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Correction of Atmospheric Refraction Errors in Radio Height Finding*

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Atmospheric refraction errors in height finding radars are studied by means of detailed refraction calculations for a wide range of meteorological conditions. For targets up to 70,000 feet above ground and 150 miles ground distance from the radar site, the mean height error was found to be as much as 5,000 feet with a standard deviation of 1,200 feet. A correction for the surface value of the refractive index at the radar site would eliminate the mean height error and reduce the maximum standard deviation to less than 900 feet. An additional correction for the initial gradient of the refractive index and the value of the refractive index at one kilometer above the surface would reduce the maximum standard deviation to less than 400 feet. Methods of correcting height errors based on available meteorological data are presented and shown to be operationally practical.

1. Introduction

As a radio ray passes through the atmosphere the length and direction of its path varies with the radio refractive index. Uncorrected radar output determines the position of a target by assuming a straight-line path at constant velocity. The difference between the straight path and the actual path results in an error which becomes increasingly significant as the distance to the target increases. The height error (the component of the position error normal to the surface of the earth) constitutes over 95 percent of the total error. Until recently, the range of height finding equipment was sufficiently limited so that the refraction errors could be either neglected, or approximated by a constant effective earth's radius correction [Schelleng, Burrows, and Ferrell, 1933].

Bauer, Mason, and Wilson [1958] obtained an equation for accurately estimating radar target heights in a specific exponential atmosphere. Beckmann [1958] presented a probability estimate of the height errors without using meteorological measurements.

The purpose of the study is to investigate the correlation between available meteorological parameters and height errors for targets of interest in terminal air traffic control and to develop height error correction procedures using these parameters. The height errors for various target positions relative to the radar site are correlated with meteorological parameters measured at or above the site to determine the predictability of height errors independently of target position. The correction procedures are developed to account for atmospheric variations and target position by combining the meteorological and geometric considerations.

2. Background

2.1. Refractive Index

The radio refractive index, n, of a propagation medium is the ratio of the free-space velocity of light, c, to the velocity in the medium, v, (i.e., $n=c/v$). Since the propagation velocity of the atmosphere is only slightly less than the free-space velocity, it is often convenient to use the scaled-up difference between the refractive index and unity. This quantity is called the refractivity and is denoted by $N=(n-1)\times 10^6$.

The refractivity is obtained from meteorological parameters by

$$N=77.6\,\frac{P}{T}+3.73\times 10^5\,\frac{e}{T^2},$$

where P is the total atmospheric pressure in millibars, T is the absolute temperature, and e is the water vapor pressure in millibars. Normally, the equation for N is dominated by the first term so that the refractivity can be approximated by an exponential function of height as shown by Bean and Thayer [1959a].

2.2. Ray Theory

If the gradient of refractive index is assumed to be normal to the surface of the smooth spherical earth and

$$\frac{dn}{dr}>-\frac{1}{r},$$

then, for frequencies greater than 100 kc/s, the path of a radio ray is determined by Snell's law for polar coordinates:

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$$nr \cos \theta = n_0 r_0 \cos \theta_0, \tag{1}$$

where θ is the local elevation angle of the ray, and r is distance from the center of the earth to a point on the ray as shown in figure 1. The bending angle, τ, is determined by [Smart, 1931]

$$\tau = -\int_{r_0}^{r_0+h} \frac{\cot \theta}{n} \frac{dn}{dr} dr. \tag{2}$$

The distance, d, along the surface of the earth is obtained by

$$d = r_0 \phi = r_0(\tau + \theta - \theta_0). \tag{3}$$

The length of the path is called the geometric range and is obtained by

$$R = \int_{r_0}^{r_0+h} \csc \Theta \, dr, \tag{4}$$

and the apparent or radio range is found by

$$R_e = \int_{r_0}^{r_0+h} n \csc \Theta \, dr = R + \int_{r_0}^{r_0+h} N \times 10^{-6} \csc \Theta \, dr. \tag{5}$$

Because the difference between R_e and the true slant range, R_0 is extremely small compared to the height error; the slant range and radio range are assumed to be identical to the geometric range, R.

The apparent height of the target, in figure 2, is obtained by solving

$$(r_0 + h_a)^2 = r_0^2 + R^2 + 2r_0 R \sin \Theta_0 \tag{6}$$

for h_a. The following form is useful for numerical calculations:

$$h_a = \frac{R(R + 2r_0 \sin \Theta_0)}{r_0 + \sqrt{r_0^2 + R(R + 2r_0 \sin \Theta_0)}}. \tag{7}$$

The height error for a target at height, h, is found by

$$\epsilon_h = h_a - h, \tag{8}$$

which will always be positive if n decreases with height.

If the refractive index is known as a function of height, the foregoing procedure is useful for determining the height error when the true height and the arrival angle of the ray are hypothesized. Unfortunately, it is not applicable for obtaining the height error from the apparent position of the target.

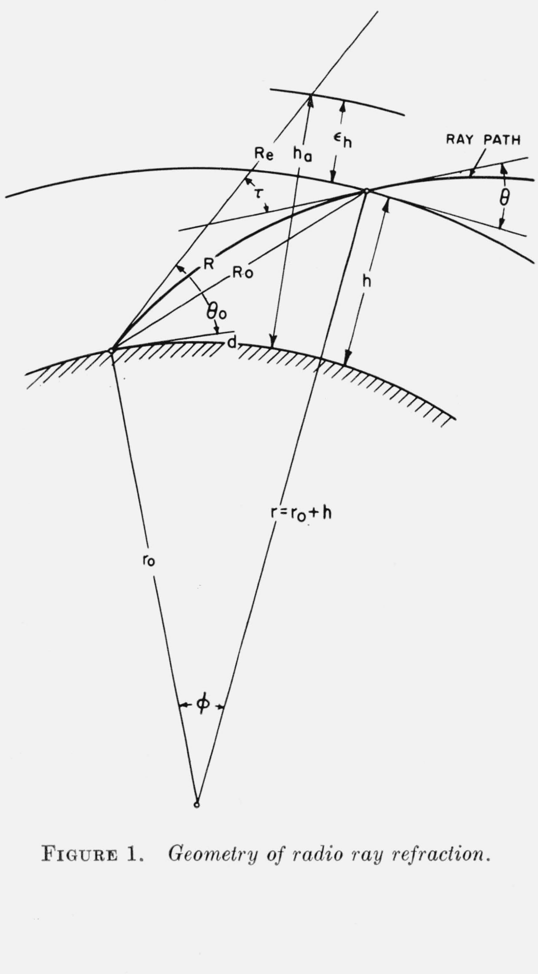

FIGURE 1. *Geometry of radio ray refraction.*

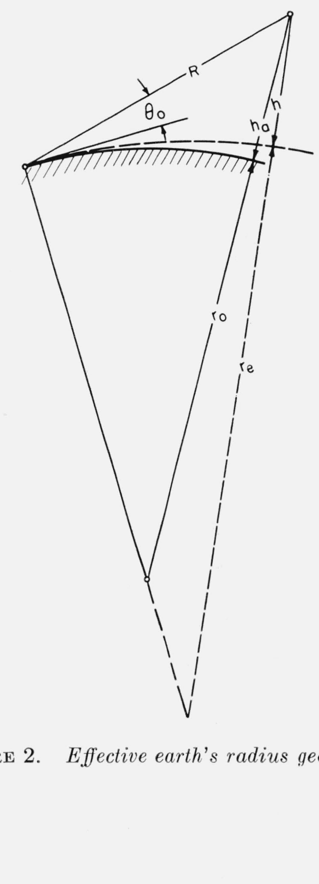

FIGURE 2. *Effective earth's radius geometry.*

2.3. Effective Earth's Radius

The inaccuracy of a constant effective earth's radius correction stems mainly from the assumption that all radio rays have the same constant curvature. The accuracy would be greatly enhanced if an "average" effective radius could be determined for each ray path.

