta\delta(x_1-x_3)\delta(x_2-x_3)+\langle\delta\mathbf{k}(x_1)\psi(x_1,\,x_3)\beta\psi(x_3,\,x_2)\delta\mathbf{k}(x_2)\rangle. \tag{5.22}$$

From (5.22), we can get B in any order of approximation by successive application of (5.11) and (4.18).

Now, substituting the integral representations (5.4), (5.15), and (4.14b), and evaluating the integrations leading to impulse function, the integral (5.14) reduces to (fig. 7)

$$\langle\psi(x_1,x_2)\Delta\mathcal{E}(x_3)\rangle=\frac{1}{(2\pi)^3}\int S_e(t_1)B(t_1,\,t_2)S_e(t_2)D((t_1-t_2)^2)e^{-i\{(t_1(x_1-x_3))+(t_2(x_3-x_2))\}}(dt_1)(dt_2). \tag{5.23}$$

Hence, (5.13) becomes

$$\delta\kappa\langle\psi(x,\,x')\rangle+\int\left[\beta\int S_e(t_1)B(t_1,\,t_2)D((t_1-t_2)^2)(dt_1)\right]\frac{1}{(2\pi)^3}\,S_e(t_2)e^{-i\{t_2(x-x')\}}(dt_2)=0.$$

Thus, applying (5.6) and (5.7), we arrive at the Fourier transform of $\delta\kappa$, i.e.,

$$-\delta\kappa(t)=\int\beta S_e(t_1)B(t_1,\,t)D((t_1-t)^2)\,(dt_1). \tag{5.24a}$$

Here the Feynman diagram of $\delta\kappa$ is represented in figure 8.

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[8] The first equation of (5.20) follows from the identity

$$\Delta\mathcal{E}(x)=\int_{\Sigma}\delta(x-x')\Delta\mathcal{E}(x')(dx').$$

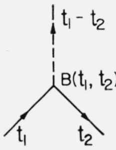

FIGURE 7. *Feynman diagram for (5.23).*

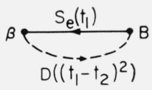

FIGURE 8. *Diagram for (5.24a).*

In the same way, we derive from (5.10) the following Fourier transform $S_e(t)$ of $\langle\psi(x'', x')\rangle$ (see (5.4)):

$$S_e(t)=S(t)-S(t)\delta\kappa(t)S_e(t)=S(t)[1-\delta\kappa(t)S_e(t)]. \tag{5.24b}$$

In a first order of approximation, $S_e(t)$ is given by

$$S_e(t)\sim S(t)[1-\delta\kappa(t)S(t)],$$

and, substituting (5.17) in (5.24a), $\delta\kappa(t)$ accordingly by

$$-\delta\kappa(t)\sim\int\beta S(t_1)\beta D((t_1-t)^2)(dt_1). \tag{5.25}$$

In the case of an isotropic medium, the $\delta\kappa(t)$ defined in (5.24a) behaves as if it were a constant whenever it operates on the solution $\xi(t)$ of the wave equation (compare (2.25))

$$[(\gamma t)-k_e]\xi(t)=0, \tag{5.26}$$

which involves $t^2=\bar{k}_e k_e=k_e'^2$. The reason for this behavior as a constant is the following: Since $\delta\kappa(t)$ should be invariant with respect to space-coordinate rotations, it should be a function of t^2, (γt), $(\gamma t)^2$, $((\gamma t)^3=t^2(\gamma t))$, and thus it can be expressed in the form

$$\delta\kappa(t)=\delta\kappa_0(t^2)+\delta\kappa_1(t^2)[(\gamma t)-k_e]+\delta\kappa_2(t^2)[(\gamma t)-k_e]^2. \tag{5.27}$$

Hence, when operating on $\xi(t)$ of (5.26), $\delta\kappa(t)$ is equivalent to the constant of $\delta\kappa_0(k_e'^2)$ which should be equal to k_o-k_e according to (5.9). Thus

$$\delta\kappa_0(k_e'^2)=k_0-k_e \tag{5.28}$$

gives the equation to determine k_e. Also in the case of anisotropic media, we would have the same conclusion that $\delta\kappa(t)$ behaves as if it is a constant matrix whenever operating on the wave function of (5.26).

