JOURNAL OF RESEARCH of the National Bureau of Standards—D. Radio Propagation
Vol. 67D, No. 1, January–February 1963

On the Theory of Radio Wave Propagation Over Inhomogeneous Earth[1]

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(Received August 10, 1962; revised August 20, 1962)

The formulas of field strength over an inhomogeneous spherical earth are obtained on the conditions that (I) the radius and the electrical properties of the earth's surface discontinuously change several times along the wave path, or (II) the surface of terrain arbitrarily changes in height along the wave path, but it is still smooth everywhere and the radius of curvature is sufficiently large as compared with the wavelength.

The case (I) is considered to be more general than those of mixed paths on a smooth earth, because the latter can be seen as special cases of the former. The case (II) corresponds to the case of multiple diffraction of radio waves by several mountains having finite radii of curvature. In both cases, the unified formulas of field strength are obtained in the form of a multiple residue series, which is reduced to the ordinary Van der Pol and Bremmer formula in the special case of homogeneous ground.

The convergence of series of the formulas is very good when the propagation distance on every section of the inhomogeneous ground is long enough or the diffraction loss is large enough, and is poor when any one of these distances is so short that the section is effectively seen as a flat plane, or the diffraction loss on the section is very small. In these cases, the flat earth or other approximations can be used, and several supplementary formulas are prepared for cases of poor convergence. Several special applications are given.

Introduction

The problems of mixed paths over a smooth earth have been investigated by many authors, especially in the case of a flat earth, and many equations and approximations have been established according to the given situation. However, the mixed paths in these cases mean the radio wave propagation path over a smooth ground, in which several sections having different electrical properties are included, and thus changes in height are not included.

In part I of this paper, these changes in height are taken into account on the assumption that the ground surface changes discontinuously along the wave path, as illustrated in figure 3. A series of papers has been established on the theory of propagation over terrain of this model [Furutsu, 1957a, 1957b, 1959a], and the unified formula of field strength was obtained in the form of a multiple residue series, which reduces to that of mixed paths in the special case of smooth earth. This model of terrain form may not be suitable for the range of very high frequency where the radius of curvature of the terrain becomes sufficiently large as compared with the wavelength, and it also may not be suitable for a completely irregular terrain where some statistical approach would be more appropriate. In the former case, the problem could be treated as a multiple diffraction by hills or mountains having large radii of curvature, in which the earth's surface could be included, as illustrated in figure 23. Also in this case, the field strength can be obtained in a unified form of multiple residue series [Furutsu, 1956], and part II of this paper is devoted to this subject.

The purposes of this paper are the survey of these formulas and the several possible applications to the practically important cases, such as the evaluations of the effects on the ground radio wave of a ridge, a cliff, and a bluff at a close line (part I) or those of several hills or mountains, taking into account their curvatures if necessary (part II).

[1] This work was sponsored by the Air Force Cambridge Research Laboratories, Office of Aerospace Research (USAF), Bedford, Massachusetts.

Part I. Radio Wave Propagation Over Inhomogeneous Spherical Earth

1. Equation Formulation

Solving Maxwell's equation in terms of the vector potential, A, the equation in the Cartesian coordinate system $(x)=(x, y, z)$ becomes (the earth surface is assumed to be flat for the time being)

$$[(\nabla k^{-2}\nabla)+1]A(x)=-k^{-2}\mu_0 I(x), \quad k^2=\omega^2\mu_0\epsilon_0(\epsilon+\sigma/i\omega), \qquad \nabla=\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{1.1}$$

Here, the time factor $e^{i\omega t}$ is omitted and I is the current density of the external system. Other notations are those ordinarily used.

In the case of the vertical dipole excitation at the arbitrary point x', we may first presuppose that the horizontally polarized wave will not be induced by the terrain of the form assumed and thus, using the z-axis in the vertical direction from the ground, put

$$A(x)=(0,\ 0,\ \psi(x, x')), \quad k^{-2}\mu_0 I(x)=(0,\ 0,\ \delta(x-x')). \tag{1.2}$$

This presupposition will be justified later. Thus, the solution can be represented by the Green function defined by the equation

$$[(\nabla k^{-2}\nabla)+1]\psi(x,x')=-\delta(x-x'). \tag{1.3}$$

Here, the boundary condition on every horizontal surface made by the medium discontinuities is found to be the continuity condition of

$$(\psi), \left(k^{-2}\frac{\partial}{\partial n}\psi\right), \tag{1.4}$$

where $\partial/\partial n=\partial/\partial z$ and the earth's surface is flat in this case.

Even when the curvature of the earth's surface is taken into account, eq (1.3) is still valid in a small but practically sufficient domain of the earth's surface; it is sufficient to assume, for any small line element having the components (dx, dy, dz), the length ds given by the metrics

$$ds^2=(z/a)^2(dx^2+dy^2)+dz^2.$$

Here, the coordinate system is taken so that the earth's surface is given by the surface $z=a$, a being the earth's radius. Further, the boundary condition (1.4) is also valid on the condition that the earth's radius is sufficiently large as compared with the wavelength.

In spite of the fact that eqs (1.3) and (1.4) are valid only in the case where the medium changes with the z-coordinate but not with the x, y-coordinates, we are going to treat the case where the height and electrical properties of the earth's surface take spatially discontinuous values, as is illustrated in figure 3. Indeed, from the definition of ψ, it is readily proven that the boundary condition (1.4) does not hold on the vertical boundary surfaces of medium discontinuity. However, as is seen in the following, this fact does not give any serious effect to the result of the assumption that eq (1.4) also holds on the vertical boundary surfaces.

For any continuous functions of ψ' and ψ'', the Green theorem can be given in the form (fig. 1)

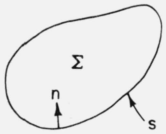


FIGURE 1. *The integration domain for eq (1.5).*

$$\int_\Sigma [\psi'\{\nabla k^{-2}\nabla+1\}\psi''-\psi''\{\nabla k^{-2}\nabla+1\}\psi']\,dv=-[\psi'(s),\ \psi''(s)]. \tag{1.5}$$

Here, the left side is the volume integral in the space Σ and

$$[\psi'(s),\psi''(s)]=-[\psi''(s),\psi'(s)]$$
$$=\int_s \left[\psi'\left(\frac{1}{k^2}\frac{\partial}{\partial n}\psi''\right)-\psi''\left(\frac{1}{k^2}\frac{\partial}{\partial n}\psi'\right)\right]ds. \tag{1.6}$$

Here, $\partial/\partial n$ is the inward normal differentiation on the surface s of the arbitrary space Σ.

It is especially to be noted that the lemma (1.5) holds even when k takes discontinuous values across the surfaces contained in the space Σ, if both of the functions ψ' and ψ'' satisfy the boundary condition of (1.4) on these surfaces.

We here introduce the solution $\psi_m(x, x')$ of (1.3) for the smooth and homogeneous earth having the propagation constant k_m. By the use of (1.5) and also the boundary condition, the symmetrical relation of $\psi_m(x, x')$ is readily derived; putting $\psi'(x)=\psi_m(x, x_1)$ and $\psi''(x)=\psi_m(x,x_2)$ in (1.5), the points x_1 and x_2 being arbitrary points in space, and taking the whole space for Σ, we have

$$\psi_m(x_1, x_2)-\psi_m(x_2, x_1)=-[\psi_m(s, x_1),\ \psi_m(s, x_2)]_{S\infty}=0,$$

or

$$\psi_m(x_1, x_2)=\psi_m(x_2, x_1). \tag{1.7}$$

It is appropriate to begin with the simple terrain as in figure 2. Here, the propagation constants k_2 and k_3 of the ground are assumed to be different.

Now in the preceding Green theorem (1.5), let $\psi'(x)=\psi_{32}(x_4, x)$ and $\psi''(x)=\psi_2(x, x_1)$. Here ψ_{32} is the Green function to be obtained, and ψ_2 is the Green function for the homogeneous ground of the elevation $z=a_2$ and of the propagation constant k_2. Taking the space Σ as the whole space except the space enclosed by the surface s_3 defined in figure 2, and the point x_1 being in Σ, we have $(a_2>a_3)$

$$\psi_{32}(x_4, x_1)=\psi_2(x_4, x_1)+[\psi_{32}(x_4, s_3),\ \psi_2(s_3, x_1)], \qquad z_4>a_2 \tag{1.8}$$

or

$$\psi_{32}(x_4, x_1)=[\psi_{32}(x_4, s_3),\ \psi_2(s_3, x_1)], \qquad z_4<a_2, \tag{1.9}$$

when the point x_4 is below the surface s_3. Here, it is noted that the integrand of the surface integral is continuous at the boundary of the medium k_2 and the atmosphere because both ψ_{32} and ψ_2 should satisfy the same boundary condition, and therefore there is no contribution to the surface integral on the right side of eqs (1.8) and (1.9).

Just in the same way, by exchanging the roles of the Green functions ψ_2 and ψ_3, we have $(a_2>a_3)$

$$\psi_{32}(x_4, x_1)=\psi_3(x_4, x_1)+[\psi_{32}(x_4, s_2),\ \psi_3(s_2, x_1)]. \tag{1.10}$$

These two equations are complementary to each other, and, by the method of successive substitution, we have the required solution in series

$$\psi_{32}(x_4, x_1)=\psi_2(x_4, x_1)+[\psi_3(x_4, s_3),\ \psi_2(s_3, x_1)]$$
$$+[\psi_2(x_4, s_2),\ \psi_3(s_2, s_3),\ \psi_2(s_3, x_1)]+\ .\ .\ ., \qquad z_4>a_2. \tag{1.11}$$

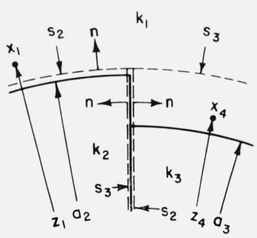


FIGURE 2. *The form of terrain and the integration domain for eqs (1.8) and (1.9).*

Here,

$$[\psi,\ \psi',\ \psi'']=[[\psi,\ \psi'],\ \psi'']=[\psi,\ [\psi',\ \psi'']],$$

and, on the vertical boundary surface, the boundary condition (1.4) is taken into account.

On the other hand, when the points x_1 and x_4 are located over the different earth media, say on the sides of the media k_2 and k_3, respectively, the term $\psi_{32}(x_4,\ s_3)$ in the bracketed terms of (1.8) and (1.9) may be given by

$$\psi_{32}(x_4,\ s_3)\simeq\psi_3(x_4,\ s_3),$$

except in the vertical domain of s_3 which is not important for the integral, because $\psi_2(s_3,\ x_1)$ rapidly tends to vanish with the distance a_2-z by the term $\exp\ [-i\sqrt{k_2^2-k_1^2}(a_2-z)]$ in the ground.