The following expression, with the effective earth's radius denoted by r_e in figure 2.

$$(r_e+h)^2=r_e^2+R^2+2r_eR\sin\Theta_0, \tag{9}$$

can be combined with (6) and (8) to obtain

$$\epsilon_h\left(1+\frac{h_a+h}{2r_e}\right)=\left(\frac{R^2-h_a^2}{2}\right)\left(\frac{1}{r_0}-\frac{1}{r_e}\right). \tag{10}$$

Because the expression multiplying ϵ_h in (10) differs from unity by less than 4×10^{-3} for all target heights ($h\leq70{,}000$ ft) and ranges ($R\leq155$ miles) to be considered, the height error can be approximated as

$$\epsilon_h\simeq\left(\frac{R^2-h_a^2}{2}\right)\left(\frac{1}{r_0}-\frac{1}{r_e}\right), \tag{11}$$

with an error never in excess of 0.4 percent. The difference between the curvature of the actual earth and the curvature of the "average" effective earth for the ray path (i.e., $1/r_0-1/r_e$) represents the "average" curvature of the ray. Thus, if the ray curvature can be determined as a function of the target position and the refractive index structure, (11) provides a simple formula for approximating the height error.

The curvature of a ray, K, at any point on the path is expressed by [Millington, 1957]

$$K=-\frac{1}{n}\frac{dn}{dr}\cos\theta, \tag{12}$$

or, from (1), replacing r by r_0+h,

$$K=-\frac{n_0\cos\theta_0}{n^2(1+h/r_0)}\frac{dn}{dh}. \tag{13}$$

From (6), ignoring the term of the order $1/r_0^2$, one obtains

$$\cos^2\theta_0\simeq\left(1+\frac{h_a}{r_0}\right)\left(\frac{R^2-h_a^2}{R^2}\right), \tag{14}$$

so that (13) becomes

$$K\simeq-\frac{n_0}{n^2}\frac{(R^2-h_a^2)^{1/2}}{R}\frac{\left(1+\frac{h_a}{r_0}\right)^{1/2}}{\left(1+\frac{h}{r_0}\right)}\frac{dn}{dh}. \tag{15}$$

The refractive index usually decreases with height so that the quantity

$$\frac{n_0}{n^2}\frac{\left(1+\frac{h_a}{r_0}\right)^{1/2}}{1+\frac{h}{r_0}}\simeq1$$

varies only slightly with height, and the curvature at a point on the ray path can be approximated by

$$K\simeq\frac{(R^2-h_a^2)^{1/2}}{R}\left|\frac{dn}{dh}\right|. \tag{16}$$

Therefore, (11) becomes

$$\epsilon_h\simeq\frac{(R^2-h_a^2)^{3/2}}{2R}g, \tag{17}$$

where g represents an average gradient on the ray path as defined in the following section.

Since g depends upon the meteorological conditions along the path, the basic problem is to determine g for a given target from the conditions at and/or near the surface.

3. Procedure

3.1. Meteorological Parameters

Measurement of the refractivity at the radar site will provide an estimate of the gradient if a model of the refractive index structure is assumed. In the exponential model, for example,

$$n(h)=1+N_s\exp(-ch)\times10^{-6},$$

where N_s is the surface refractivity and c is a constant, the gradient

$$\frac{dn}{dh}=-cN_s\exp(-ch)\times10^{-6}.$$

For a target at a height h_t the simple *average gradient* along the ray path from radar to target is

$$g=-\frac{1}{h_t}\int_0^{h_t}\frac{dn}{dh}dh, \tag{18}$$

which for the exponential model is

$$g=\frac{N_s}{h_t}[1-\exp(-ch_t)]\times10^{-6},$$

but, since h_t is not known, g must be approximated as a function of the apparent height.

Additional meteorological measurements at a sufficient height above the surface to obtain values significantly different from the surface values can be used to determine the initial gradient of refractivity,

$$G_0=\left.\frac{dN}{dh}\right|_{h=0};$$

assuming the initial layer to be exponential yields

$$G_0 = \frac{N_s}{H} \log\left(\frac{N_H}{N_s}\right), \tag{19}$$

where N_H is the refractivity at the height, H, in kilometers of the above surface measurements. The initial gradient provides a boundary condition for estimating g as a function of the apparent height. The average gradient for the ray path determined with the initial gradient and the true height for the exponential model is

$$g = \frac{G_0}{ch_t} [\exp(-ch_t) - 1] \times 10^{-6}.$$

For the purposes of this study the average (per kilometer) gradient of the first kilometer of the atmosphere is the only prediction parameter used which will require upper air measurements. The average 1-km gradient,

$$\Delta N = N_1 - N_s, \tag{20}$$

where N_1 is refractivity at 1 km above the surface, was selected because climatological summaries [Bean, Horn, and Ozanich, 1960] can be used to estimate the height error when meteorological measurements are unobtainable.

3.2. Calculation and Correlation of Height Errors

Bean, Cahoon, and Thayer [1960] selected refractive index profiles, determined from radiosonde observations at thirteen climatically distinct locations, which represent a wide variety of mutually exclusive profile types. This profile sample was used for the present study because it represents a complete range of meteorological conditions. The ray paths at arrival angles varying from 0 to near 90° were determined for each profile by numerical evaluation of (1) through (5) using methods similar to those described by Bean and Thayer [1959b]. The height errors were calculated with (7) and (8) at selected height intervals to 70,000 ft for each ray path. Newton's method of interpolation with divided differences was used to determine height errors for fixed ground distances to 150 miles. The limits of height and distance were chosen to extend beyond the current needs in terminal air traffic control, but are sufficiently restricted to allow some of the previous assumptions.

The prediction parameters, N_s, G_0, and ΔN, were obtained from each of the refractive index profiles. Linear and multiple regression analyses were employed to obtain least squares estimates of the height error at each height and distance for each prediction parameter and for various combinations of the parameters.

3.3. Estimation of the Average Gradient

Based on the correlations, the following forms suggested by (18) were selected for approximating g:

$$g_1 = \frac{N_s}{h_a} f_{11}(h_a), \tag{21}$$

$$g_2 = \frac{N_s}{h_a} f_{21}(h_a) + \frac{G_0}{h_a} f_{22}(h_a), \tag{22}$$

or

$$g_3 = \frac{N_s}{h_a} f_{31}(h_a) + \frac{G_0}{h_a} f_{32}(h_a) + \frac{\Delta N}{h_a} f_{33}(h_a), \tag{23}$$

where g_1 is an estimate of the average gradient if only surface observations are available, g_2 is an improved estimate obtained with additional tower measurements, and g_3 is an estimate obtained with the addition of upper air measurements such as radiosonde observations.

To obtain a direct estimate of the height error, (21) through (23) were combined with (17), and the functions $f_{ij}(i \geq j = 1, 2, 3)$ were determined as least squares polynomials.

4. Results

4.1. Regression Analysis

The volume of data processed is of sufficient magnitude that it is impractical to include it all in this report. Therefore, certain information obtained from the regression analysis was selected as being the most significant.

The mean height error is representative of average meteorological conditions, and, therefore, provides the best general estimate obtainable if meteorological data are not available at the radar site under consideration. In figure 3, the mean height error was plotted for each target position and then contour lines were drawn to display the mean height error as a function of the true height and distance.

The standard deviation (about the mean) of the height errors provides a measure of the residual error if the mean is used as an estimate, since 68 percent of the observed height errors are expected to be within ± 1 standard deviation of the mean height error if the observations are normally distributed. The standard deviation is displayed as a function of target position in figure 4. The construction of figure 4 and subsequent figures is similar to that of figure 3.