5.1. Evaluation of the Effective Medium Constant for an Isotropic Medium (in First Order of Approximation)

It will be interesting to compute the $\delta\kappa$ as defined by (5.9), especially its imaginary part, in a first order of approximation in the case of an isotropic medium. Then, from (5.25), applying (2.29),

$$-\delta\kappa(t)=\int_{-\infty}^{\infty}(t_1^2-k'^2)^{-1}\beta g(t_1)\beta D((t_1-t)^2)(dt_1). \tag{5.29}$$

Here, when the medium fluctuation is due to the dielectric constant, we can derive the following relations with the aid of (4.6b) and (2.29):

$$\beta=\tfrac{1}{2}(1+\rho_3)(\omega/c),\quad \beta^2=(\omega/c)\beta,\quad k\beta=(\omega/c)\mathcal{E}\beta,\quad \beta g(t_1)\beta=\mathcal{E}^{-1}\{(t_1\gamma)^2-t_1^2+k'^2\}\beta, \tag{5.30}$$

and thus (5.29) takes the form

$$\delta\kappa(t)-\beta\mathcal{E}^{-1}\int_{-\infty}^{\infty}D((t_1-t)^2)(dt_1)=-\beta\mathcal{E}^{-1}\int_{-\infty}^{\infty}(t_1^2-k'^2)^{-1}(\gamma t_1)^2D((t_1-t)^2)(dt_1),\qquad \mathrm{Im}(k')<0. \tag{5.31}$$

Since the correlation function $D(x)$ is an even real function of the space coordinates x, the Fourier-transformed function $D(t^2)$ is a real function of t. Also $\beta g(t_1)\beta$ is real in so far as it is Hermitian, and thus has only real eigenvalues.

On the other hand, k' and k'^2 should have an infinitesimal negative imaginary part (see the end of sec. 2). Hence, in the integrations,

$$(t_1^2-k'^2)^{-1}=P(t_1^2-k'^2)^{-1}-\pi i\delta(t_1^2-k'^2). \tag{5.32a}$$

Here, P denotes the Cauchy principal part. Thus

$$\mathrm{Im}\,[\delta\kappa(t)]=\beta\pi\mathcal{E}^{-1}\int_{-\infty}^{\infty}\delta(t_1^2-k'^2)(\gamma t_1)^2 D((t_1-t)^2)(dt_1),$$

$$\mathrm{Re}\,[\delta\kappa(t)]-\beta\mathcal{E}^{-1}\int_{-\infty}^{\infty}D((t_1-t)^2)(dt_1)=-\beta\mathcal{E}^{-1}\int_{-\infty}^{\infty}P(t_1^2-k'^2)^{-1}(\gamma t_1)^2 D((t_1-t)^2)(dt_1). \tag{5.32b}$$

The integral for Im $[\delta\kappa(t)]$ should be a function of t^2, (γt) and $(\gamma t)^2$ in view of the same invariance discussed when dealing with (5.27); thus, when operating on the wave function $\xi(t)$ of (5.26), it behaves as a constant. In this case, the factor $(\gamma t_1)^2$ of the integrand can be replaced by

$$[\xi^*(t)(\gamma t_1)^2\xi(t)]/[\xi^*(t)\xi(t)]=\tfrac{1}{2}t_1^2(1+\cos^2\theta). \tag{5.33}$$

This result is easily obtained by using the method to be described in Part III. The right side of (5.33) constitutes the mean value for the two independent polarizations of $\xi(t)$, and θ is the angle between t_1 and t. Thus, putting $(dt_1)=|t_1|^2 d|t_1|d\Omega$ and $d\Omega=2\pi\sin\theta d\theta$, (5.32b) yields, when evaluating the integration with respect to $|t_1|$,

$$\mathrm{Im}\,(\delta\kappa)=\beta\frac{\pi}{4}k'^3\mathcal{E}^{-1}\int(1+\cos^2\theta)D((2k'\sin\theta/2)^2)d\Omega. \tag{5.34}$$

Hence, from (5.28), we find

$$\mathrm{Im}\,(k_e)=-\beta k'^{-1}\mathcal{E}\int\frac{1}{2}(1+\cos^2\theta)\sigma(\theta,k')d\Omega. \tag{5.35}$$

Here

$$\sigma(\theta,k')=(\pi/2)k'^4\mathcal{E}^{-2}D((2k'\sin\theta/2)^2), \tag{5.36}$$

which is interpreted usually as the scattering cross section per unit volume from the view point of scattering loss.