Thus

$$\begin{aligned}\psi_{32}(x_4,\ x_1)&\simeq\psi_2(x_4,\ x_1)+[\psi_3(x_4,\ s_3),\ \psi_2(s_3,\ x_1)], \qquad z_4>a_2\\ &\simeq[\psi_3(x_4,\ s_3),\ \psi_2(s_3,\ x_1)], \qquad z_4<a_2.\end{aligned}$$

The above result is just the equation obtained from (1.11) by the omission of terms of higher order than the third in the series. The higher order terms become important only when the point x_4 or x_1 is located in the immediate vicinity of the vertical boundary surface. Also the surface integral over the vertical boundary surface can be neglected because of the large attenuation of $\psi_2(s_3,\ x_1)$ in the ground, and so far the presupposition that the horizontally polarized wave will not be induced is self-consistent.

There may be another problem of singularity of solution at the diffracting edge or uniqueness of solution for the form of boundary assumed. But it is known [Born and Wolf, 1959] that, insofar as the field components can be expressed as convergent Fourier integrals, which is the situation in our case, they are free from having singularities of too high an order, and their uniqueness is assured.

The above result can easily be extended to the general case where there are several boundaries of the earth medium discontinuities, as in figure 3. As is illustrated there, we successively assume, along the wave path from the point x_1 to x_{n+1}, the values $k_2,\ k_3,\ k_4,\ \ldots,\ k_n$ for the propagation constants of the different sections of inhomogeneous earth, and the values a_2, $a_3,\ a_4,\ \ldots,\ a_n$ for the radii of the surfaces of the respective sections, and $r_2,\ r_3,\ r_4,\ \ldots,\ r_n$ for the propagation distances, respectively. Also, the solution in this case will be denoted by $\psi_{n,\ldots,2}(x_{n+1},\ x_1)$ and thus the solution $\psi_{n-1,\ldots,2}(x_{n+1},\ x_1)$ will be the one to be obtained from $\psi_{n,\ldots,2}(x_{n+1},\ x_1)$ by setting $k_n\rightarrow k_{n-1}$ and $a_n\rightarrow a_{n-1}$.

On referring to figure 3, we now set $\psi'(x)=\psi_{n,\ldots,2}(x_{n+1},\ x)$, $\psi''(x)=\psi_{n-1,\ldots,2}(x,\ x_1)$ in eq (1.5), and take, for the space Σ, the whole space excluding the space of the medium k_n and the atmosphere above it in the range $z<a_{n-1}$, the point x_1 being assumed to be within Σ. Then, as in eq (1.8), we have

$$\psi_{n,\ldots,2}(x_{n+1},\ x_1)=\psi_{n-1,\ldots,2}(x_{n+1},\ x_1)+[\psi_{n,\ldots,2}(x_{n+1},\ s_n),\ \psi_{n-1,\ldots,2}(s_n,\ x_1)], \qquad z_{n+1}>a_{n-1} \tag{1.12}$$

or

$$\psi_{n,\ldots,2}(x_{n+1},\ x_1)=[\psi_{n,\ldots,2}(x_{n+1},\ s_n),\ \psi_{n-1,\ldots,2}(s_n,\ x_1)], \qquad z_{n+1}<a_{n-1}, \tag{1.13}$$

depending on whether the point x_{n+1} is within or without the space Σ, respectively.

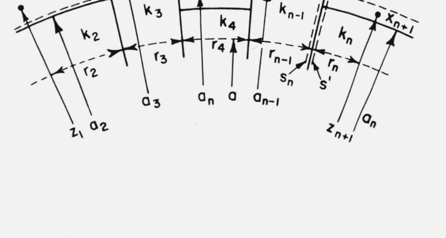


FIGURE 3. *The form of terrain and the domains of integrations for eqs (1.12), (1.13), and (1.14).*

In the same way, by letting $\psi'(x)=\psi_{n,\ldots,2}(x_{n+1}, x)$ and $\psi''(x)=\psi_n(x, x_1)$ in eq (1.5), we have

$$\psi_{n,\ldots,2}(x_{n+1}, x_1)=\psi_n(x_{n+1}, x_1)+[\psi_{n,\ldots,2}(x_{n+1}, s'), \psi_n(s', x_1)]. \tag{1.14}$$

Here s' is the whole upper surface of the media k_m for which $a_m \geq a_n$, $m=2, 3, \ldots, n-1$, plus the surface of $z=a_n$ above the media for which $a_m < a_n$, plus the vertical boundary surface of the medium k_n, as is illustrated in figure 3, and both the points x_1 and x_{n+1} are to be outside the surface s'.

When the solution $\psi_{n-1,\ldots,2}(x_{n+1}, x_1)$ is known, we can, by the successive substitution of eqs (1.12) and (1.14), express the solution $\psi_{n,\ldots,2}(x_{n+1}, x_1)$ in a form of series similar to eq (1.11):

$$\begin{aligned}\psi_{n,\ldots,2}(x_{n+1}, x_1)=\psi_{n-1,\ldots,2}(x_{n+1}, x_1)&+[\psi_n(x_{n+1}, s_n), \psi_{n-1,\ldots,2}(s_n, x_1)]\\ &+[\psi_{n-1,\ldots,2}(x_{n+1}, s'), \psi_n(s', s_n), \psi_{n-1,\ldots,2}(s_n, x_1)]+\ldots,\end{aligned} \tag{1.15}$$

or, in just the same way,

$$\begin{aligned}\psi_{n,\ldots,2}(x_{n+1}, x_1)=&[\psi_n(x_{n+1}, s_n), \psi_{n-1,\ldots,2}(s_n, x_1)]\\ &+[\psi_n(x_{n+1}, s_n), \psi_{n-1,\ldots,2}(s_n, s'), \psi_n(s', s_n''), \psi_{n-1,\ldots,2}(s_n'', x_1)]+\ldots,\end{aligned} \tag{1.16}$$

depending on whether the point x_{n+1} is within or without the space Σ, respectively. Thus, eq (1.15) gives a recurrence formula for $\psi_{n,\ldots,2}$.

As in eq (1.11), we can neglect the terms of higher order than the third or second in the series above except for the range immediately near the boundary.

2. Formula of Field Strength in the General Case

The attenuation coefficient A is here defined by

$$\psi_{n,\ldots,2}(x_{n+1}, x_1)=2A(z_{n+1}|r_n, r_{n-1},\ldots, r_2|z_1)\psi_0(x_{n+1}, x_1),$$

$$\psi_0(x_{n+1}, x_1)=\frac{k_1^2}{4\pi r}e^{-ik_1 r}, \qquad r=r_n+r_{n-1}+\ldots+r_2, \tag{2.1}$$

where $\psi_0(x_{n+1}, x_1)$ may be regarded as the solution in free space. Then, on the assumption that the boundary surfaces between the different sections of the inhomogeneous earth are all vertical and parallel with each other, the result of evaluation according to eq (1.15) is as follows (figure 3):

$$\begin{aligned}A(z_{n+1}|r_n, r_{n-1},\ldots, r_2|z_1)=\sum_{t_n, t_{n-1},\ldots, t_2}&(r/r_n)^{\frac{1}{2}}A(z_{n+1}|r_n)_{t_n}T(r_{n-1})_{t_n, t_{n-1}}\\ &\times T(r_{n-2})_{t_{n-1}, t_{n-2}}\ldots T(r_3)_{t_4, t_3}T(r_2)_{t_3, t_2}f_{t_2}(z_1, a_2).\end{aligned} \tag{2.2}$$

Here, t_m stands for the set of roots of the equation

$$W'(t)-q_m W(t)=0, \tag{2.3}$$

where

$$W(-t)=\exp(-i2\pi/3)(\pi t/3)^{\frac{1}{2}}H_{1/3}^{(2)}\left(\frac{2}{3}t^{3/2}\right),$$

$$iq_m=(k_1 a/2)^{\frac{1}{3}}\times\begin{cases}k_1\sqrt{k_m^2-k_1^2}/k_m^2, & \text{Vert. Pol.}\\ \sqrt{k_m^2-k_1^2}/k_1, & \text{Horiz. Pol.}\end{cases} \tag{2.4}$$

These notations are similar to those of Fock [1946].

The factor $f_{t_2}(z_1, a_2)$ is the ordinarily defined height-gain function for the end point x_1:

$$f_{t_2}(z_1, a_2)=W(t_2-y_1)/W(t_2),\ \ y_1=(2/k_1 a)^{\frac{1}{3}}k_1(z_1-a_2), \tag{2.5}$$

and the other two kinds of terms that appear in the preceding formula for attenuation coefficient A are as follows:

$$A(z_{n+1}|r_n)_{t_n}=\sqrt{(\pi/2)k_1r_n}(2/k_1a)^{\frac{1}{3}}(t_n-q_n^2)^{-1}f_{t_n}(z_{n+1},a_n)\exp\left[-i\{(r_n/a)(k_1(a_n-a)+(k_1a/2)^{\frac{1}{3}}t_n)+\pi/4\}\right], \tag{2.6}$$

$$T(r_n)_{t_m,t_n}=\{k_1(a_m-a_n)(2/k_1a)^{\frac{1}{3}}+t_m-t_n\}^{-1}(t_n-q_n^2)^{-1}$$
$$\times\exp\left[-i(r_n/a)\{k_1(a_n-a)+(k_1a/2)^{\frac{1}{3}}t_n\}\right]\times\begin{cases} q_mf'_{t_m}(a_n,a_m)-q_nf_{t_m}(a_n,a_m), & a_n\geq a_m\\ q_mf_{t_n}(a_m,a_n)-q_nf'_{t_n}(a_m,a_n), & a_m\geq a_n.\end{cases} \tag{2.7}$$

The term $A(z_{n+1}|r_n)_{t_n}$ depends only on the electrical properties of the nth section through q_n and t_n and on the propagation distance r_n and the elevation a_n of the nth section, but does not depend on those quantities of other sections.

On the other hand, the term $T(r_n)_{t_m,t_n}$ depends on both the mth and nth sections through t_m and t_n and also through q_m and q_n, and therefore it serves as the coupling term between the two sections. Besides, it depends of course on the height difference between the two sections through the ordinary height-gain function $f_{t_n}(a_m,a_n)$ and also through another height-gain function $f'_{t_n}(a_m,a_n)$ defined by

$$q_mf'_{t_m}(a_n,a_m)=-(k_1a/2)^{\frac{1}{3}}\frac{\partial}{\partial(k_1a_n)}f_{t_m}(a_n,a_m), \tag{2.8}$$

which is proportional to the first-order derivative of the ordinary height-gain function.