The standard error of estimate establishes the same confidence limits for prediction with a regression as the standard deviation does for the mean. Thus, the standard error provides a measure of the residual error if the height errors are estimated by a regression equation involving meteorological parameters. The standard error of estimate was determined for each of the following regression equations

$$\epsilon_h = b_1 N_s + a, \tag{24}$$

$$\epsilon_h = b_1 N_s + b_2 G_0 + a, \tag{25}$$

and

$$\epsilon_h = b_1 N_s + b_2 G_0 + b_3 \Delta N + a. \tag{26}$$

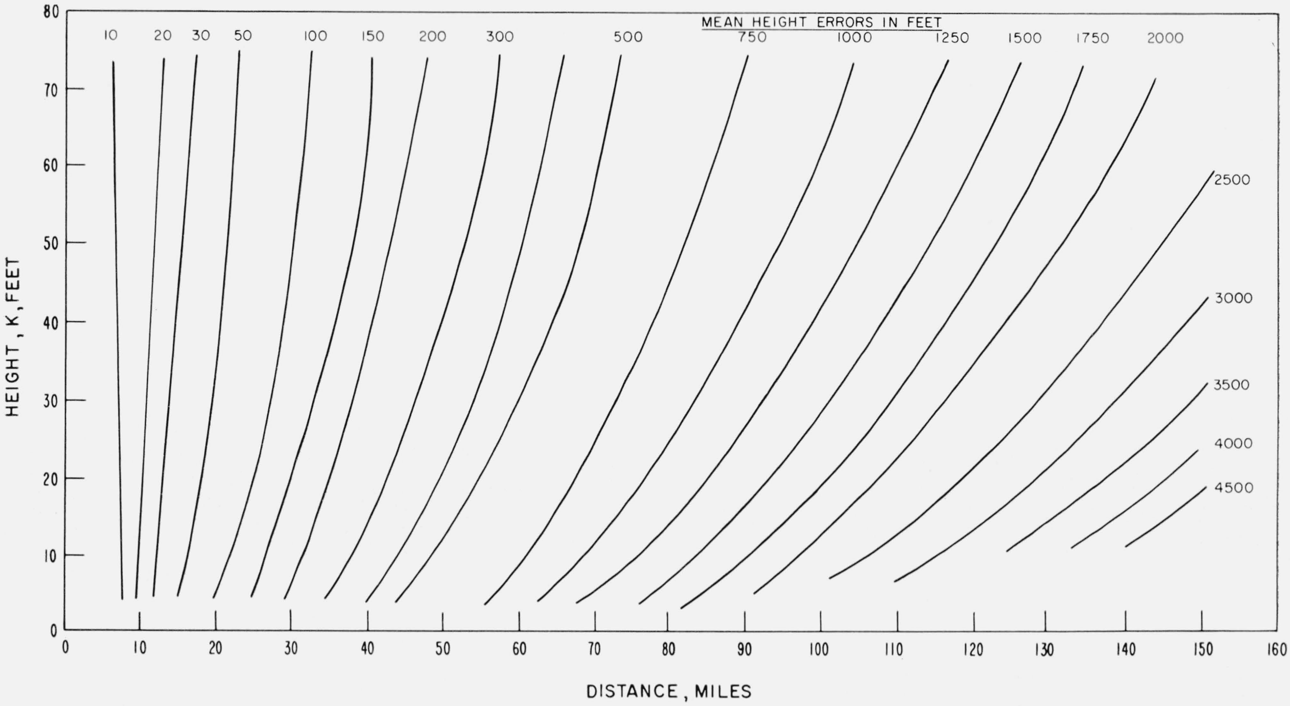

FIGURE 3. *Mean height errors in feet.*

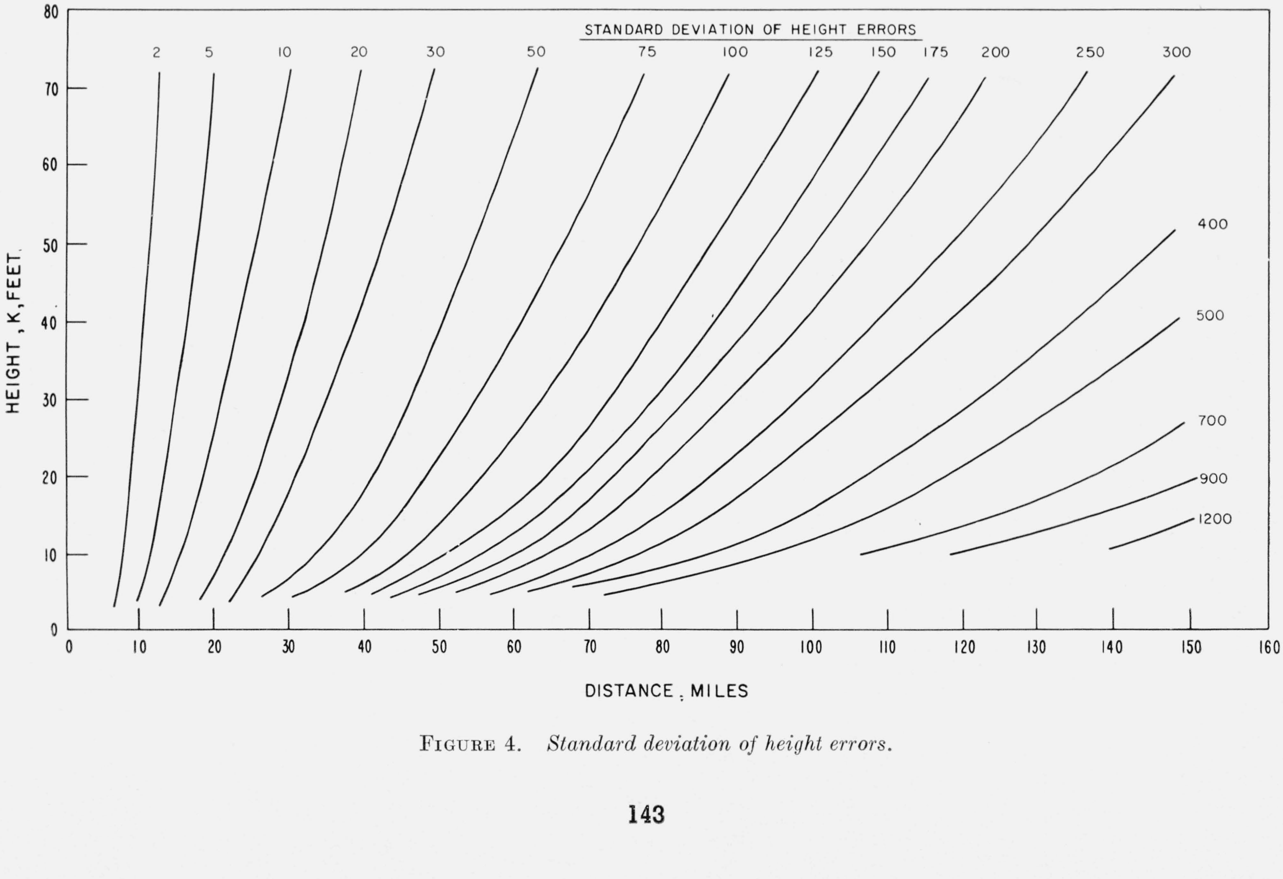

FIGURE 4. *Standard deviation of height errors.*

The standard error of estimate with (24) is displayed in figure 5. Comparison of figures 4 and 5 indicates the improvement, that is, the reduction in residual error, if surface meteorological observations are used in place of the mean to predict the height error.

The standard errors of estimate with (25) and (26) are shown in figures 6 and 7, respectively. These figures demonstrate how each additional parameter, obtained from tower or upper air measurements, enhances the accuracy of the estimate.

