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Titheridge Coefficients for the Polynomial Method of Deducing Electron Density Profiles From Ionograms

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Sets of Titheridge polynomial coefficients suitable for the conversion into electron density profiles of smooth virtual height frequency curves, such as those observed regularly at night, are presented for a series of values of the magnetic dip angle and gyrofrequency. These magnetic conditions have been chosen so that the coefficients are suitable for the analysis, to a reasonable degree of accuracy, of ordinary and extraordinary ray records obtained anywhere in the world.

The coefficients may be used for the analysis of topside-sounder data to a first order of accuracy if the plasma frequency at the satellite is small.

1. Introduction

The information contained in ionograms about the distribution of electrons with height continues to provide a very important source of our knowledge of the ionosphere and of how it changes with time. A new method for deriving the electron density real height variation from the observed $h'(f)$ curve, known as the "polynomial" method, has recently been independently, and almost simultaneously, developed by Titheridge [1961], Unz [1961], and Van Zandt [1961].

The electron density profile corresponding to the observed virtual height curve is obtained by this method on the assumption that the complete real height curve can be represented by a single polynomial in plasma frequency, f_N. This polynomial is the one of lowest degree in f_N which can produce the observed virtual heights, so that the method produces the smoothest possible real height curve which is consistent with the experimental results.

The principle of the method and the details of the calculation have been fully described by Titheridge [1961] who shows that, in certain circumstances, the method may be applied either manually or in an electronic digital computer in a very simple way, although the procedure becomes complicated and rather unsuitable for manual application if the ionograms correspond to profiles in which the electron density N does not increase monotonically with height. For the analysis of smooth $h'(f)$ curves (such as those observed regularly at night when the electron density in the E region is very much lower than that in the F region) however, the simple polynomial method is the most accurate and rapid method to employ. It seems likely, too, that this method may be of importance for the analysis of topside sounder data [Knecht, Van Zandt, and Watts, 1961]. This application is considered in some detail in a companion paper [Long and Thomas, 1963].

It is the purpose of this note to present sets of polynomial coefficients calculated for a series of values of f_H (the gyrofrequency) and I (the magnetic dip angle) which are such that the coefficients are suitable for the analysis, to a reasonable degree of accuracy, of records obtained anywhere in the world.

The calculation of the coefficients was made using a digital computer program written for the purpose by Titheridge [1961] for EDSAC II, the computer in the Mathematical Laboratory of the University of Cambridge.

2. Calculation of the Coefficients

It is assumed that the real height h is given as a function of plasma frequency f_N by the relation

$$h=\sum_{j=1}^{n}\alpha_j f_N{}^{j+1}$$

where α is a constant. This can be written in matrix notation as

$$\mathbf{h}=\mathbf{A}\boldsymbol{\alpha}.$$

It is possible to define the virtual height, h', from the polynomial expression for h in a similar form (see Titheridge, 1961, p. 3):

$$\mathbf{h}'=\mathbf{B}\boldsymbol{\alpha}.$$

Combining these two equations gives

$$\mathbf{h}=(\mathbf{A}\mathbf{B}^{-1})\mathbf{h}'$$

where the matrix $(\mathbf{AB}^{-1})$ depends on the plasma frequency and the magnitude and direction of the earth's magnetic field only. A matrix may therefore be calculated once and for all for a particular location and for a given set of predetermined frequencies at which the $h'(f)$ record is to be read.

Tests have shown that a sixth-order polynomial ($n=6$) gives the most accurate and useful results. The tables presented in this paper give sets of 6×6 matrices for the Ordinary and Extraordinary reflected waves calculated for 12 magnetic zones chosen in such a way that records for any location may, with the appropriate matrix coefficients, be reduced to provide an $N(h)$ profile which is accurate to about 5 percent [Schmerling and Ventrice, 1960; Wright and Norton, 1959]. Figure 1 shows the magnetic locations for which matrices are computed in relation to the worldwide dip-gyrofrequency variation. The matrix to be used in a particular calculation is selected after reference to table 1 and figure 1 as the one for which the values of f_H and dip angle correspond most closely to those for the actual location at which the record was obtained. The way in which the coefficients are used is discussed in the next section.