There is a clear one-to-one correspondence between the terms of the formula (2.2) and the respective sections of the inhomogeneous earth, i.e., the height-gain function $f_{t_2}(z_1,a_2)$ expresses the effect of the elevation of the point x_1 from the ground, of course, and the term $T(r_2)_{t_3,t_2}$ expresses the effect of propagation along the surface of the section of No. 2. In the same way, the term $T(r_3)_{t_4,t_3}$ corresponds to the propagation along the section of No. 3, the term $T(r_4)_{t_5,t_4}$ corresponds to the propagation along the section of No. 4, and so on, and finally, the term $A(z_{n+1}|r_n)_{t_n}$ corresponds to the propagation along the nth section, over which the point x_{n+1} exists.

The convergence of the multiple series of the attenuation coefficient A is very good when the propagation distance of every section of the inhomogeneous ground is sufficiently long, and in this case, the first term of the residue series is a sufficiently good approximation. This situation is just the same as in the ordinary Van der Pol and Bremmer formula for a homogeneous spherical earth.

On the other hand, the convergence of the series becomes poor when the propagation distance over one of the sections is very small. In this case, the flat-earth approximation or other proper approximation can be suitably used.

In the extreme case where the width of the one of the sections tends to zero, it will represent a ridge on the ground, as in figure 5. In this case, the responsible series in the formula takes the asymptotic form as r_m tends to zero (fig. 4),

$$\lim_{r_m\to 0}\Sigma_{t_m}T(r_m)_{t_l,t_m}T(r_n)_{t_m,t_n}=T^{(m)}(r_n)_{t_l,t_n}\equiv\{q_lf'_{t_l}(a_m,a_l)f_{t_n}(a_m,a_n)-q_nf_{t_l}(a_m,a_l)f'_{t_n}(a_m,a_n)\}$$
$$\times\{k_1(a_l-a_n)(2/k_1a)^{\frac{1}{3}}+t_l-t_n\}^{-1}(t_n-q_n^2)^{-1}\exp\left[-i(r_n/a)\{k_1(a_n-a)+(k_1a/2)^{\frac{1}{3}}t_n\}\right],\ (a_m\geq a_n,a_l). \tag{2.9}$$

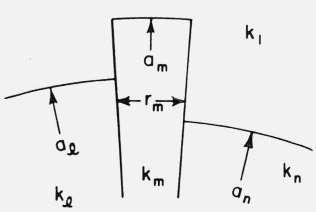


FIGURE 4. *The form of terrain and the notations for eq (2.9).*

The above result is applicable to the case of diffraction by ridges on spherical earth, as it will be seen in the example of the following section.

It may be remarked that, in the special case of $a_n = a_{n-1} = \ldots = a_2$, the formula (2.2) reduces to that previously obtained for mixed paths on a smooth earth [Furutsu, 1955].

3. A Few Examples of Applications

The simplest application of the foregoing formula (2.2) will be to the case of one ridge on a spherical ground, where the electrical properties on each side of the ridge could be different, as illustrated in figure 5. By the use of the result (2.9), the attenuation coefficient A is then given by

$$A = \Sigma_{t_4, t_2} (r/r_4)^{\frac{1}{2}} A(z_5|r_4)_{t_4} T^{(3)}(r_2)_{t_4, t_2} f_{t_2}(z_1, a_2). \tag{3.1}$$

One of the immediate applications of this formula will be the estimation of the effect of a ridge on a spherical smooth earth (fig. 6), and another will be the similar estimation of the effect of a cliff on the ground radio wave (fig. 7). In the case in which the propagation distances r_4 and r_2 on both sides of the ridge (or cliff) are long enough and the height h of the ridge (or cliff) is low enough, i.e.,

$$k_1(r_4 - \sqrt{2ah})(k_1 a)^{-\frac{2}{3}} \gg 1, \qquad k_1(r_2 - \sqrt{2ah})(k_1 a)^{-\frac{2}{3}} \gg 1,$$

so that simply the first term of the series would give a good approximation, then the effect of the ridge (or cliff) can be expressed by multiplying the field strength in the case of smooth earth by the factor $F_{42}^{(3)}$. Here, using the notation t_4^0 and t_2^0 for the first values of the set of values t_4 and t_2,

$$F_{42}^{(3)} = \{ q_4 f'_{t_4^0}(a_3, a_4) f_{t_2^0}(a_3, a_2) - q_2 f_{t_4^0}(a_3, a_4) f'_{t_2^0}(a_3, a_2) \} \times \{ k_1(a_4 - a_2)(2/k_1 a)^{\frac{1}{3}} + t_4^0 - t_2^0 \}^{-1} (t_2^0 - q_2^2)^{-1} \tag{3.2}$$

or, when $a_4 = a_2$ and $q_4 = q_2$,

$$F_{22}^{(3)} = [1 - k_1(a_3 - a_2)(2/k_1 a)^{\frac{1}{3}} (t_2^0 - q_2^2)^{-1}] f_{t_2^0}^2(a_3, a_2). \tag{3.3}$$

Thus, depending on whether the boundary is a ridge or a cliff, the field strength is given in the form (figs. 6 and 7)

$$E = (E)_{h=0} F_{KK'}(\rho), \quad E = (E)_{h=0} F_{KK'}^{\pm}(\rho), \tag{3.4}$$

respectively. Here, K and K' correspond to k and k', respectively, and according to K. A. Norton, K and b are defined by

$$K = |q|^{-1} 2^{-1/3}, \quad -b = \pi/2 + 2 \arg(q), \tag{3.5}$$

and

$$\rho = k_1 h (k_1 a)^{-1/3}, \quad k_1 = \omega/c. \tag{3.6}$$

Hence ρ is proportional to the height of the ridge or cliff.

The set of graphs in figure 8 shows the numerical values of the factor F_{KK}, giving the effect of a ridge on the ground waves, and it is displayed as a function of ρ, which is proportional to the height of the ridge, for a useful range of values of K and b.

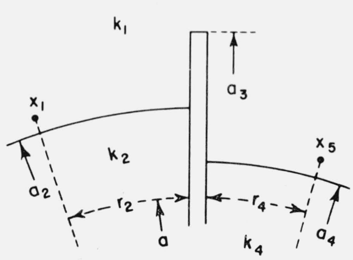


FIGURE 5. *The form of terrain and the notations for eq (3.1).*

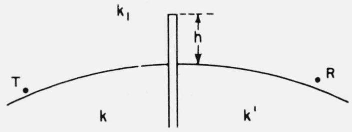


FIGURE 6. *The form of terrain for eq (3.3) and figure 8.*

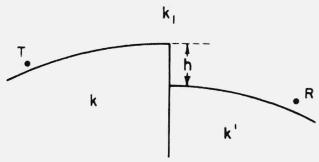


FIGURE 7. *The form of terrain for figure 9.*

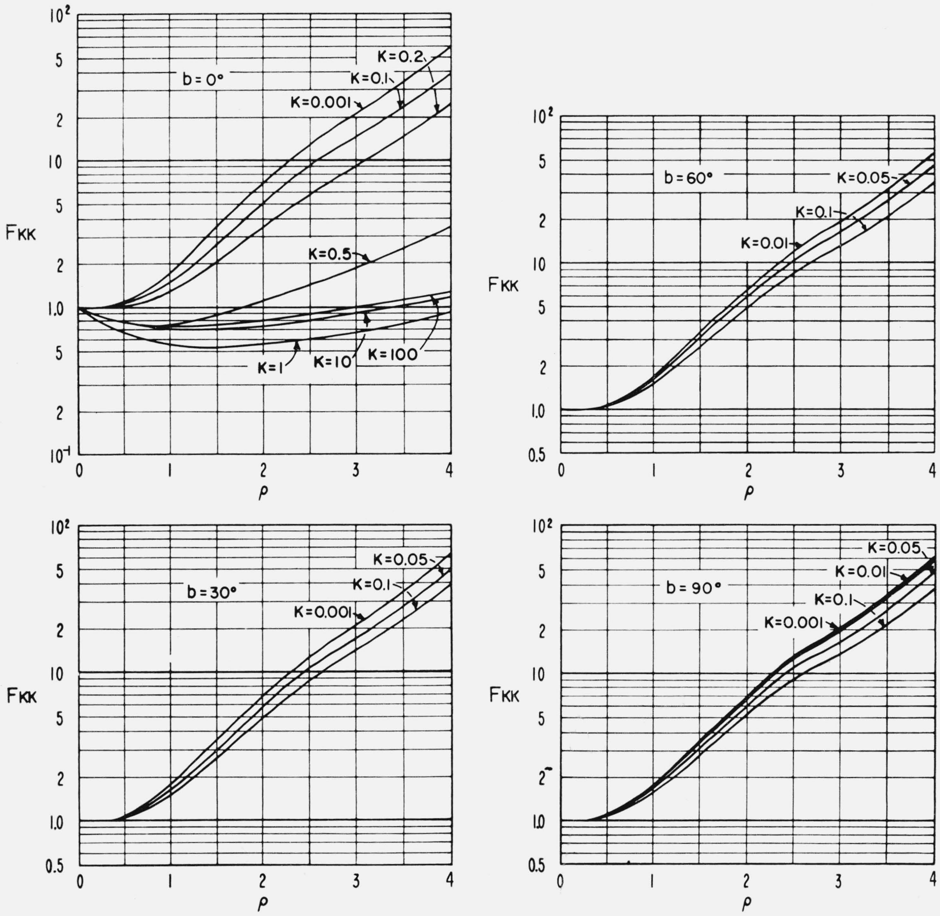


FIGURE 8. *Ridge effect on ground radio wave as a function of* ρ *and* b *in the case of* K′=K.

In the same way, the set of graphs in figure 9 shows the numerical values of the factor $F^{\pm}_{KK}$ giving the effect of a cliff on the ground wave, and it is also displayed here as a function of ρ for the same range of K and b.

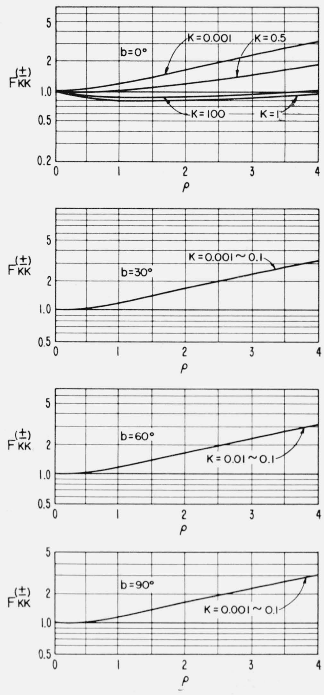


FIGURE 9. *Effect of cliff on ground radio wave as a function of ρ and b in the case of K′=K.*

4. A Few Examples in Flat-Earth Approximation

The results of eq (3.4) are not available in the case where the propagation distance on one or both sides of the ridge is very short. Then the effect of the ridge depends on the propagation distances from the ridge, so the result is not so simple as the preceding cases.