5.2. Dispersion Relation

A well defined dispersion relation existing between $\delta\kappa_{\mathrm{I}}$, the anti-Hermitian part of $\delta\kappa$, and $\delta\kappa_{\mathrm{R}}$, the Hermitian part, follows from (5.32b). This relation makes it possible to express $\delta\kappa_{\mathrm{R}}$ in terms of $\delta\kappa_{\mathrm{I}}$; by the change of variable $t_1-t=t'$, the right side of (5.31) takes the following form, when Im (k') tends to zero:

$$\text{Equation (5.31)}=-\beta\mathcal{E}^{-1}\int_{-\infty}^{\infty}\frac{\{(t+t')\cdot\gamma\}^2}{(t+t')^2-k'^2+i\epsilon}D(t'^2)(dt'). \tag{5.37}$$

Here, ϵ is an infinitesimal positive real number. From now on, the integral (5.37) will be understood as operating on $\xi(t)$ of (5.26). Hence, the term $\{(t+t')\cdot\gamma\}^2$ in the integrand can be replaced by its mean value for the two independent polarizations of $\xi(t)$, and thus by the right side of (5.33) with $t^2=k'^2$.

Since $\epsilon>0$, we recognize the integral (5.37) as an analytic function of k'^2 in the lower half complex plane. Further, taking into account that $\delta\kappa_{\mathrm{I}}$, which is given by (5.34), will become the leading term of the left-hand member of (5.31) for $k'^2\to+\infty$, we see that, for $k'\to\infty$, this member tends to

$$i\beta(\pi/2)k'^3\mathcal{E}^{-1}\int D((k'\theta)^2)d\Omega\bigg|_{k'\to\infty}=i\beta\pi^2k'\mathcal{E}^{-1}\int_0^{\infty}D(t^2)tdt\sim k'.$$

On the other hand, for $k'\to 0$, (5.37) becomes

$$-\beta\mathcal{E}^{-1}\frac{2}{3}\int_{-\infty}^{\infty} D(t'^2)(dt')+0[k'].$$

Hence, in view of (5.31), it follows that

$$\left(\delta\kappa-\frac{1}{3}\beta\mathcal{E}^{-1}\int_{-\infty}^{\infty} D(t^2)(dt)\right)/k'^2\equiv\delta f(k'^2) \tag{5.38}$$

is a function of k'^2 which is analytic in the lower half plane and also on the positive real axis, while the singularity at the origin, if any, is integrable. Also, this function tends to zero if $|k'|^2\to\infty$ in the mentioned domain. Hence

$$\delta f(k'^2)=\frac{-1}{2\pi i}\int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{\delta f(s)}{s-k'^2}\,ds,\qquad 0>-\epsilon>\mathrm{Im}\,(k'^2),$$

or, using (5.32a) with $\mathrm{Im}\,(k'^2)\to-\epsilon$,

$$\delta f(k'^2)=-\frac{1}{\pi i}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} P\,(s-k'^2)^{-1}\delta f(s)ds. \tag{5.39}$$

Hence, taking the Hermitian parts on both sides of (5.39) in the limit $\epsilon\to 0$,

$$\delta f_{\mathrm{R}}(k'^2)=-\frac{1}{\pi}\int_{0}^{\infty} P(s-k'^2)^{-1}\delta f_{\mathrm{I}}(s)ds,\qquad \epsilon\to 0. \tag{5.40}$$

Here δf_{R} and δf_{I} are the Hermitian and anti-Hermitian parts of δf, respectively, and $\delta f_{\mathrm{I}}(s)$ is zero for $s<0$ because of (5.32b). By the substitution of (5.38) into (5.40), we obtain

$$\delta\kappa_{\mathrm{R}}(k'^2)-\frac{1}{3}\beta\mathcal{E}^{-1}\int_{-\infty}^{\infty} D(t^2)(dt)=-\frac{k'^2}{\pi}\int_{0}^{\infty} P(s-k'^2)^{-1}\frac{\delta\kappa_{\mathrm{I}}(s)}{s}\,ds. \tag{5.41}$$

Equation (5.41) is the dispersion relation which gives $\delta\kappa_{\mathrm{R}}$ in terms of the known $\delta\kappa_{\mathrm{I}}$-spectrum.