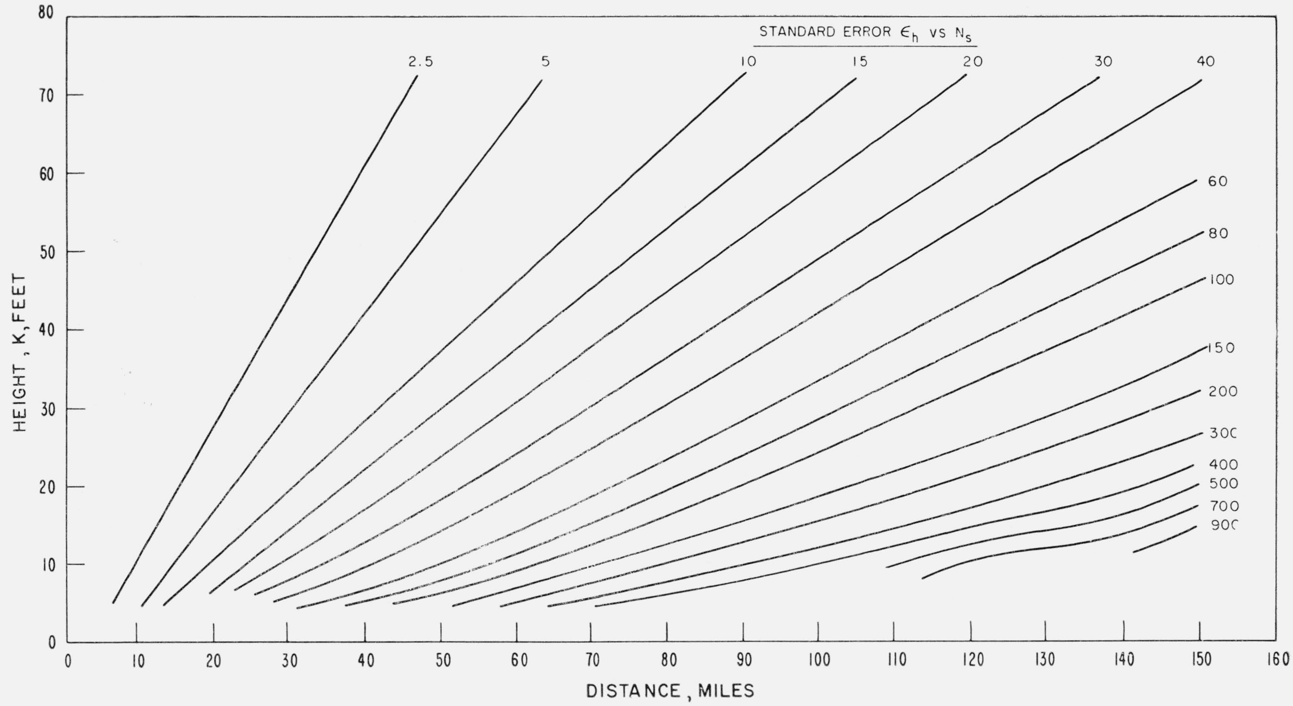

FIGURE 5. *Standard error ϵ_h versus N_s.*

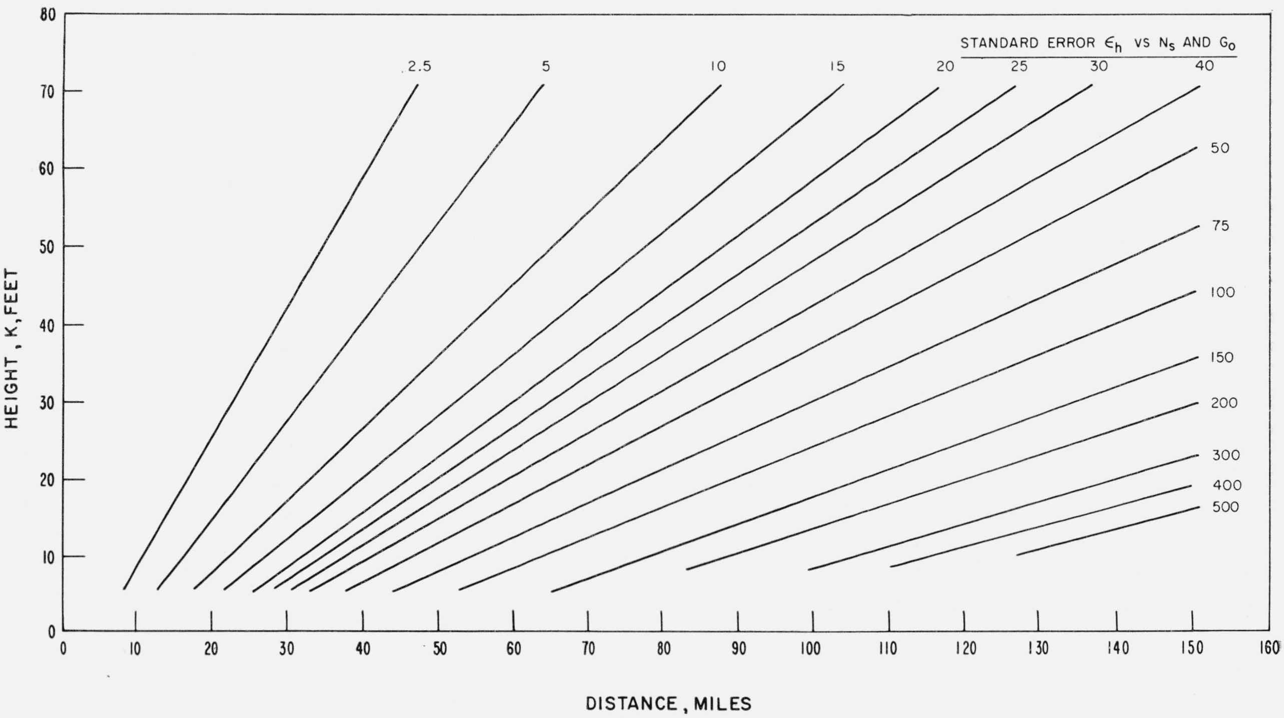

FIGURE 6. *Standard error ϵ_h versus N_s and G_0.*

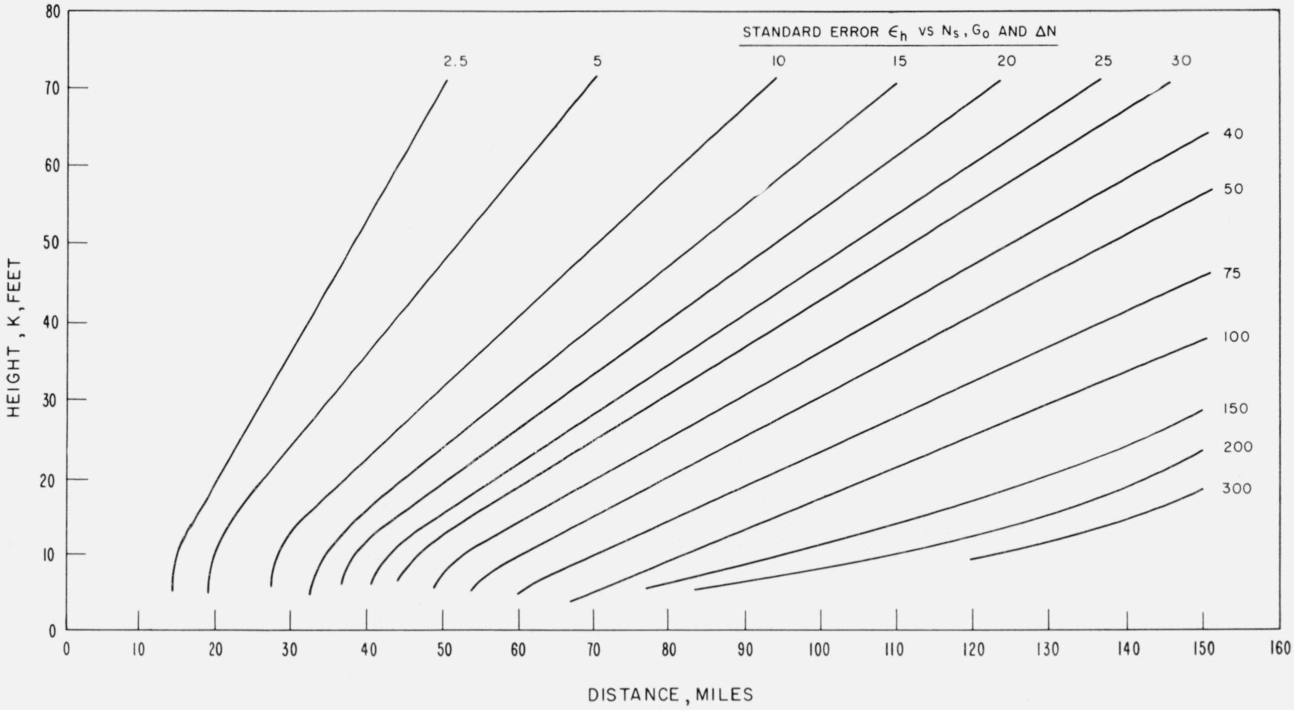

FIGURE 7. *Standard error ϵ_h versus N_s, G_0, and ΔN.*

The parameters G_0 and ΔN applied individually, that is,

$$\epsilon_h = bG_0 + a$$

and

$$\epsilon_h = b\Delta N + a,$$

were of significant value only for targets at low heights ($h \leq 10{,}000$ ft). Examination of the figures shows that prediction of ϵ_h with N_s provides significant improvement over the mean for target heights above 15,000 ft. The addition of G_0 improves the estimate for heights below 15,000 ft and the addition of ΔN provides a slight overall improvement. These results are, perhaps, more clearly illustrated by figures 8 and 9 in which the mean height error, the standard deviation, and the standard errors of estimate are displayed as functions of distance for fixed heights of 15,000 feet and 30,000 feet, respectively.

In figures 3 through 7 the contours do not extend below 15,000 ft for distances greater than 120 miles and 10,000 ft for distances greater than 80 miles. Correlations were not calculated for these target positions because, for certain refractive index profiles they are beyond the radio horizon, and for certain other profiles the arrival angle is too low for the ray to penetrate a trapping layer. If a target at 5,000 ft height and 150 miles distance is visible to radar, with $dn/dr > -1/r$ along the ray path, the resulting height error would be about 10,000 ft.

As an aid to further studies, the coefficients for (24) through (26) are listed in the appendix.

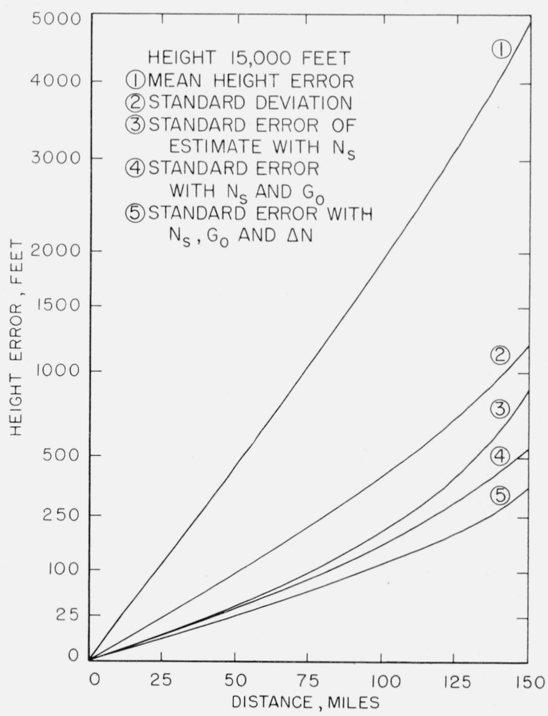

FIGURE 8. *Height error statistics for fixed height of 15,000 feet.*

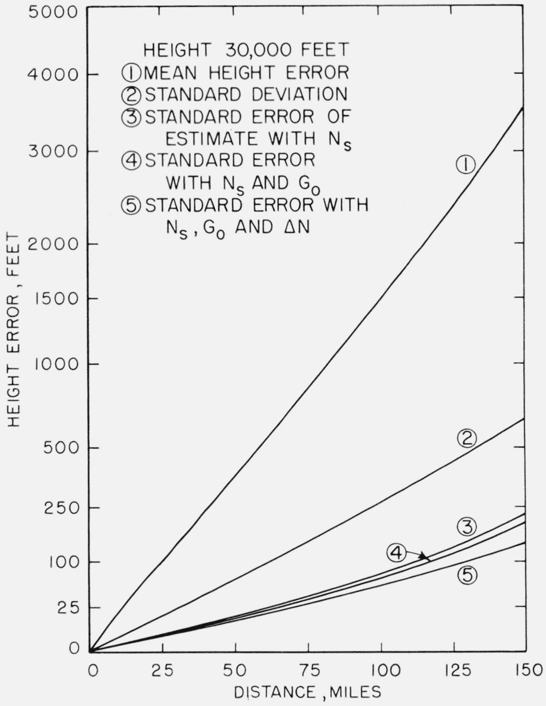

FIGURE 9. *Height error statistics for fixed height of 30,000 feet.*

4.2. Height-Error Equations

The equations for approximating ϵ_h were determined as

$$\epsilon_{h_1}=a_{10}\frac{D}{h_a^2}+\frac{D^{3/2}}{R}g_1+k, \tag{27}$$