In this case, the effect of the earth curvature may be neglected and the flat earth approximation is appropriate; the convergence of series of the formula (3.1) becomes very poor in this case. However, there is a general rule to overcome this difficulty which will be treated in part II. The results in the flat-earth approximation are then obtained as the asymptotic forms for $a \to \infty$, i.e., the infinite radius of earth. The formula (3.1) in the case of a flat earth becomes as follows:

Referring to figure 10, it is convenient first to introduce the notations that will be used later:

$$\begin{aligned} d_4 &= -i(k_1/k_4')^2 k_1 r_4/2, & d_2 &= -i(k_1/k_2')^2 k_1 r_2/2, \\ f_5 &= (z_5 - a_4)\sqrt{k_1/2r_4}\,e^{i\pi/4}, & f_1 &= (z_1 - a_2)\sqrt{k_1/2r_2}\,e^{i\pi/4}, \\ f_4 &= (a_3 - a_4)\sqrt{k_1/2r_4}\,e^{i\pi/4}, & f_2 &= (a_3 - a_2)\sqrt{k_1/2r_2}\,e^{i\pi/4}, \\ k_1 &= \frac{\omega}{c},\ k_n' = \begin{cases} k_n^2/\sqrt{k_n^2 - k_1^2}, & \text{Vert. Pol.} \\ k_1^2/\sqrt{k_n^2 - k_1^2}, & \text{Horiz. Pol.} \end{cases} & & (n=2,4). \end{aligned} \tag{4.1}$$

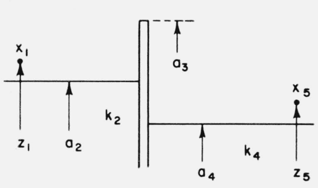


FIGURE 10. *The form of terrain and the notations for eqs (4.2) and (4.4).*

The attenuation coefficient in this case will conveniently be expressed by $A^{(3)}(z_5|r_4,\ r_2|z_1)$, which means the attenuation for the wave propagation from the point x_1 at the height z_1 to the point x_5 at the height z_5 across the ridge of the elevation a_3 which exists at the distance r_2 from the point x_1 and the distance r_4 from the point x_5.

In the special case where both the points x_1 and x_5 are on the ground, i.e., when $z_5=a_4$, $z_1=a_2$ or $f_5=f_1=0$,

$$
\begin{aligned}
A^{(3)}(a_4|r_4, r_2|a_2)=F(d_4,f_4|d_2,f_2) \equiv e^{-(f_4^2+f_2^2)}[\mathscr{E}(\sqrt{n_2}f_4+\sqrt{n_4}f_2)-i\sqrt{\pi}(\sqrt{d_4/n_4}+\sqrt{d_2/n_2})^{-1} \\
\times[\sqrt{d_4d_2/n_4n_2}\,\mathscr{E}(f_4+i\sqrt{d_4})\mathscr{E}(f_2+i\sqrt{d_2})+(d_4/n_4)\{\mathscr{E}(\sigma_4,(\sqrt{n_2}f_4+\sqrt{n_4}f_2)/\sigma_4) \\
-\mathscr{E}(\rho_4,(f_4+i\sqrt{d_4})/\rho_4)\}+(d_2/n_2)\{\mathscr{E}(\sigma_2,(\sqrt{n_2}f_4+\sqrt{n_4}f_2)/\sigma_2)-\mathscr{E}(\rho_2,(f_2+i\sqrt{d_2})/\rho_2)\}]]. \quad (4.2)
\end{aligned}
$$

Here

$$
\mathscr{E}(z,n/z)=\left(\frac{2}{\sqrt{\pi}}\right)^2 e^{z^2+n^2}\int_z^{\infty} dx\, e^{-x^2}\int_{(n/z)x}^{\infty} e^{-y^2}dy,
$$

$$
\sigma_4=\sqrt{n_4}f_4-\sqrt{n_2}f_2+i\sqrt{d_4/n_4}, \qquad \rho_4=i\sqrt{d_4n_2/n_4}-f_2,
$$

$$
\sigma_2=\sqrt{n_2}f_2-\sqrt{n_4}f_4+i\sqrt{d_2/n_2}, \qquad \rho_2=i\sqrt{d_2n_4/n_2}-f_4. \quad (4.3)
$$

In the general case of $z_5\neq a_4$ and $z_1\neq a_2$,

$$
\begin{aligned}
A^{(3)}(z_5|r_4,\ r_2|z_1)=F(d_4,f_4+f_5|d_2,f_2+f_1)+\tfrac{1}{2}\{F(d_4,f_4+f_5|0,f_2-f_1) \\
-F(d_4,f_4+f_5|0,f_2+f_1)+F(0,f_4-f_5|d_2,f_2+f_1)-F(0,f_4+f_5|d_2,f_2+f_1)\} \\
+\tfrac{1}{4}\{F(0,f_4+f_5|0,f_2+f_1)+F(0,f_4-f_5|0,f_2-f_1)-F(0,f_4+f_5|0,f_2-f_1)-F(0,f_4-f_5|0,f_2+f_1)\}. \\
(4.4)
\end{aligned}
$$

Immediate applications of the formula (4.2) are possible for the effect of a ridge on the ground wave, and the effect of a cliff and the effect of a bluff along a coast line, as illustrated in figures 11 to 13, some numerical results of which will be displayed in the following. The notations that will be used are:

h : Height of ridge, bluff, or cliff, in meters

f : Frequency in megacycles per second

σ : Conductivity of land in millimho per meter

$$
\left.\begin{matrix} D \\ d \end{matrix}\right\}=\frac{5\pi f^2}{27\sigma}\times\left\{\begin{matrix} r_1+r_2, \\ r_1 \end{matrix}\right. \qquad Y=\frac{\pi}{900}f\sqrt{\frac{f}{2\sigma}}\,h,
$$

$$
\left.\begin{matrix} L \\ S \end{matrix}\right\}=\frac{5\pi f^2}{27\sigma}\times\left\{\begin{matrix} r_2, \\ r_1 \end{matrix}\right. \qquad f\ll 18\sigma/\epsilon. \quad (4.5)
$$

Case 1. One Ridge on Homogeneous Ground

The ridge is assumed to have the height h from the flat ground and the transmitting and receiving points to be at the distance r_2 and r_1 from the ridge on opposite sides, as illustrated in figure 11.

The two curves in figures 14 (a), (b) represent respectively the attenuation coefficient and the phase delay in the case of a homogeneous earth. Here, the abscissa is the Sommerfeld numerical distance for the whole propagation distance. On the other hand, the curves in

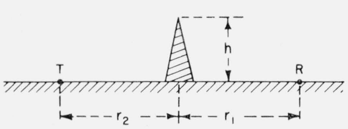


FIGURE 11. *The form of ridge and the notations for figures 15 (a) and (b).*

figures 15 (a), (b) represent the same attenuation coefficient and the phase delay in the case where one ridge is present between the transmitter and receiver. Here, the parameter d is the numerical distance on one side of the ridge, and Y is the numerical height introduced in eq (4.5) which is proportional to the height of the ridge.

Case 2. A Bluff at Coast Line

The bluff is assumed to have the height h along a coast line, and the transmitter and receiver to be at the distance r_1 on the sea side and r_2 on the land side from the coast line, as illustrated in figure 12. The sea is assumed to be a perfectly conducting plane, while the land has the finite conductivity σ.

The curves in figures 16 (a), (b) represent the values of the attenuation coefficient and the phase delay in the case where there is no bluff at all at a coast line. The abscissa S is the Sommerfeld numerical distance for the propagation distance on the sea side measured in land conductivity, and the parameter L is the numerical distance on the land side.

On the other hand, figures 17 (a), (b) show the values in the case where there is a bluff at the coast line. Here Y is again the numerical height proportional to the height of the bluff.

Case 3. A Cliff

The cliff is assumed to have the height h on the flat ground, and the transmitter and receiver to be at the distance r_1 on the lower side and r_2 on the higher side, as illustrated in figure 13.

The curves in figure 18 display the corresponding values in this case.

In case 1, ridge diffraction, the magnitude A of the attenuation coefficient decreases with the height h of the ridge in most cases, but for the large transmission distances of $D \gg 1$ it ceases to decrease at some height and then tends to gradually increase with the height. This fact may be interpreted by noticing that the diffraction loss by the ridge is rather smaller than the transmission loss along the dissipative ground at large transmission distances; a ridge could give an obstacle-gain even on a flat earth.

In case 2, where the radio waves propagate across a coast line having a bluff, the rate of change of the relative phase with the distance r_1 (which is proportional to the numerical distance S in figure 17) from the coast line becomes larger as the height of the bluff increases and is sometimes much more than that without the bluff.

Besides the subjects just mentioned, it is also possible to evaluate the diffracted wave by a ridge of finite thickness, as illustrated in figure 19.

In summary of part I, it may be remarked that the general formula (2.2) for the attenuation coefficient, given in the residue series, takes the unified form independently of the number of sections of the inhomogeneous earth, but the corresponding formula in the flat earth approximation is probably more difficult to derive and would take a more complicated form. Equation (4.2) or (4.4) is an example of the latter for two sections. Practically, the case of three sections will be the limitation in which the flat earth approximation is possible, unless some special assumptions about the propagation distances are made. Also, the effect of reflecting waves from the boundaries of discontinuities on the transmitted waves are completely neglected here. But they are generally believed to be very small in most cases. In fact, the result has been proven to be exact at least for the trapped wave (or surface wave) mode [Furutsu, 1959b].