Although this dispersion relation is derived for $\delta\kappa$ in the first order of approximation, the corresponding formula would hold also for the exact $\delta\kappa$ independent of any approximation.

6. Differential Equation for the Cross-Correlation Function

Another approach to the multiple scattering problem is as follows: Let us introduce the function $G(x_2, x_2'|x_1, x_1')$ defined by

$$G(x_2, x_2'|x_1, x_1')=\langle\psi(x_2, x_1)\cdot\psi_1^*(x_2', x_1')\rangle. \tag{6.1}$$

Here the sign * indicates the complex conjugate, and the product $a\cdot b$ the direct product of two matrices a and b. Hence, the function G is a $6\times 6=36$ matrix, the elements of which equal the products of two corresponding elements of its matrix factors. Then, from (4.4), $\langle W_{21}\rangle$ is given by

$$\langle W_{21}\rangle=\iiiint\bar{\xi}_{\sigma_2}(x_2)\bar{\xi}^*_{\sigma_2}(x_2')G(x_2, x_2'|x_1, x_1')\xi_{\sigma_1}(x_1)\xi^*_{\sigma_1}(x_1')(dx_2)(dx_2')(dx_1)(dx_1'), \tag{6.2}$$

provided the vectors are suitably multiplied to G.

Now, applying the wave equations (5.1) together with (2.3) and (2.9), we find

$$\begin{aligned}[(\gamma\partial_2)-k_e]\cdot[(\gamma\partial_2')-k_e^*]G(x_2, x_2'|x_1, x_1')&=\langle\delta\mathbf{k}(x_2)\psi(x_2, x_1)\cdot\delta\mathbf{k}^*(x_2')\psi^*(x_2', x_1')\rangle\\ &+\delta(x_2'-x_1')\langle\delta\mathbf{k}(x_2)\psi(x_2, x_1)\rangle+\delta(x_2-x_1)\langle\delta\mathbf{k}^*(x_2')\psi^*(x_2', x_1')\rangle+\delta(x_2-x_1)\delta(x_2'-x_1'). \end{aligned}\tag{6.3}$$

Here, according to (5.8) and (5.12),

$$\langle \delta\mathbf{k}(x_2)\psi(x_2,\, x_1)\rangle = 0,\ \langle \delta\mathbf{k}^*(x_2')\psi^*(x_2',\, x_1')\rangle = 0,\ \delta\mathbf{k}(x) = \beta\Delta\mathcal{E}\,(x) + \delta\kappa. \tag{6.4}$$

If the medium fluctuations satisfy a multivariate normal distribution, we can use the lemma (5.18) and thus get for the first term on the right side of (6.3)

$$\begin{aligned}\langle \delta\mathbf{k}(x_2)\psi(x_2, x_1)\cdot\delta\mathbf{k}^*(x_2')\psi^*(x_2', x_1')\rangle &= [\beta\cdot\beta^* D(x_2 - x_2') + \delta\kappa\cdot\delta\kappa^*]\langle\psi(x_2, x_1)\cdot\psi^*(x_2', x_1')\rangle \\ &+ \int [\delta\kappa\cdot\beta^* D(x_2' - x) + \beta\cdot\delta\kappa^* D(x_2 - x)]\left\langle \frac{\delta}{\delta\Delta\mathcal{E}\,(x)}\{\psi(x_2, x_1)\cdot\psi^*(x_2', x_1')\}\right\rangle(dx) \\ &+ \beta\cdot\beta^*\int D(x_2 - x)D(x_2' - x')\left\langle \frac{\delta^2}{\delta\Delta\mathcal{E}\,(x)\delta\Delta\mathcal{E}\,(x')}\{\psi(x_2, x_1)\cdot\psi^*(x_2', x_1')\}\right\rangle(dx)(dx'). \end{aligned}\tag{6.5}$$