$$\epsilon_{h_2}=a_{20}\frac{D}{h_a^2}+\frac{D^{3/2}}{R}g_2+k, \tag{28}$$

and

$$\epsilon_{h_3}=a_{30}\frac{D}{h_a^2}+\frac{D^{3/2}}{R}g_3+k, \tag{29}$$

where

$$D=R^2-0.03587h_a^2,$$

g_1, g_2, and g_3 are obtained from (21) through (23) with

$$f_{i1}(h_a)=a_{i1}+a_{i2}h_a+a_{i3}h_a^2 \qquad (i=1, 2, 3), \tag{30}$$

$$f_{i2}(h_a)=a_{i4}+a_{i5}h_a \qquad (i=2, 3), \tag{31}$$

$$f_{33}(h_a)=a_{36}+a_{37}h_a, \tag{32}$$

for R in miles and h_a in thousands of feet. The term in D/h_a^2 was introduced to account for, in part, a large negative constant term which tended to produce negative height errors for ranges less than 30 miles. Furthermore, the inclusion of this term increased the accuracy of the estimate of ϵ_{hi} by about 2 percent. An additional term in h_a^3 for (30) increased the accuracy by about 1 percent but introduced a fictitious minimum near 60,000 ft, while a term in h_a^2 for (31) and (32) increased the accuracy of (28) and (29) by less than 0.1 percent. The relative improvement of (28) over (27) is about 3 percent and of (29) over (28) about 1 percent.

The coefficients a_{ij} are listed in table A. The constant term, k, which would vanish if the equations were exact, is about -70 for a least squares approximation.

5. Conclusions

Height-error correction can be significantly improved by accounting for the surface refractivity at the radar site. The use of the initial gradient, in addition to the surface refractivity, yields a significant improvement only for targets beyond about 60 miles and below 15,000 ft. In this case, G_0 is important not only to improve the accuracy but to determine if the assumption in section 2.2 has been violated, namely, if $G_0 \leq -10^6/r_0$. The still further improvement obtained with the use of ΔN would not, in general, justify the trouble and expense of measuring this parameter.

If the distance to the target exceeds about 50 miles, the normal decrease with height of the gradient should be accounted for in a height error correction.