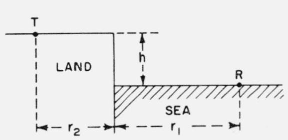


FIGURE 12. *The form of coast line and the notations for figures 17 (a) and (b).*

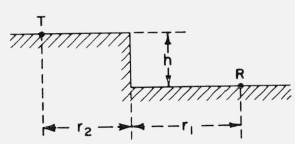
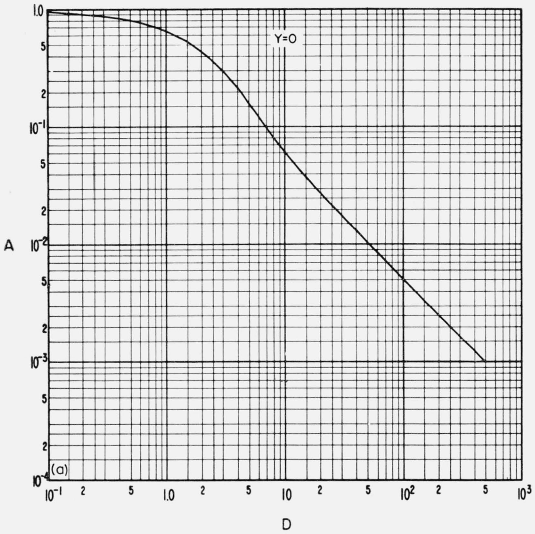


FIGURE 13. *The form of cliff and the notations for figure 18.*



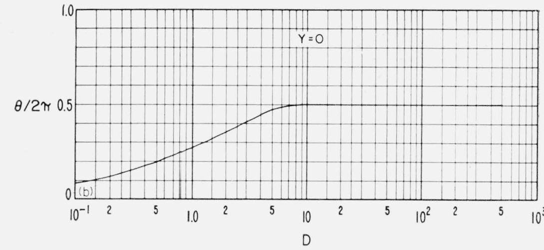


FIGURE 14. *The attenuation coefficient* (a) *and the phase delay* (b) *in the case of a homogeneous earth.*

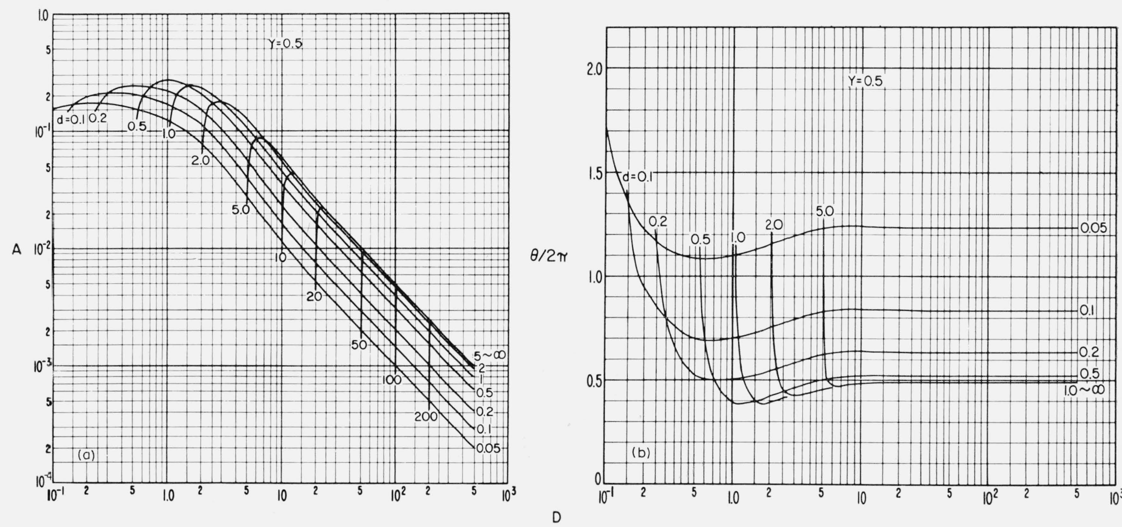


FIGURE 15. *An example of the attenuation coefficient* (a) *and the phase delay* (b) *in the case of one ridge present*

Part II. Multiple Diffraction of Electromagnetic Wave by Surfaces Having Finite Radii of Curvature

1. Statement of Problem and the Result in the Simplest Case of One Diffracting Surface

Many authors have discussed the diffraction of electromagnetic waves by a single object such as sphere, cylinder, paraboloid, ellipsoid, plane, etc. However, there are many difficulties when trying to calculate the diffraction loss by actual mountains or hills, because they usually can not be seen as a single smooth diffracting obstacle, but more or less have irregularities. Strictly speaking, the actual diffraction of radio waves by mountains is a problem in multiple diffraction. But it is, of course, impossible to derive the formula of diffraction loss for such complicated obstacles, and different models have usually been adopted for the diffracting mountains according to the different range of frequency. However, if we could assume that every diffracting mountain has a smooth surface whose radius of curvature is sufficiently large as compared with the wavelength, the problem can be treated on fairly general conditions, and we can get the result in a unified form, independent of the number of diffracting mountains.

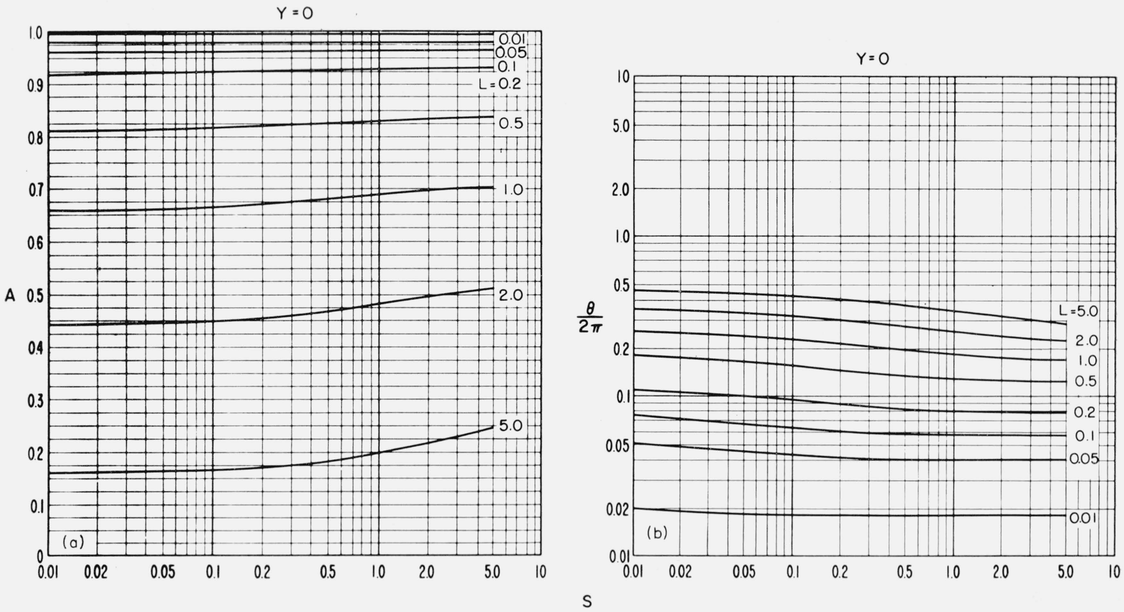


FIGURE 16. *An example of the attenuation coefficient* (a) *and the phase delay* (b) *in the case of no bluff at coast line.*

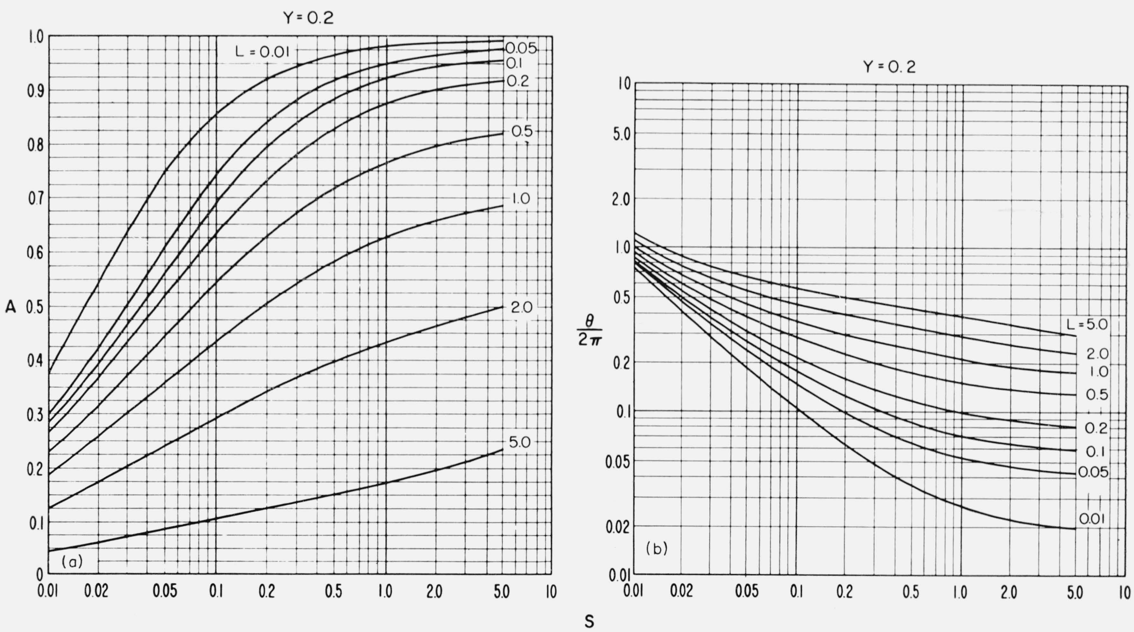
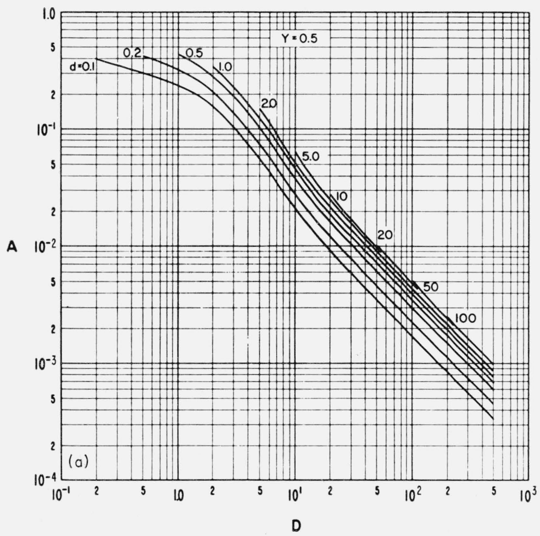


FIGURE 17. *An example of the corresponding attenuation coefficient* (a) *and the phase delay* (b) *in the case of a bluff present.*

In the simplest case of one diffracting surface, the result can be deduced from the ordinary Van der Pol and Bremmer formula for diffraction by a large spherical surface. Referring to figure 20, the attenuation coefficient A, which is the ratio of the field strength to that in free space, is given by

$$A=\sqrt{2\pi k_1 r}(2/k_1 a)^{1/3}\sum(t_s-q^2)^{-1}\exp[-i\{(r/a)(k_1 a/2)^{1/3}t_s+\pi/4\}]\times f_s(y_1)f_s(y_2). \tag{1.1}$$