Here

$$D(x - x') = \langle \Delta\mathcal{E}\,(x)\Delta\mathcal{E}\,(x')\rangle.$$

As we have seen in section 5, $D(x)$ and $\delta\kappa$ are of the second or higher order with respect $\Delta\mathcal{E}$. Hence, all the terms except the first on the right side of (6.5) are of the fourth order in $\Delta\mathcal{E}$, and thus may reasonably be neglected in most cases as compared with the first term which is of the second order. Hence

$$\langle \delta\mathbf{k}(x_2)\psi(x_2,\, x_1)\cdot\delta\mathbf{k}^*(x_2')\psi(x_2',\, x_1')\rangle \simeq \beta\cdot\beta^* D(x_2 - x_2')G(x_2,\, x_2'|x_1,\, x_1'), \tag{6.6}$$

and thus (6.3) becomes, remembering the vanishing of the second and third terms of its right side:

$$[(\gamma\partial_2) - k_e]\cdot[(\gamma\partial_2') - k_e^*]G(x_2,\, x_2'|x_1,\, x_1') \sim \beta\cdot\beta^* D(x_2 - x_2')G(x_2,\, x_2'|x_1, x_1') + \delta(x_2 - x_1)\cdot\delta(x_2' - x_1'). \tag{6.7}$$

This constitutes the differential equation to be solved to get the function G. In the case of scalar waves, the corresponding equation reads:

$$[\Delta_2 + k_e^2][\Delta_2' + k_e^{*2}]G(x_2,\, x_2'|x_1,\, x_1') = cD(x_2 - x_2')G(x_2,\, x_2'|x_1,\, x_1') + \delta(x_2 - x_1)\cdot\delta(x_2' - x_1'). \tag{6.8}$$

The approximation used to derive (6.7) is equivalent to taking into account all the contributions from the graphs of ladder type like figure 9a but not those from graphs like figure 9b

The investigation of the solution of (6.7) or (6.8) is also a powerful approach to the multiple scattering problem.[9]

[9] The differential equation of this kind may be solved relatively easily by the use of the relative coordinates $u = x_2 - x_1$ and those of averages $u = (x_2 + x_1)/2$.

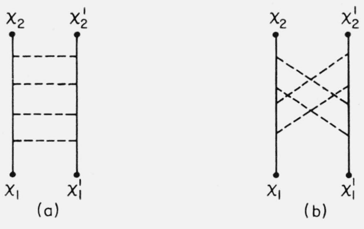

FIGURE 9(a) *Diagram for ladder type interaction;*

(b) *diagram for nonladder type interaction.*

7. Summary and Discussion

In this paper, the treatment of a statistical theory of electromagnetic fields in a fluctuating medium is primarily based on the graphical method introduced by Feynman [1949] for quantum electrodynamics. The latter also has been one of the most convenient methods in general quantum field theory. The theory is developed in parallel with that of Feynman as much as possible, and a perfect correspondence between them is found in almost all kinds of problems occurring in our scattering theory, especially when the medium fluctuations follow the Gaussian distribution. For instance, the problem of getting the correlation between the fields at two different points in space (sec. 6) is equivalent to solving the Schrödinger equation for two mutually interacting particles, whose interaction term can be obtained in the same way as in quantum field theory [Salpeter and Bethe, 1951; Gell-Mann and Low, 1951]. The latter has a perfect correspondence with our theory especially when the medium fluctuations satisfy the above mentioned conditions. Many methods and approximations used in the quantum field theory [Lévy, 1952a, 1952b; Klein, 1953] will therefore be available for solving the partial differential equation (6.7) for the most relevant correlation function; also the theory of renormalization treated in section 5 has an almost perfect correspondence with that of the quantum field theory.

The correspondence in question refers to the behavior of the electromagnetic field in a randomly changing medium, and to that of the quantized electron field which is in interaction with the quantized electromagnetic field; $\Delta\mathcal{E}(x)$ behaves just like a quantized Boson field, especially when $\Delta\mathcal{E}(x)$ has a Gaussian distribution so that (4.18) will hold. This might not be accidental, because the quantum theory states the problem and the answer only in terms of probability or statistics; any quantized field cannot be free from its own fluctuation. In Part II of this series of papers, the fundamental statistical theory for the electromagnetic field in a fluctuating media will be extended. The quantum-field theoretical treatment will inevitably be introduced there, and the basic relationship between the statistical theory of electromagnetic fields and the quantum field theory will be derived; many powerful methods used in the latter are expected to be available in the former in view of this relationship. In Part III, a few applications to tropospheric scattering will be discussed, taking into account the boundary conditions.