6. Appendix. Coefficients for the Regression Equations

Constant term, a:
- Equation (24) Table 1
- Equation (25) Table 2
- Equation (26) Table 3

Coefficient of N_s, b_1:
- Equation (24) Table 4
- Equation (25) Table 5
- Equation (26) Table 6

Coefficient of G_o, b_2:
- Equation (25) Table 7
- Equation (26) Table 8

Coefficient of ΔN, b_3:
- Equation (26) Table 9

TABLE A. *Coefficients* a_{ij}

i \ j	0	1	2	3	4	5	6	7
1	−19.596	0.014096	0.77906×10^{-4}	0.67545×10^{-6}				
2	−17.849	.011202	$.13665\times10^{-3}$	$.58925\times10^{-7}$	-0.64975×10^{-2}	0.12340×10^{-3}		
3	−15.319	.006388	$.18549\times10^{-3}$	$.39074\times10^{-7}$	$-.55818\times10^{-2}$	$.12671\times10^{-3}$	−0.023980	-0.22547×10^{-4}

TABLE 1

DISTANCE MILES	HEIGHT KFT 5	10	15	20	25	30	35	40	50	60	70
5	-3.	-3.	-2.	-1.	-.	.	1.	1.	2.	2.	2.
10	-14.	-13.	-10.	-8.	-6.	-4.	-3.	-2.	-.	1.	1.
15	-31.	-29.	-24.	-19.	-15.	-12.	-9.	-7.	-4.	-2.	-1.
20	-55.	-53.	-44.	-35.	-28.	-23.	-18.	-14.	-9.	-5.	-3.
25	-86.	-83.	-69.	-56.	-45.	-37.	-29.	-24.	-15.	-10.	-7.
30	-123.	-119.	-100.	-81.	-66.	-54.	-44.	-35.	-23.	-15.	-11.
35	-166.	-162.	-136.	-111.	-91.	-74.	-60.	-49.	-32.	-22.	-16.
40	-215.	-211.	-178.	-146.	-120.	-98.	-80.	-65.	-43.	-30.	-22.
45	-269.	-267.	-226.	-186.	-153.	-125.	-102.	-83.	-55.	-39.	-29.
50	-327.	-329.	-280.	-231.	-190.	-156.	-127.	-104.	-70.	-49.	-36.
60	-454.	-470.	-404.	-335.	-277.	-228.	-187.	-153.	-103.	-73.	-54.
70	-586.	-633.	-552.	-461.	-381.	-315.	-259.	-212.	-144.	-102.	-76.
80	-709.	-812.	-722.	-607.	-505.	-418.	-345.	-284.	-193.	-137.	-103.
90	.	-1000.	-913.	-776.	-648.	-539.	-446.	-367.	-252.	-179.	-134.
100	.	-1186.	-1122.	-965.	-812.	-678.	-563.	-465.	-320.	-228.	-171.
110	.	-1352.	-1342.	-1176.	-997.	-836.	-697.	-578.	-399.	-286.	-214.
120	.	-1473.	-1565.	-1404.	-1203.	-1014.	-850.	-707.	-491.	-352.	-264.
130	.	.	-1774.	-1644.	-1429.	-1215.	-1022.	-852.	-597.	-429.	-322.
140	.	.	-1946.	-1889.	-1672.	-1434.	-1214.	-1019.	-718.	-518.	-389.
150	.	.	-2048.	-2123.	-1928.	-1674.	-1428.	-1203.	-855.	-620.	-465.

TABLE 2

DISTANCE MILES	HEIGHT KFT 5	10	15	20	25	30	35	40	50	60	70
5	-3.	-3.	-2.	-1.	-.	.	1.	1.	2.	2.	2.
10	-12.	-12.	-10.	-8.	-6.	-4.	-3.	-2.	-.	1.	1.
15	-28.	-29.	-24.	-19.	-15.	-12.	-9.	-7.	-4.	-2.	-1.
20	-50.	-51.	-43.	-35.	-28.	-22.	-18.	-14.	-9.	-5.	-3.
25	-77.	-80.	-67.	-55.	-45.	-36.	-29.	-23.	-15.	-10.	-7.
30	-110.	-115.	-97.	-80.	-65.	-53.	-43.	-35.	-23.	-15.	-11.
35	-148.	-156.	-133.	-110.	-90.	-73.	-60.	-48.	-32.	-22.	-16.
40	-190.	-204.	-174.	-144.	-118.	-97.	-79.	-64.	-43.	-30.	-22.
45	-235.	-257.	-221.	-183.	-151.	-124.	-101.	-82.	-55.	-38.	-28.
50	-284.	-316.	-273.	-227.	-187.	-154.	-126.	-103.	-69.	-48.	-36.
60	-383.	-449.	-394.	-329.	-273.	-225.	-185.	-151.	-102.	-72.	-54.
70	-473.	-600.	-536.	-452.	-376.	-311.	-257.	-210.	-143.	-101.	-76.
80	-537.	-762.	-700.	-595.	-497.	-413.	-341.	-281.	-191.	-136.	-102.
90	.	-925.	-880.	-757.	-637.	-531.	-441.	-363.	-249.	-178.	-133.
100	.	-1076.	-1075.	-940.	-797.	-667.	-556.	-460.	-317.	-226.	-170.
110	.	-1189.	-1275.	-1141.	-976.	-822.	-687.	-571.	-395.	-283.	-212.
120	.	-1236.	-1468.	-1355.	-1174.	-996.	-837.	-697.	-486.	-349.	-262.
130	.	.	-1637.	-1576.	-1390.	-1189.	-1005.	-840.	-590.	-425.	-319.
140	.	.	-1748.	-1794.	-1619.	-1400.	-1192.	-1003.	-708.	-512.	-385.
150	.	.	-1765.	-1989.	-1855.	-1630.	-1399.	-1183.	-843.	-612.	-460.

TABLE 3

DISTANCE MILES	HEIGHT KFT 5	10	15	20	25	30	35	40	50	60	70
5	-.	-2.	-1.	-1.	-.	.	1.	1.	2.	2.	2.
10	-2.	-8.	-8.	-6.	-5.	-3.	-2.	-2.	-.	.	1.
15	-4.	-18.	-18.	-15.	-12.	-10.	-8.	-6.	-3.	-2.	-1.
20	-7.	-33.	-33.	-28.	-23.	-19.	-15.	-12.	-7.	-5.	-3.
25	-9.	-51.	-51.	-44.	-37.	-31.	-25.	-20.	-13.	-9.	-6.
30	-12.	-73.	-74.	-64.	-54.	-45.	-37.	-30.	-20.	-13.	-10.
35	-13.	-98.	-100.	-87.	-74.	-62.	-51.	-42.	-28.	-19.	-14.
40	-13.	-125.	-130.	-114.	-97.	-81.	-67.	-55.	-37.	-26.	-19.
45	-10.	-156.	-164.	-144.	-123.	-104.	-86.	-71.	-48.	-33.	-25.
50	-5.	-187.	-201.	-178.	-152.	-129.	-107.	-88.	-60.	-42.	-31.
60	21.	-254.	-285.	-256.	-221.	-187.	-156.	-129.	-88.	-62.	-47.
70	70.	-318.	-378.	-346.	-302.	-257.	-216.	-179.	-123.	-87.	-66.
80	152.	-369.	-478.	-448.	-395.	-338.	-285.	-237.	-164.	-117.	-88.
90	.	-396.	-578.	-560.	-499.	-431.	-366.	-306.	-213.	-152.	-115.
100	.	-383.	-669.	-677.	-615.	-536.	-457.	-384.	-269.	-194.	-146.
110	.	-310.	-741.	-795.	-739.	-652.	-560.	-473.	-334.	-241.	-182.
120	.	-163.	-776.	-906.	-867.	-776.	-673.	-573.	-408.	-296.	-224.
130	.	.	-754.	-998.	-996.	-909.	-797.	-682.	-491.	-359.	-272.
140	.	.	-649.	-1055.	-1117.	-1044.	-929.	-804.	-585.	-430.	-327.
150	.	.	-436.	-1057.	-1217.	-1179.	-1068.	-933.	-690.	-510.	-389.