3. Use of the Coefficients

The equation to be solved for the calculation of the real height, h_i, at a plasma frequency, $f_N=f_i$, is

$$h_i=\sum_{j=1}^{n}(\mathbf{AB}^{-1})_{ij}h'_j.$$

For the *Ordinary ray*, $\Delta f=f_i-f_{i-1}$ has been chosen to be constant and equal to 1 Mc/s, and f_1 to be 1 Mc/s, so that the virtual heights are read at 1, 2, 3,

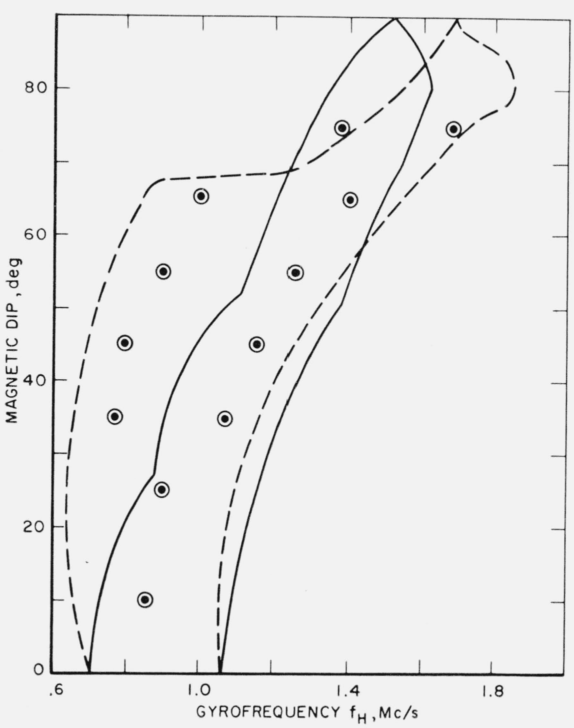

FIGURE 1. *The limits of magnetic dip I and gyrofrequency* f_H *at 200 km (after Wright and Norton [1959]).*

— North of the magnetic equator
— — South of the magnetic equator
◉ ◉ Magnetic location for which polynomial coefficients are presented.

TABLE 1

Gyrofrequency f_H	Magnetic dip I
Mc/s	*degrees*
0.86	10
.9	25
.77	35
1.07	35
0.8	45
1.15	45
0.9	55
1.25	55
1.0	65
1.4	65
1.38	75
1.68	75

The magnetic conditions (gyrofrequency and dip) for which polynomial coefficients are presented in the following tables.

4, 5, and 6 Mc/s. To calculate the real height at, say, a plasma frequency $f_N=f_5=5$ Mc/s, the virtual heights at these six frequencies are multiplied by the coefficients in row 5 of the appropriate matrix. For example, for a dip of 65° and gyrofrequency 1.4 Mc/s,

$$h_5=0.1291h'_1+0.1951h'_2+0.1108h'_3+0.3424h'_4 +0.1602h'_5-0.0035h'_6.$$

For the *Extraordinary ray*, the frequencies at which the virtual heights are to be read is given by f_{xj} where

$$f^2_{xj}=f^2_N+f_Hf_{xj}.$$

In the example quoted above, the virtual heights would be read, on the extraordinary ray trace, at 1.92, 2.82, 3.78, 4.76, 5.75, and 6.74 Mc/s, and the real height at the plasma frequency $f_N=5$ Mc/s, i.e., a wave frequency of 5.75 Mc/s would be given by

$$h_5=0.1188h'_{1x}+0.1408h'_{2x}+0.0866h'_{3x}+0.2819h'_{4x} +0.1985h'_{5x}-0.0036h'_{6x}.$$

It is necessary to calculate the values of f_x for the value of f_H at the actual location at which the ionogram was obtained in order to read the extraordinary ray trace virtual heights at the correct frequencies.

With the aid of these matrices, it is possible to reduce an $h'(f)$ curve to an $N(h)$ curve manually in less than 10 min, by merely reading six virtual heights and using six slide-rule settings.