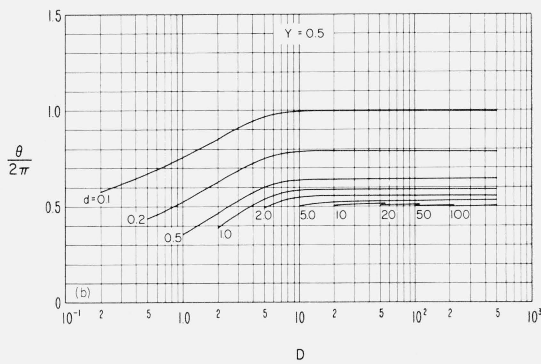


FIGURE 18. *An example of the attenuation coefficient* (a) *and the phase delay* (b) *in the case of one cliff present.*

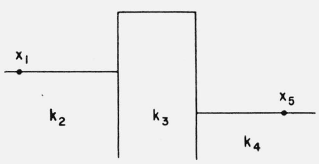
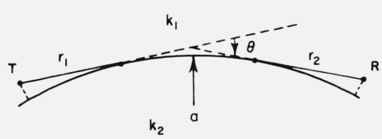


FIGURE 19. *The form of ridge having finite thickness.*

FIGURE 20. *The spherical diffracting surface and the notations for eq (1.1).*



Here t_s's are the roots of

$$W'(t)-qW(t)=0,\quad W(-t)=\sqrt{\pi/3}\,e^{-i2\pi/3}t^{1/2}H^{(2)}_{1/3}\left(\tfrac{2}{3}t^{3/2}\right), \tag{1.2}$$

and

$$f_s(y)=W(t_s-y)/W(t_s),\qquad y_i=(2/k_1a)^{1/3}k_1h_i,\qquad (h_i\ll a)$$

$$iq=(k_1a/2)^{1/3}\times\begin{cases} k_1\sqrt{k_2^2-k_1^2}/k_2^2, & \text{Vert. Pol.}\\ \sqrt{k_2^2-k_1^2}/k_1, & \text{Horiz. Pol.}\end{cases} \tag{1.3}$$

For $y\gg1$, the height gain function $f_s(y)$ takes the asymptotic form

$$f_s(y)\simeq y^{-1/4}\exp\left[-i\left(\tfrac{2}{3}y^{3/2}-t_sy^{1/2}+\tfrac{1}{4}y^{-1/2}t_s^2+\pi/4\right)\right]/W(t_s), \tag{1.4}$$

where the argument of the exponential function is expanded with respect to t_S and the terms involving powers higher than 2 are neglected. Thus, when the heights of either or both the transmitter and receiver take sufficiently large values as y_1, $y_2\gg1$, the attenuation coefficient A of (1.1) takes the following forms:

Case 1: $y_1 \gg 1,\ y_2 \lesssim 1$

$$A \simeq \exp\,[-i(k_1a/3)(r_1/a)^3]2\sqrt{\pi}\sqrt{r/r_1}\sum_s (t_s-q^2)^{-1}W(t_s)^{-1}\times\exp\,[-i\{\theta(k_1a/2)^{1/3}t_s+\pi/2\}]f_s(y_2),$$

$$\theta=(r-r_1)/a,\quad r_1=\sqrt{2ah_1},\quad r_2=\sqrt{2ah_2}. \tag{1.5}$$

Case 2: $y_1 \gg 1,\ y_2 \gg 1$

$$A \simeq \exp\left[-i\,\frac{k_1a}{3}\left\{\left(\frac{r_1}{a}\right)^3+\left(\frac{r_2}{a}\right)^3\right\}\right]\sqrt{\frac{r}{k_1r_1r_2}}\sum_s 2\sqrt{2\pi}\,(k_1a/2)^{1/3}\,(t_s-q^2)^{-1}W(t_s)^{-2}$$

$$\times\exp\,[-i\{\theta(k_1a/2)^{1/3}t_s+(2k_1)^{-1}(k_1a/2)^{2/3}(1/r_1+1/r_2)t_s^2+3\pi/4\}],$$

$$\theta=(r-r_1-r_2)/a. \tag{1.6}$$

In case 2, the series is convergent for the whole range of θ. On the other hand, in case 1, the series diverges for negative values of θ, and hence it must be analytically continued, as will be shown later. Thus the analytically continued function A of θ is regular on the whole range of θ, including negative values. However, in case 2, if the square term of t_s^2 were neglected in the exponential function, the function A of θ would have a pole at $\theta=0$, even though it is analytically continued. Thus the square term of t_s^2 cannot be neglected in case 2.

The convergence of series of these formulas is good for large diffraction angles. But, otherwise, it becomes poor. However, there is some general rule to overcome this poor convergence, as will be treated in the following section.

Since the above results have been developed according to the original Van der Pol and Bremmer formula given in residue series, they could be valid only when both the transmitter and receiver are not far from the surface, and it is not immediately clear whether they are also valid even when calculating the field far from the surface.

But it can be proven [Furutsu, 1956; Wait and Conda, 1959] that the results (1.5) and (1.6) are correct even when the transmitter and receiver are at great distances from the diffraction surface as compared with the radius of curvature, if we reinterpret $r_1(\neq\sqrt{2ah_1})$ and $r_2(\neq\sqrt{2ah_2})$ as the lengths of the parts of wave path from the transmitter and receiver to the first contacting points on the diffracting surface, respectively, as illustrated in figure 20. The important part of the mountain surface which decisively contributes to the diffracting waves is the very small part of the surface in the vicinity of wave path, and the other part is not important, provided that the radius of curvature of surface is sufficiently large as compared with the wavelength.

On the other hand, in this small part of the diffracting surface, the surface could be expressed as a surface of second degree having some finite radius of curvature such as a sphere, cylinder, paraboloid, etc., and therefore it follows that we could solve the wave equation exactly in the range of the important part of the diffracting surface. In fact, the Green function usually happens to be the same in this small range, independently of the kinds of surfaces adopted, provided that the radius of curvature of the diffracting surfaces is defined along the wave path.

2. Methods for the Case of Poor Convergence

For kinds of series such as (1.5) and (1.6), there are some general rules to overcome the difficulty when the convergence of series is poor. Taking into account

$$\mathrm{Res}_{t=t_s}\left[\frac{(t-q^2)\,W(t)}{W'(t)-qW(t)}\right]=1, \tag{2.1}$$

we have for the arbitrary series $\Sigma_s a(t_s)$

$$\sum_s a(t_s)=\sum_s \frac{1}{2\pi i}\int_{C_s}\frac{(t-q^2)W(t)}{W'(t)-qW(t)}\,a(t)dt, \tag{2.2}$$

where C_s is the infinitesimal contour integration path around the sth pole.

Hence, if the integrand thus formulated does not have any pole besides the poles t_s's, as in case 1 of eq (1.5), the sum of the contour paths is equivalent to the contour path C around the set of poles, as illustrated in figure 21. Further, if the integrand tends to zero sufficiently rapidly at infinity, we can deform it to the path C_1+C_2 in figure 21; these paths are proven to be the best paths for numerical integration in the meaning that the integrand decreases most rapidly on these paths. Furthermore, in the case of (1.5) the integral converges for negative values of θ, even when the original series diverged. Thus

$$\sum_s \int_{C_s}=\int_C=\int_{C_1+C_2}. \tag{2.4}$$

On the other hand, however, if the integrand has extra poles, such as those of the function $W(t)$, besides the necessary poles of t_s, the foregoing method cannot be used, and another integrand must be sought.

From the Wronskian identity, we see that

$$\{qv(t_s)-v'(t_s)\}W(t_s)=1,\quad v(-t)=\frac{1}{2}\sqrt{\frac{\pi}{3}}\left\{e^{-i\pi/6}t^{1/2}H_{1/3}^{(2)}\left(\frac{2}{3}t^{3/2}\right)+e^{i\pi/6}t^{1/2}H_{1/3}^{(1)}\left(\frac{2}{3}t^{3/2}\right)\right\}. \tag{2.5}$$

Hence, in principle, we could multiply the integrand by this function to any power we wished. For example, by multiplying it once, eq (2.2) is replaced by

$$\sum_s a(t_s)=\frac{-1}{2\pi i}\sum_s\int_{C_s}\left(\frac{v'(t)-qv(t)}{W'(t)-qW(t)}\right)(t-q^2)W^2(t)a(t)dt. \tag{2.6}$$

Thus, even if the old integrand had the undesirable extra poles of the function $W(t)$, the new integrand would not have them. Thus we have the contour integration path C and, further, the path C_1+C_2 if the integrand decreases sufficiently rapidly at infinity. This situation actually occurs in the series of (1.6).

Here the question may occur whether it is possible to multiply the integrand by the square of the function (2.5) and to deform the integration path to the path C_1+C_2. The answer is no, because, though the integrand has no extra pole besides t_s, it diverges at infinity on the way deforming the path from the C to the path C_1+C_2. Hence, generally, there exists only one integrand for which the integration path C_1+C_2 is available.

Usually the Kirchhoff approximation terms appear as the leading terms of these integrals, and are obtained from the asymptotic forms of the integrands for large magnitudes of t; for instance,

$$\frac{-1}{2\pi i}\left\{\frac{v'(t)-qv(t)}{W'(t)-qW(t)}\right\}\sim-\frac{1}{4\pi} \tag{2.7}$$

on the path C_1 or any path of $(0, \infty e^{-i\beta})$ in the range $\pi>\beta>\pi/3$, and it tends to zero as $|t|\rightarrow\infty$ on the path C_2.

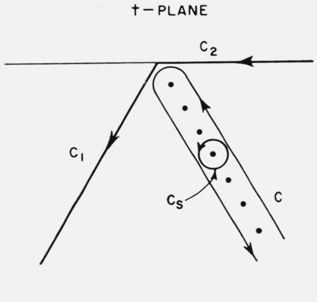


FIGURE 21. *The infinitesimal contour path* C_s *and the integration paths* C_1 *and* C_2 *for eqs (2.2) and (2.4).*

Using these asymptotic forms in the integrand of A for case 2, it becomes

$$A \simeq \exp[\,]\sqrt{\frac{r}{k_1 r_1 r_2}}\,(k_1 a/2)^{1/3}\frac{1}{\sqrt{2\pi}}\int_0^{\infty e^{-i\pi/2}} dt \exp\left[-i\{\theta(k_1 a/2)^{1/3}t+(2k_1)^{-1}(1/r_1+1/r_2)(k_1 a/2)^{2/3}t^2\}\right]$$

$$=\exp[\,]\frac{1}{\sqrt{\pi}}e^{i\zeta^2}\int_{\zeta e^{i\pi/4}}^{\infty} e^{-t^2}dt, \qquad \zeta=\theta\sqrt{k_1 r_1 r_2/2(r_1+r_2)}, \qquad (2.8)$$

with

$$\exp[\,]=\exp\left[-i(k_1 a/3)\{(r_1/a)^3+(r_2/a)^3\}\right].$$

The result of exact evaluation takes the form

$$A=\exp[\,]\left\{\frac{1}{\sqrt{\pi}}e^{i(\xi/\eta)^2}\int_{(\xi/\eta)e^{i\pi/4}}^{\infty}e^{-t^2}dt-\eta G(\xi)\right\},$$

$$\xi=\theta(k_1 a/2)^{1/3}, \qquad \eta=(k_1 a/2)^{1/3}\sqrt{(2/k_1)(1/r_1+1/r_2)}. \qquad (2.9)$$

Here, the numerical values of the function $G(\xi)$ have been calculated by Logan [1959] for a surface of perfect conductor, and by Wait and Conda [1959] for a wide range of surface impedance. It may be remarked that the corresponding numerical integration method for the case of poor convergence has been used more previously by Fock [1946] and Rice [1954].