TABLE 4

DISTANCE MILES	HEIGHT KFT 5	10	15	20	25	30	35	40	50	60	70
5	.0318	.0313	.0296	.0281	.0267	.0257	.0248	.0243	.0236	.0231	.0225
10	.1153	.1059	.0944	.0848	.0770	.0705	.0652	.0608	.0543	.0498	.0462
15	.2545	.2303	.2025	.1795	.1608	.1454	.1326	.1218	.1056	.0943	.0858
20	.4495	.4046	.3541	.3123	.2784	.2504	.2271	.2074	.1776	.1568	.1413
25	.7005	.6291	.5494	.4835	.4300	.3858	.3489	.3177	.2704	.2373	.2127
30	1.0078	.9040	.7889	.6934	.6159	.5519	.4983	.4531	.3841	.3359	.3003
35	1.3714	1.2295	1.0728	.9424	.8366	.7489	.6756	.6136	.5190	.4529	.4042
40	1.7917	1.6060	1.4018	1.2311	1.0923	.9774	.8812	.7998	.6754	.5885	.5244
45	2.2688	2.0336	1.7762	1.5599	1.3837	1.2377	1.1154	1.0119	.8536	.7429	.6613
50	2.8030	2.5129	2.1968	1.9295	1.7113	1.5305	1.3788	1.2503	1.0539	.9163	.8150
60	4.0426	3.6268	3.1787	2.7938	2.4780	2.2155	1.9952	1.8085	1.5223	1.3216	1.1738
70	5.5119	4.9494	4.3513	3.8300	3.3981	3.0380	2.7355	2.4788	2.0845	1.8072	1.6032
80	7.2127	6.4794	5.7236	5.0453	4.4788	4.0046	3.6053	3.2666	2.7442	2.3769	2.1060
90	.	8.2133	7.2966	6.4464	5.7273	5.1225	4.6119	4.1779	3.5078	3.0342	2.6854
100	.	10.1439	9.0736	8.0412	7.1529	6.4005	5.7634	5.2207	4.3804	3.7846	3.3456
110	.	12.2529	11.0533	9.8363	8.7649	7.8482	7.0684	6.4028	5.3687	4.6329	4.0909
120	.	14.5240	13.2284	11.8360	10.5679	9.4723	8.5370	7.7337	6.4808	5.5863	4.9257
130	.	.	15.5836	14.0399	12.5772	11.2895	10.1777	9.2192	7.7254	6.6512	5.8573
140	.	.	18.0866	16.4440	14.7931	13.3002	12.0051	10.8816	9.1125	7.8354	6.8904
150	.	.	20.7053	19.0333	17.2171	15.5220	14.0251	12.7148	10.6504	9.1488	8.0339

TABLE 5

HEIGHT KFT

DISTANCE MILES	5	10	15	20	25	30	35	40	50	60	70
5	.0285	.0303	.0292	.0279	.0267	.0257	.0249	.0244	.0237	.0232	.0227
10	.1014	.1015	.0922	.0835	.0762	.0701	.0649	.0607	.0543	.0498	.0463
15	.2226	.2202	.1972	.1763	.1588	.1441	.1317	.1212	.1053	.0942	.0857
20	.3915	.3862	.3444	.3065	.2746	.2479	.2253	.2061	.1768	.1563	.1410
25	.6072	.5997	.5340	.4742	.4239	.3817	.3459	.3155	.2690	.2365	.2122
30	.8686	.8603	.7661	.6797	.6069	.5457	.4939	.4497	.3821	.3346	.2994
35	1.1741	1.1680	1.0411	.9234	.8240	.7403	.6694	.6089	.5162	.4511	.4028
40	1.5217	1.5225	1.3591	1.2056	1.0755	.9658	.8729	.7935	.6716	.5860	.5226
45	1.9089	1.9235	1.7204	1.5268	1.3619	1.2227	1.1046	1.0037	.8486	.7396	.6589
50	2.3325	2.3701	2.1254	1.8873	1.6836	1.5114	1.3651	1.2400	1.0475	.9121	.8119
60	3.2717	3.3976	3.0671	2.7287	2.4355	2.1864	1.9743	1.7928	1.5126	1.3152	1.1691
70	4.2984	4.5945	4.1841	3.7339	3.3361	2.9957	2.7052	2.4560	2.0705	1.7981	1.5965
80	5.3577	5.9423	5.4790	4.9073	4.3907	3.9449	3.5629	3.2348	2.7249	2.3643	2.0968
90	.	7.4110	6.9435	6.2511	5.6047	5.0400	4.5537	4.1346	3.4816	3.0171	2.6730
100	.	8.9558	8.5676	7.7696	6.9854	6.2887	5.6849	5.1625	4.3456	3.7620	3.3294
110	.	10.5049	10.3305	9.4588	8.5356	7.6975	6.9635	6.3258	5.3231	4.6035	4.0699
120	.	11.9716	12.1954	11.3115	10.2579	9.2713	8.3974	7.6323	6.4215	5.5484	4.8989
130	.	.	14.1047	13.3104	12.1560	11.0200	9.9945	9.0868	7.6489	6.6027	5.8234
140	.	.	15.9660	15.4244	14.2189	12.9405	11.7614	10.7087	9.0139	7.7737	6.8477
150	.	.	17.6672	17.6007	16.4316	15.0401	13.7056	12.4899	10.5245	9.0705	7.9804

TABLE 6

HEIGHT KFT

DISTANCE MILES	5	10	15	20	25	30	35	40	50	60	70
5	.0091	.0231	.0259	.0265	.0265	.0262	.0260	.0258	.0255	.0252	.0248
10	.0217	.0689	.0744	.0720	.0687	.0652	.0619	.0589	.0540	.0504	.0474
15	.0418	.1447	.1551	.1478	.1391	.1303	.1218	.1141	.1016	.0923	.0850
20	.0682	.2497	.2677	.2539	.2378	.2214	.2058	.1915	.1683	.1511	.1378
25	.0991	.3830	.4119	.3902	.3646	.3386	.3138	.2912	.2542	.2268	.2058
30	.1325	.5430	.5873	.5567	.5198	.4821	.4462	.4133	.3595	.3196	.2891
35	.1660	.7281	.7933	.7532	.7033	.6520	.6030	.5579	.4843	.4295	.3878
40	.1971	.9358	1.0291	.9795	.9151	.8483	.7843	.7253	.6287	.5567	.5020
45	.2232	1.1641	1.2939	1.2354	1.1553	1.0712	.9903	.9155	.7929	.7014	.6318
50	.2418	1.4090	1.5863	1.5206	1.4239	1.3209	1.2213	1.1290	.9772	.8637	.7775
60	.2501	1.9361	2.2485	2.1771	2.0459	1.9009	1.7588	1.6263	1.4070	1.2423	1.1170
70	.2231	2.4816	2.9993	2.9439	2.7804	2.5892	2.3986	2.2192	1.9202	1.6944	1.5223
80	.2005	3.0016	3.8207	3.8126	3.6248	3.3863	3.1423	2.9101	2.5191	2.2224	1.9953
90	.	3.4452	4.6780	4.7689	4.5741	4.2912	3.9913	3.7012	3.2075	2.8288	2.5384
100	.	3.7639	5.5316	5.7994	5.6251	5.3042	4.9474	4.5952	3.9877	3.5170	3.1545
110	.	3.9201	6.3291	6.8710	6.7588	6.4182	6.0094	5.5936	4.8629	4.2900	3.8464
120	.	3.9312	7.0067	7.9475	7.9627	7.6279	7.1738	6.6971	5.8369	5.1516	4.6173
130	.	.	7.4937	8.9769	9.2092	8.9193	8.4423	7.9028	6.9123	6.1053	5.4713
140	.	.	7.7293	9.8930	10.4569	10.2723	9.7952	9.2175	8.0909	7.1551	6.4118
150	.	.	7.7099	10.6160	11.6542	11.6633	11.2325	10.6218	9.3770	8.3049	7.4432