The circumstances under which the tables may be applied and the errors which may arise from their application form the subject matter of the companion paper already referred to [Long and Thomas, 1963] and are also discussed by Titheridge [1961].

The coefficients presented here were calculated using the method devised by Dr. J. E. Titheridge (Seagrove Radio Research Station, University of Auckland), from whom permission was obtained to use a program written by him for EDSAC II, the digital computer in the Mathematical Laboratory of the University of Cambridge.

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	Ordinary ray						Extraordinary ray					
i \ f_N	1 Mc/s	2 Mc/s	3 Mc/s	4 Mc/s	5 Mc/s	6 Mc/s	1 Mc/s	2 Mc/s	3 Mc/ s	4 Mc/s	5 Mc/s	6 Mc/s
f_H=0.86 Mc/s; I=10°												
1	0.7291	−0.1475	0.0759	−0.0301	0.0075	−0.0008	0.5432	−0.1840	0.1053	−0.0441	0.0113	−0.0013
2	.4531	.4458	−.0426	.0118	−.0024	.0002	.4218	.2911	−.0153	.0032	.0003	−.0001
3	.2183	.3393	.3615	−.0295	.0056	−.0006	.2354	.2873	.2928	−.0323	.0067	−.0007
4	.2097	.1218	.3394	.2768	−.0099	.0007	.2040	.1161	.3250	.2109	−.0075	.0005
5	.1135	.1920	.0721	.3305	.2303	−.0038	.1300	.1629	.0975	.3039	.1819	−.0035
6	.3216	−.1962	.4798	−.1612	.4048	.1810	.2421	−.1292	.4163	−.1133	.3730	.1412
f_H=0.9 Mc/s; I=25°												
1	0.7200	−0.1509	0.0783	−0.0312	0.0078	−0.0009	0.5397	−0.1829	0.1048	−0.0438	0.0112	−0.0013
2	.4656	.4297	−.0425	.0117	−.0024	.0002	.4130	.2956	−.0151	.0011	.0003	−.0001
3	.2255	.3462	.3459	−.0291	.0055	−.0006	.2304	.2822	.2997	−.0326	.0067	−.0007
4	.2113	.1294	.3429	.2627	−.0097	.0007	.2013	.1110	.3231	.2174	−.0076	.0005
5	.1166	.1923	.0804	.3312	.2177	−.0037	.1282	.1608	.0932	.3033	.1883	−.0036
6	.3122	−.1799	.4670	−.1450	.4011	.1702	.2429	−.1367	.4215	−.1220	.3752	.1468
f_H=0.77 Mc/s; I=35°												
1	0.7141	−0.1581	0.0829	−0.0333	0.0083	−0.0009	0.5724	−0.1790	0.1007	−0.0418	0.0106	−0.0012
2	.4759	.4175	−.0430	.0119	−.0024	.0002	.4210	.3224	−.0203	.0031	−.0003	−.0000
4	.2292	.3533	.3350	−.0295	.0056	−.0006	.2290	.2912	.3127	−.0322	.0065	−.0007
4	.2118	.1324	.3492	.2521	−.0097	.0007	.2034	.1110	.3259	.2303	−.0081	.0006
5	.1178	.1924	.0837	.3365	.2078	−.0037	.1261	.1662	.0867	.3092	.1979	−.0037
6	.3039	−.1713	.4622	−.1394	.4048	.1610	.2570	−.1508	.4351	−.1342	.3829	.1548
f_H=1.07 Mc/s; I=35°												
1	0.7111	−0.1561	0.0816	−0.0328	0.0082	−0.0009	0.5061	−0.1820	0.1058	−0.0445	0.0114	−0.0013