3. General Formula for Diffraction by Two or More Surfaces

As illustrated in figures 22 and 23, we now consider the general case where the waves propagate over several mountains with the subscripts 2, 3, . . ., n along the wave path from the point x_1 to x_{n+1}, and the radii of curvature (along the wave path) a_2, a_3, . . ., a_n, and the propagation constants k_2, k_3, . . ., k_n, respectively. The diffraction angles of the respective mountains will be denoted by θ_2, θ_3, . . ., θ_n, and thus $d_m=a_m\theta_m (m=2, 3, \ldots, n)$ will be the distance of that part of the wave path contacting the mth mountain.

Now, on the assumption that the whole wave path lies in a plane profile, the result of evaluation gives the following expressions for the attenuation coefficient A which are just the generalizations of (1.5) and (1.6). Using the notation t_m for the set of roots of eq (1.2) in the case of $q=q_m$, they are [Furutsu, 1956]:

Case 1: $\quad (a_m/r_{m,m\pm1})^{3/4}(k_1 a_m)^{-1/4}\ll 1 \qquad (2\leq m\leq n)$

$$A=\{(r_{n+1,n}+d_n+\ldots+r_{32}+d_2+r_{21})/k_1^{n-1}r_{n+1,n}r_{n,n-1}\ldots r_{32}r_{21}\}^{1/2}$$

$$\times\sum_{t_n,\ldots,t_3,t_2}T(r_{n+1,n})_{0,t_n}T(\xi_n)_{t_n}\ldots T(\xi_3)_{t_3}T(r_{32})_{t_3,t_2}T(\xi_2)_{t_2}T(r_{21})_{t_2,0}. \qquad (3.1)$$

Case 2: $\quad (a_2/r_{21})^{3/4}(k_1 a_2)^{-1/4}\geq 1, \qquad (a_2/r_{32})^{3/4}(k_1 a_2)^{-1/4}\ll 1,$

$\qquad (a_m/r_{m,m\pm1})^{3/4}(k_1 a_m)^{-1/4}\ll 1 \qquad (3\leq m\leq n)$

$$A=\{(r_{n+1,n}+d_n+\ldots+r_{32}+d_2)/k_1^{n-2}r_{n+1,n}r_{n,n-1}\ldots r_{43}r_{32}\}^{1/2}$$

$$\times\sum_{t_n\ldots,t_3,t_2}T(r_{n+1,n})_{0,t_n}T(\xi_n)_{t_n}\ldots T(r_{43})_{t_4,t_3}T(\xi_3)_{t_3}T(r_{32})_{t_3,t_2}T(\xi_2,z_1)_{t_2}. \qquad (3.2)$$

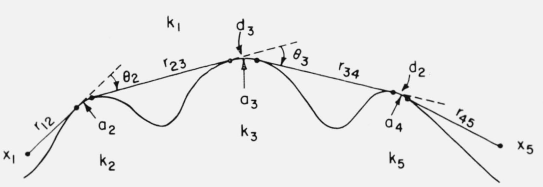


FIGURE 22. *The form of diffracting surfaces and the notations for eq (3.1).*

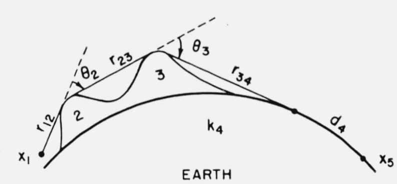


FIGURE 23. *The form of terrain and the notations for eq (3.2).*

$$\text{Case 3:} \quad (a_2/r_{21})^{3/4}(k_1a_2)^{-1/4} \geq 1, \qquad (a_n/r_{n+1,n})^{3/4}(k_1a_n)^{-1/4} \geq 1$$

$$(a_m/r_{m,m\pm 1})^{3/4}(k_1a_m)^{-1/4} \ll 1 \qquad (3 \leq m \leq n-1)$$

$$A = \{(d_n + r_{n,n-1} + \ldots + d_3 + r_{32} + d_2)/k_1^{n-3} r_{n,n-1} r_{n-1,n-2} \ldots r_{43} r_{32}\}^{1/2}$$

$$\times \sum_{t_n, \ldots, t_3, t_2} T(\xi_n, z_{n+1})_{t_n} T(r_{n,n-1})_{t_n, t_{n-1}} T(\xi_{n-1})_{t_{n-1}} \ldots T(\xi_3)_{t_3} T(r_{32})_{t_3, t_2} T(\xi_2, z_1)_{t_2}. \quad (3.3)$$

Here, $r_{m,n}$ is the distance between the mth and nth mountains and

$$\xi_m = \theta_m (k_1 a_m/2)^{1/3},$$

$$T(\xi)_{t_m} = 2\sqrt{2\pi}(k_1 a_m/2)^{1/3}(t_m - q_m^2)^{-1} W(t_m)^{-2} e^{-i(\xi t_m + 3\pi/4)},$$

$$T(r)_{t_m, t_n} = \exp\left[-i(2k_1 r)^{-1}\{(k_1 a_m/2)^{1/3} t_m - (k_1 a_n/2)^{1/3} t_n\}^2\right],$$

$$T(\xi, z)_{t_m} = 2\sqrt{\pi}(t_m - q_m^2)^{-1} W(t_m)^{-1} f_{t_m}(z, a_m) e^{-i(\xi t_m + \pi/2)}. \quad (3.4)$$

In case 1, both the transmitter and receiver are sufficiently apart from the diffracting mountains (fig. 22). The formula (3.1) in this case consists of two kinds of terms, one of which, $T(\xi_m)_{t_m}$, depends only on the diffraction angle, the surface impedance and the radius of curvature of the mth mountain, but does not depend on those quantities of other diffracting mountains.

On the other hand, the other kind of term $T(r_{m,n})_{t_m, t_n}$ depends on the radii of curvature of the mth and nth mountains and their electrical properties through t_m and t_n and also the propagation distance between them, but does not depend on other quantities. Therefore, it serves as the coupling term between the mth and nth mountains. There is no other kind of term.

This fact facilitates the understanding of this formula considerably; there is a clear one-to-one correspondence between the terms in the formula and the respective parts of the wave path: for instance, $T(r_{21})_{t_2, 0}$ corresponds to the wave path from the point x_1 to the second diffracting surface, the term $T(\xi_2)_{t_2}$ corresponds to the wave path along the same diffracting surface, and $T(r_{32})_{t_3, t_2}$ corresponds to the propagation in free space from the second to the third diffracting surface, etc.

As already stated, this formula is essentially the generalization of the formula (1.6) for one diffracting surface which was derived from the original Van der Pol and Bremmer formula on the restrictive condition that both the transmitter and the receiver are not so far from the diffracting surface as compared with the radius of curvature. But here it is derived from this general formula as the special case of one diffracting surface, on the more general condition.

In case 2, one of either transmitter or receiver, say the point x_1, is on or near the diffracting surface (fig. 23). The only difference from that of case 1 is in the last two terms; they were replaced here by the term $T(\xi_2, z_1)_{t_2}$, and the others are just the same. Here, $f_{t_m}(z, a_m)$ is the ordinarily defined height-gain function. Again, the special case of this formula agrees with (1.5).

In case 3, both the transmitter and receiver are on or near the respective diffracting surfaces. The change to be made in the preceding formulas is so evident that it might not be necessary to mention it here explicitly.

Sometimes it may be necessary to take into account the contributions of the waves which are reflected from the surfaces between the diffracting mountains. Mathematically speaking, these reflecting points on the surfaces are just the phase stationary points of the integrand, and we could get the result from these formulas by multiplying the reflection coefficients of the surfaces and also adjusting the diffraction angles so that they correspond to the reflected wave paths.

4. Summation of the Series

Finally, there remains the problem of convergence of the formulas (3.1) to (3.3). There are two kinds of series in the above formulas; one is of the form

$$\sum_{t_m} T(r_{m+1,m})_{t_{m+1}, t_m} T(\xi_m)_{t_m} T(r_{m,m-1})_{t_m, t_{m-1}}, \quad (4.1)$$

and its convergence is poor when ξ_m is near or smaller than 1. Now the treatment for the case of poor convergence is exactly the same as in section 2. Using the same method, we have the result

$$\text{eq (4.1)}=T(r_{m+1,m})_{t_{m+1},0}T(r_{m,m-1})_{0,t_{m-1}}\sqrt{2}(k_1a_m/2)^{1/3}F(\xi_m-\xi'_m,\eta_m,q_m). \tag{4.2}$$

Here, on referring to eq (2.9),

$$F(\xi,\eta,q)=\frac{1}{\sqrt{\pi}\eta}e^{i(\xi/\eta)^2}\int_{(\xi/\eta)e^{i\pi/4}}^{\infty}e^{-t^2}dt-G(\xi),$$

$$\eta_m=(k_1a_m/2)^{1/3}\sqrt{(2/k_1)\ (1/r_{m+1,m}+1/r_{m,m-1})},$$

$$\xi'_m=(k_1a_m/2)^{1/3}\{(k_1r_{m+1,m})^{-1}(k_1a_{m+1}/2)^{1/3}t_{m+1}+(k_1r_{m,m-1})^{-1}(k_1a_{m-1}/2)^{1/3}t_{m-1}\}, \tag{4.3}$$

and ξ'_m depends on the preceding and succeeding diffracting surfaces through $t_{m\pm1}$, and thus it serves as the only coupling variable between the diffracting surfaces.