TABLE 7

DISTANCE MILES	HEIGHT KFT 5	10	15	20	25	30	35	40	50	60	70
5	-.0065	-.0019	-.0008	-.0003	-.0000	.0001	.0002	.0002	.0003	.0003	.0004
10	-.0268	-.0084	-.0043	-.0024	-.0015	-.0009	-.0005	-.0003	-.0000	.0001	.0002
15	-.0617	-.0196	-.0102	-.0060	-.0039	-.0026	-.0018	-.0012	-.0006	-.0003	-.0001
20	-.1123	-.0356	-.0187	-.0112	-.0073	-.0050	-.0035	-.0026	-.0015	-.0008	-.0005
25	-.1807	-.0571	-.0299	-.0179	-.0118	-.0081	-.0057	-.0043	-.0025	-.0016	-.0011
30	-.2695	-.0846	-.0441	-.0264	-.0175	-.0120	-.0086	-.0064	-.0039	-.0025	-.0018
35	-.3820	-.1191	-.0615	-.0368	-.0244	-.0168	-.0120	-.0091	-.0055	-.0036	-.0026
40	-.5228	-.1616	-.0827	-.0493	-.0326	-.0224	-.0161	-.0122	-.0075	-.0048	-.0035
45	-.6968	-.2132	-.1080	-.0642	-.0423	-.0291	-.0209	-.0158	-.0097	-.0063	-.0047
50	-.9109	-.2765	-.1382	-.0817	-.0537	-.0369	-.0265	-.0200	-.0123	-.0081	-.0059
60	-1.4924	-.4437	-.2160	-.1260	-.0822	-.0564	-.0404	-.0305	-.0187	-.0123	-.0090
70	-2.3492	-.6870	-.3238	-.1860	-.1201	-.0820	-.0586	-.0441	-.0270	-.0177	-.0130
80	-3.5911	-1.0399	-.4737	-.2672	-.1704	-.1156	-.0822	-.0616	-.0374	-.0245	-.0179
90	.	-1.5533	-.6835	-.3782	-.2375	-.1596	-.1128	-.0839	-.0507	-.0330	-.0239
100	.	-2.3002	-.9795	-.5258	-.3243	-.2163	-.1520	-.1125	-.0674	-.0437	-.0314
110	.	-3.3840	-1.3993	-.7309	-.4438	-.2917	-.2030	-.1491	-.0884	-.0568	-.0406
120	.	-4.9414	-1.9997	-1.0155	-.6002	-.3891	-.2703	-.1962	-.1148	-.0734	-.0518
130	.	.	-2.8631	-1.4124	-.8152	-.5219	-.3547	-.2562	-.1482	-.0939	-.0658
140	.	.	-4.1054	-1.9740	-1.1116	-.6963	-.4717	-.3347	-.1909	-.1194	-.0827
150	.	.	-5.8817	-2.7735	-1.5206	-.9329	-.6185	-.4353	-.2437	-.1515	-.1037

TABLE 8

DISTANCE MILES	HEIGHT KFT 5	10	15	20	25	30	35	40	50	60	70
5	-.0032	-.0006	-.0002	-.0001	-.0000	.0000	.0000	.0000	-.0000	-.0000	.0000
10	-.0133	-.0029	-.0013	-.0005	-.0002	-.0001	-.0000	-.0000	-.0000	.0000	.0000
15	-.0309	-.0067	-.0031	-.0012	-.0005	-.0002	-.0001	-.0000	-.0000	.0000	.0000
20	-.0573	-.0124	-.0056	-.0022	-.0011	-.0004	-.0002	-.0001	-.0000	.0000	.0000
25	-.0942	-.0202	-.0091	-.0036	-.0017	-.0008	-.0003	-.0001	-.0000	.0001	.0000
30	-.1442	-.0306	-.0136	-.0055	-.0027	-.0012	-.0005	-.0002	-.0001	.0001	-.0000
35	-.2105	-.0442	-.0194	-.0079	-.0038	-.0017	-.0007	-.0004	-.0001	.0001	-.0000
40	-.2974	-.0618	-.0265	-.0109	-.0053	-.0024	-.0010	-.0006	-.0002	.0001	-.0000
45	-.4101	-.0840	-.0354	-.0146	-.0072	-.0034	-.0015	-.0008	-.0002	.0002	-.0001
50	-.5553	-.1130	-.0464	-.0193	-.0095	-.0045	-.0020	-.0012	-.0004	.0002	-.0001
60	-.9784	-.1951	-.0767	-.0322	-.0159	-.0078	-.0037	-.0022	-.0007	.0001	-.0002
70	-1.6559	-.3276	-.1222	-.0516	-.0256	-.0129	-.0065	-.0038	-.0014	-.0000	-.0003
80	-2.7138	-.5396	-.1916	-.0809	-.0401	-.0206	-.0107	-.0063	-.0024	-.0004	-.0006
90	.	-.8786	-.2981	-.1261	-.0622	-.0323	-.0171	-.0102	-.0040	-.0010	-.0010
100	.	-1.4169	-.4631	-.1907	-.0929	-.0488	-.0266	-.0160	-.0065	-.0020	-.0017
110	.	-2.2638	-.7186	-.2906	-.1416	-.0741	-.0406	-.0246	-.0101	-.0035	-.0026
120	.	-3.5736	-1.1171	-.4432	-.2097	-.1095	-.0622	-.0371	-.0153	-.0059	-.0039
130	.	.	-1.7384	-.6752	-.3139	-.1646	-.0906	-.0548	-.0229	-.0093	-.0059
140	.	.	-2.7042	-1.0330	-.4716	-.2424	-.1372	-.0810	-.0339	-.0142	-.0085
150	.	.	-4.1878	-1.5853	-.7079	-.3584	-.1978	-.1175	-.0485	-.0212	-.0123

TABLE 9

DISTANCE MILES	HEIGHT KFT 5	10	15	20	25	30	35	40	50	60	70
5	-.0806	-.0301	-.0136	-.0060	-.0011	.0021	.0043	.0057	.0073	.0083	.0087
10	-.3321	-.1360	-.0739	-.0481	-.0313	-.0201	-.0127	-.0075	-.0012	.0024	.0044
15	-.7533	-.3146	-.1754	-.1189	-.0819	-.0575	-.0411	-.0296	-.0154	-.0077	-.0029
20	-1.3470	-.5687	-.3196	-.2191	-.1535	-.1104	-.0813	-.0608	-.0355	-.0218	-.0132
25	-2.1168	-.9026	-.5084	-.3498	-.2468	-.1793	-.1336	-.1015	-.0616	-.0401	-.0265
30	-3.0664	-1.3217	-.7448	-.5125	-.3628	-.2648	-.1986	-.1519	-.0940	-.0628	-.0430
35	-4.1993	-1.8327	-1.0320	-.7091	-.5028	-.3679	-.2768	-.2126	-.1330	-.0901	-.0627
40	-5.5178	-2.4440	-1.3744	-.9418	-.6680	-.4894	-.3690	-.2842	-.1788	-.1222	-.0860
45	-7.0221	-3.1634	-1.7769	-1.2136	-.8604	-.6308	-.4761	-.3672	-.2320	-.1592	-.1127
50	-8.7091	-4.0036	-2.2458	-1.5276	-1.0819	-.7935	-.5992	-.4625	-.2929	-.2016	-.1434
60	-12.5871	-6.0883	-3.4103	-2.2978	-1.6230	-1.1893	-.8979	-.6937	-.4402	-.3038	-.2171
70	-16.9766	-8.8019	-4.9354	-3.2912	-2.3149	-1.6931	-1.2773	-.9865	-.6263	-.4320	-.3092
80	-21.4838	-12.2504	-6.9080	-4.5605	-3.1905	-2.3271	-1.7521	-1.3526	-.8573	-.5909	-.4227
90	.	-16.5206	-9.4378	-6.1745	-4.2931	-3.1194	-2.3426	-1.8054	-1.1417	-.7843	-.5608
100	.	-21.6282	-12.6475	-8.2077	-5.6667	-4.1012	-3.0720	-2.3636	-1.4910	-1.0207	-.7284
110	.	-27.4311	-16.6690	-10.7799	-7.4021	-5.3291	-3.9747	-3.0501	-1.9168	-1.3061	-.9311
120	.	-33.4941	-21.6151	-14.0135	-9.5614	-6.8460	-5.0972	-3.8960	-2.4355	-1.6531	-1.1731
130	.	.	-27.5400	-18.0521	-12.2761	-8.7510	-6.4662	-4.9325	-3.0686	-2.0720	-1.4666
140	.	.	-34.3121	-23.0424	-15.6717	-11.1154	-8.1908	-6.2123	-3.8447	-2.5769	-1.8159
150	.	.	-41.4796	-29.0964	-19.9016	-14.0671	-10.3024	-7.7822	-4.7800	-3.1896	-2.2378

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