2	.4777	.4136	−.0423	.0117	−.0024	.0002	.3910	.2807	−.0104	−.0008	.0007	−.0001
3	.2322	.3534	.3307	−.0290	.0055	−.0006	.2232	.2660	.2990	−.0331	.0069	−.0007
4	.2128	.1363	.3468	.2488	−.0095	.0007	.1950	.1043	.3161	.2166	−.0072	.0005
5	.1193	.1929	.0876	.3331	.2052	−.0037	.1260	.1541	.0922	.2958	.1898	−.0036
6	.3017	−.1638	.4555	−.1311	.3990	.1591	.2322	−.1350	.4163	−.1229	.3694	.1489
f_H=0.8 Mc/s; I=45°												
1	0.7084	−0.1669	0.0884	−0.0358	0.0090	−0.0010	0.5689	−0.1766	0.0992	−0.0411	0.0104	−0.0012
2	.4881	.4030	−.0435	.0120	−.0025	.0002	.4137	.3256	−.0199	.0030	−.0002	−.0000
3	.2342	.3610	.3221	−.0301	.0058	−.0006	.2258	.2865	.3174	−.0321	.0065	−.0007
4	.2129	.1366	.3557	.2393	−.0097	.0007	.2017	.1080	.3230	.2352	−.0081	.0006
5	.1195	.1931	.0882	.3413	.1962	−.0037	.1251	.1648	.0843	.3069	.2027	−.0037
6	.2964	−.1624	.4574	−.1328	.4081	.1503	.2570	−.1542	.4368	−.1382	.3820	.1593
f_H=1.15 Mc/s; I=45°												
1	0.7026	−0.1636	0.0865	−0.0350	0.0088	−0.0010	0.4925	−0.1801	0.1051	−0.0443	0.0114	−0.0013
2	.4918	.3958	−.0424	.0117	−.0024	.0002	.3783	.2785	−.0089	−.0014	.0009	−.0001
3	.2394	.3616	.3144	−.0291	.0056	−.0006	.2177	.2577	.3023	−.0333	.0070	−.0007
4	.2147	.1427	.3526	.2334	−.0094	.0007	.1911	.0996	.3120	.2201	−.0071	.0005
5	.1222	.1935	.0947	.3364	.1912	−.0036	.1240	.1508	.0897	.2923	.1940	−.0036
6	.2927	−.1502	.4466	−.1190	.3993	.1466	.2283	−.1375	.4164	−.1271	.3677	.1529
f_H=0.9 Mc/s; I=55°												
1	0.7024	−0.1762	0.0945	−0.0385	0.0097	−0.0011	0.5496	−0.1754	0.0992	−0.0412	0.0105	−0.0012
2	.5028	.3859	−.0437	.0121	−.0025	.0002	.4001	.3186	−.0172	.0020	.0000	−.0000
3	.2413	.3693	.3066	−.0304	.0059	−.0006	.2216	.2763	.3186	−.0323	.0066	−.0007
4	.2149	.1423	.3622	.2244	−.0096	.0007	.1981	.1038	.3179	.2365	−.0079	.0005
5	.1226	.1935	.0944	.3456	.1826	−.0037	.1240	.1608	.0832	.3019	.2053	−.0037
6	.2883	−.1508	.4507	−.1232	.4100	.1379	.2523	−.1551	.4350	−.1398	.3783	.1621
f_H=1.25 Mc/s; I=55°												
1	0.6927	−0.1745	0.0936	−0.0382	0.0096	−0.0011	0.4744	−0.1782	0.1048	−0.0442	0.0114	−0.0013
2	.5065	.3747	−.0418	.0114	−.0023	.0002	.3637	.2735	−.0068	−.0022	.0011	−.0002
3	.2475	.3691	.2973	−.0295	.0057	−.0006	.2117	.2481	.3043	−.0335	.0071	−.0008
4	.2165	.1491	.3589	.2166	−.0092	.0007	.1864	.0948	.3073	.2222	−.0070	.0004
5	.1253	.1940	.1022	.3400	.1761	−.0036	.1217	.1468	.0876	.2880	.1972	−.0036
6	.2813	−.1359	.4378	−.1066	.3996	.1329	.2233	−.1395	.4156	−.1305	.3651	.1563