The author expresses his cordial thanks to Dr. H. Bremmer for his great interest and encouragement extended to this paper.

8. Appendix 1

Introducing the matrices $\beta_i (i=1, 2, 3)$ by

$$\beta_1=2s_1^2-1,\ \beta_2=2s_2^2-1,\ \beta_3=2s_3^2-1, \tag{A1}$$

these prove to satisfy the relations

$$\beta_i s_j+s_j\beta_i=\begin{cases}0 & (i\neq j) \\ 2s_j & (i=j),\end{cases} \qquad \begin{array}{l}\beta_i\beta_j-\beta_j\beta_i=0, \\ \beta_1\beta_2\beta_3=1,\ \ \beta_1^2=1, \text{ etc.}\end{array} \tag{A2}$$

Hence, applying the algebraic relations (2.10) and also the relation $Tr(ab)=Tr(ba)$ for arbitrary matrices a and b,

$$Tr'(s_1)=Tr'(\beta_1 s_1)=Tr'(s_1\beta_1)=\tfrac{1}{2}Tr'(\beta_1 s_1+s_1\beta_1)=0,$$

$$Tr'(s_1s_2)=Tr'(\beta_1s_1s_2)=-Tr'(s_1s_2\beta_1)=-Tr'(\beta_1s_1s_2)=-Tr'(s_1s_2)=0,$$

$$Tr'(s_1^2)=\tfrac{1}{3}Tr'(s_1^2+s_2^2+s_3^2)=2,$$

$$Tr'(s_is_js_k)=Tr'(-s_ks_js_i+\delta_{ij}s_k+\delta_{jk}s_i)=-Tr'(s_ks_js_i)=\frac{1}{2}\,Tr'[(s_is_j-s_js_i)s_k]=\frac{1}{2i}\,\epsilon_{ijl}Tr'(s_ls_k)=i^{-1}\epsilon_{ijk}. \tag{A3}$$

Here, the commutation relation in (2.10) is used. In the same way, since the trace is invariant for the matrix transposition and $s_i^T=-s_i$,

$$Tr'(s_is_js_ks_l)=Tr'(s_ls_ks_js_i)=\tfrac{1}{2}\,Tr'[(s_is_js_k+s_ks_js_i)s_l]=\tfrac{1}{2}\,Tr'[\delta_{ij}s_ks_l+\delta_{jk}s_is_l]=\delta_{ij}\delta_{kl}+\delta_{jk}\delta_{il}. \tag{A4}$$

Generally, using the commutation relations for s_i and β_i, the trace of any function of s_i can be expressed in terms of the following traces:

$$Tr'(s_i)=Tr'(s_i\beta_j)=Tr'(s_is_j\beta_k)=Tr'(s_i\beta_j\beta_k)=0 \qquad (i\neq j\neq k),$$

$$Tr'(s_is_j)=2\delta_{ij},\ Tr'(s_is_js_k)=i^{-1}\epsilon_{ijk},\ Tr'(\beta_i)=+1,\ Tr'(\beta_i\beta_j)=-1 \qquad (i\neq j).$$

9. Appendix 2

The second equation of (2.14) is derived from the first one as follows. The Green's function $\psi(x, x')$ is defined by the first equation

$$[(\gamma\partial)-k]\psi(x, x')=\delta(x-x'),$$

which gives, operating $[-(\gamma\overleftarrow{\partial}')-k]$ from the right side,

$$[(\gamma\partial)-k]\psi(x, x')[-(\gamma\overleftarrow{\partial}')-k]=\delta(x-x')[-(\gamma\overleftarrow{\partial}')-k]=[(\gamma\partial)-k]\delta(x-x').$$

Hence

$$[(\gamma\partial)-k]\{\psi(x, x')[-(\gamma\overleftarrow{\partial}')-k]-\delta(x-x')\}=0,$$

which has the general solution

$$\psi(x, x')[-(\gamma\overleftarrow{\partial}')-k]=\delta(x-x')+\phi(x, x').$$

Here $\phi(x, x')$ is a solution of the homogeneous equation

$$[(\gamma\partial)-k]\phi(x, x')=0,$$

which has to satisfy all given boundary conditions in space. Generally, there exists no such solution of a homogeneous equation. Thus $\phi(x, x')=0$ and the second equation of (2.14) is obtained.

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