	Ordinary ray						Extraordinary ray					
f_N / i	1 Mc/s	2 Mc/s	3 Mc/s	4 Mc/s	5 Mc/s	6 Mc/s	1 Mc/s	2 Mc/s	3 Mc/ s	4 Mc/s	5 Mc/s	6 Mc/s
f_H=1.0 Mc/s; I=65°												
1	0. 6967	−0. 1870	0. 1016	−0. 0418	0. 0106	−0. 0012	0. 5301	−0. 1745	0. 0995	−0. 0414	0. 0105	−0. 0012
2	. 5176	. 3676	−. 0435	. 0120	−. 0024	. 0002	. 3869	. 3109	−. 0147	. 0010	. 0003	−. 0001
3	. 2488	. 3772	. 2910	−. 0309	. 0060	−. 0006	. 2172	. 2667	. 3190	−. 0325	. 0067	−. 0007
4	. 2168	. 1482	. 3686	. 2092	−. 0096	. 0007	. 1944	. 0997	. 3133	. 2372	−. 0077	. 0005
5	. 1252	. 1942	. 1009	. 3497	. 1688	−. 0037	. 1227	. 1569	. 0822	. 2973	. 2072	−. 0037
6	. 2798	−. 1391	. 4440	−. 1132	. 4116	. 1253	. 2453	−. 1540	. 4319	−. 1408	. 3748	. 1643
f_H=1.4 Mc/s; I=65°												
1	0. 6850	−0. 1844	0. 1002	−0. 0413	0. 0105	−0. 0012	0. 4460	−0. 1763	0. 1051	−0. 0446	0. 0115	−0. 0013
2	. 5230	. 3538	−. 0412	. 0112	−. 0022	. 0002	. 3444	. 2617	−. 0035	−. 0035	. 0014	−. 0002
3	. 2571	. 3772	. 2787	−. 0297	. 0058	−. 0006	. 2040	. 2351	. 3033	−. 0340	. 0072	−. 0008
4	. 2195	. 1572	. 3642	. 1990	−. 0091	. 0007	. 1797	. 0891	. 3014	. 2215	−. 0067	. 0004
5	. 1291	. 1951	. 1108	. 3424	. 1602	−. 0035	. 1188	. 1408	. 0866	. 2819	. 1985	−. 0036
6	. 2719	−. 1204	. 4281	−. 0924	. 3985	. 1187	. 2135	−. 1383	. 4114	−. 1320	. 3608	. 1580
f_H=1.38 Mc/s; I=75°												
1	0. 6827	−0. 1962	0. 1081	−0. 0449	0. 0115	−0. 0013	0. 4518	−0. 1751	0. 1039	−0. 0440	0. 0113	−0. 0013
2	. 5362	. 3386	−. 0412	. 0111	−. 0022	. 0002	. 3447	. 2682	−. 0045	−. 0031	. 0013	−. 0002
3	. 2631	. 3842	. 2664	−. 0304	. 0060	−. 0006	. 2033	. 2358	. 3075	−. 0338	. 0072	−. 0008
4	. 2212	. 1610	. 3710	. 1866	−. 0091	. 0007	. 1800	. 0883	. 3011	. 2259	−. 0068	. 0004
5	. 1314	. 1952	. 1154	. 3474	. 1489	−. 0035	. 1182	. 1419	. 0843	. 2825	. 2022	−. 0036
6	. 2660	−. 1134	. 4251	−. 0870	. 4026	. 1081	. 2156	−. 1414	. 4143	−. 1355	. 3621	. 1613
f_H=1.68 Mc/s; I=75°												
1	0. 6746	−0. 1940	0. 1069	−0. 0444	0. 0114	−0. 0013	0. 3936	−0. 1731	0. 1064	−0. 0458	0. 0119	−0. 0014
2	. 5396	. 3290	−. 0395	. 0105	−. 0021	. 0002	. 3121	. 2353	. 0028	−. 0061	. 0021	−. 0003
3	. 2692	. 3834	. 2580	−. 0295	. 0058	−. 0006	. 1908	. 2135	. 2960	−. 0349	. 0076	−. 0008
4	. 2237	. 1670	. 3672	. 1797	−. 0087	. 0006	. 1676	. 0805	. 2919	. 2147	−. 0061	. 0004
5	. 1343	. 1963	. 1219	. 3417	. 1431	−. 0034	. 1132	. 1302	. 0867	. 2713	. 1960	−. 0036
6	. 2616	−. 1009	. 4150	−. 0733	. 3929	. 1036	. 1927	−. 1296	. 3984	−. 1294	. 3520	. 1568

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