When the diffraction angle of the mth surface is small and/or the radius of curvature is sufficiently small, the second correction term becomes very small in most cases and can be neglected as compared with the first leading term; in the case of ξ_m, $\eta_m \ll 1$ the series (4.1) takes the simple form

$$\sum_{t_m}\simeq T(r_{m+1,m})_{t_{m+1},0}T(r_{m,m-1})_{0,t_{m-1}}\tfrac{1}{2}\mathscr{E}((\xi_m-\xi'_m)/\eta_m e^{i\pi/4})\sqrt{k_1/(1/r_{m+1,m}+1/r_{m,m-1})},\qquad \xi_m\ll1,$$

$$\mathscr{E}(z)=\frac{2}{\sqrt{\pi}}e^{z^2}\int_z^{\infty}e^{-t^2}dt. \tag{4.3a}$$

Another special case is the case in which the diffraction angles of the $m\pm1$th mountains are sufficiently large, as $\xi_{m\pm1}/\eta_{m\pm1}\gg1$. It follows then that $\xi'_m/\eta_m\ll1$ and thus ξ'_m in (4.2) can be neglected:

$$\sum_{t_m}=T(r_{m+1,m})_{t_{m+1},0}T(r_{m,m-1})_{0,t_{m-1}}\sqrt{2}(k_1a_m/2)^{1/3}F(\xi_m,\eta_m,q_m). \tag{4.3b}$$

Similarly, on the condition of $\xi_m/\eta_m\gg1$ and $\xi_l/\eta_l\geq-1(l\neq m)$,

$$\sqrt{2}(k_1a_m/2)^{1/3}F(\xi_m-\xi'_m,\eta_m,q_m)\simeq\sqrt{2}(k_1a_m/2)^{1/3}\left[\frac{1}{2\sqrt{\pi}\xi_m}e^{-i\pi/4}-G(\xi_m)\right]\equiv M(\xi_m,a_m),\qquad \xi_m\gg|\xi'_m|. \tag{4.3c}$$

Here, the function $M(\xi_m, a_m)$ is introduced for later convenience.

The other kind of series is

$$\sum_{t_2}T(r_{32})_{t_3,t_2}T(\xi_2,z_1)_{t_2}\simeq T(r_{32})_{t_3,0}\sum_{t_2}T(\xi_2-\xi'_2,z_1)_{t_2}, \tag{4.4}$$

which always occurs with the height-gain function of receiver or transmitter. Using the same method as in section 2, it takes the form, on the condition that $\xi_l/\eta_l\gg1(l\neq2)$ and $\xi_2/\eta_2\geq-1$,

$$\sum_{t_2}T(\xi_2-\xi'_2,z_1)_{t_2}\simeq g(\xi_2)\{1+i(k_1^2/k'_2)z_1\},\qquad (2/k_1a_2)^{1/3}k_1z_1\ll1,$$

$$k_1/k'_2=\begin{cases}k_1\sqrt{k_2^2-k_1^2}/k_2^2, & \text{Vert. Pol.}\\ \sqrt{k_2^2-k_1^2}/k_1, & \text{Horiz. Pol.}\end{cases} \tag{4.5}$$

Here, z_1 is the height of the point x_1 from the surface, and the function $g(\xi)$ has been numerically calculated by Logan [1959] for the case of a perfect conductor and by Wait and Conda [1959] for a wide range of surface impedance.

Special applications of the supplemental formulas will be as follows:

Case 1: $\quad \xi_m/\eta_m\gg1 \qquad (n\geq m\geq2).$

This is the case where the diffraction angle of every diffracting surface is sufficiently large. On using (4.3c) in the formula (3.1), the attenuation coefficient A takes the form

$$A=\{(r_{n+1,\,n}+d_n+\ .\ .\ .\ +r_{32}+d_2+r_{21})/k_1^{n-1}r_{n+1,n}r_{n,\,n-1}\ .\ .\ .\ r_{32}r_{21}\}^{1/2} \times M(\xi_n,\, a_n)M(\xi_{n-1},\, a_{n-1})\ .\ .\ .\ M(\xi_2,\, a_2), \quad (4.6a)$$

or, when either or both transmitter or receiver is on or near the diffracting surface, the formulas (3.2) and (3.3), respectively, give on using (4.5),

$$A=\{r_{n+1,\,n}+d_n+\ .\ .\ .\ +r_{32}+d_2)/k_1^{n-2}r_{n+1,\,n}r_{n,\,n-1}\ .\ .\ .\ r_{43}r_{32}\}^{1/2} \times M(\xi_n,\, a_n)\ .\ .\ .\ M(\xi_3,\, a_3)g(\xi_2)f(z_1,\, a_2), \quad (4.6b)$$

$$A=\{(d_n+r_{n,\,n-1}+\ .\ .\ .\ +r_{32}+d_2)/k_1^{n-3}r_{n,\,n-1}\ .\ .\ .\ r_{43}r_{32}\}^{1/2} \times f(z_{n+1},\, a_n)g(\xi_n)M(\xi_{n-1},\, a_{n-1})\ .\ .\ .\ M(\xi_3,\, a_3)g(\xi_2)f(z_1,\, a_2). \quad (4.6c)$$

These results just correspond to the so-called multiplication rule in diffraction.

Another application that may be of interest will be that in which one of the diffracting surfaces does not serve as a diffracting obstacle but simply as a reflecting surface, as in figure 24. In this case the coupling between the preceding and succeeding surfaces becomes very serious. From the leading term of formula (4.3a) we have

Case 2: $\quad \xi_m/\eta_m \ll -1$

$$\sum_{t_m} T\,(r_{m+1,\,m})_{t_{m+1},\,t_m}T(\xi_m)_{t_m}T(r_{m,\,m-1})_{t_m,\,t_{m-1}}$$

$$\simeq T(r_{m+1,\,m})_{t_{m+1},\,0}T(r_{m,\,m-1})_{0,\,t_{m-1}}\sqrt{k_1/(1/r_{m+1,\,m}+1/r_{m,\,m-1})}\ \exp\,[i\,(\xi_m-\xi_m')^2/\eta_m^2]$$

$$=T(r_{m+1,\,m}+r_{m,\,m-1})_{\,t_{m+1},\,t_{m-1}}\times\exp\,[i\{(\xi_m/\eta_m)^2-\theta_m(r_{m,\,m-1}/(r_{m+1,\,m}+r_{m,\,m-1}))\,(k_1a_{m+1}/2)^{1/3}t_{m+1}$$

$$-\theta_m(r_{m+1,\,m}/(r_{m+1,\,m}+r_{m,\,m-1}))\,(k_1a_{m-1}/2)^{1/3}t_{m-1}\}], \qquad \xi_m/\eta_m \ll -1. \quad (4.7)$$

When this result is substituted in (3.1), (3.2), and (3.3), they indicate what would happen if there were no diffracting surface at all between the $m\pm 1$th diffracting surfaces: the phase terms of $\theta_m\{r_{m,\,m\mp1}/(r_{m+1,\,m}+r_{m,\,m-1})\}(k_1a_{m\pm1}/2)^{1/3}t_{m\pm1}$ are respectively combined with the terms of $T(\xi_{m\pm1})_{t_{m\pm1}}$ to give the new diffraction angles $\theta'_{m\pm1}$ due to the omission of the mth surface (fig. 24). The correction to the leading term can be proven to give the reflected wave from the mth surface.

In the same way, when the convergence of the double series

$$\sum_{t_m,\,t_{m+1}} T(r_{m+2,\,m+1})_{t_{m+2},\,t_{m+1}}T(\xi_{m+1})_{t_{m+1}}T(r_{m+1,\,m})_{t_{m+1},\,t_m}T(\xi_m)_{t_m}T(r_{m,\,m-1})_{t_m,\,t_{m-1}} \quad (4.8)$$

is poor with respect to both t_m and t_{m+1}, we have the leading term (fig. 25)

$$T(r_{m+2,\,m+1})_{t_{m+2},\,0}T(r_{m,\,m-1})_{0,\,t_{m-1}}\{k_1^2r_{m+2,\,m+1}r_{m+1,\,m}r_{m,\,m-1}/(r_{m+2,\,m+1}+r_{m+1,\,m}+r_{m,\,m-1})\}^{1/2} \times \tfrac{1}{4}\{\mathscr{E}\,(\zeta_{m+1}e^{i\pi/4},\,\beta_{m+1}/\zeta_{m+1})+\mathscr{E}\,(\zeta_m e^{i\pi/4},\,\beta_m/\zeta_m)\}, \quad (4.9)$$

which is valid for $\eta_m \ll 1$, $\eta_{m+1} \ll 1$. Here,

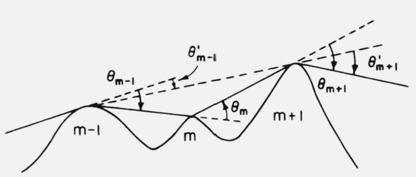


FIGURE 24. *The situation in the case of eq (4.7).*

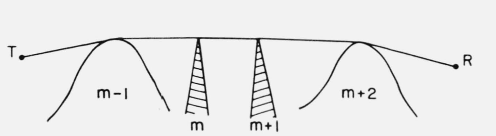


FIGURE 25. *The forms of diffracting surfaces for eqs (4.8) and (4.9).*

$$\beta_m = \{\xi_m - (k_1 r_{m,\,m-1})^{-1}(k_1 a_m/2)^{1/3}(k_1 a_{m-1}/2)^{1/3} t_{m-1}\}/\eta_m,$$

$$\zeta_m = \{\gamma\beta_m + \beta_{m+1}\}[(r_{m+2,\,m+1}+r_{m+1,\,m})(r_{m+1,\,m}+r_{m,\,m-1})/r_{m+1,\,m}(r_{m+2,\,m+1}+r_{m+1,\,m}+r_{m,\,m-1})]^{1/2},$$

$$\gamma = \{r_{m+2,\,m+1}r_{m,\,m-1}/(r_{m+1,\,m}+r_{m,\,m-1})(r_{m+2,\,m+1}+r_{m+1,\,m})\}^{1/2}, \text{ etc.},$$

$$\mathscr{E}(z, n) = \left(\frac{2}{\sqrt{\pi}}\right)^2 e^{(1+n^2)z^2}\int_z^\infty dt_2 e^{-t_2^2}\int_{nt_2}^\infty e^{-t_1^2} dt_1. \tag{4.10}$$

The result of (4.9) corresponds to (4.3a) and can be applied to the formulas (3.1), (3.2), and (3.3) for the conditions $\xi_m \ll 1$, $\xi_{m+1} \ll 1$.

When $\xi_{m-1}/\eta_{m-1} \gg 1$ and $\xi_{m+2}/\eta_{m+2} \gg 1$, the terms of t_{m-1} and t_{m+2} in (4.10) can be neglected as in (4.3b), and it follows that

$$\zeta_m = \{(\theta_{m+1}+\theta_m)r_{m,\,m-1}+\theta_{m+1}r_{m+1,\,m}\}\{k_1 r_{m+2,\,m+1}/2(r_{m+1,\,m}+r_{m,\,m-1})(r_{m+2,\,m+1}+r_{m+1,\,m}+r_{m,\,m-1})\}^{1/2},$$

$$\beta_m = \theta_m\sqrt{k_1/2(1/r_{m+1,\,m}+1/r_{m,\,m-1})}, \text{ etc.} \tag{4.11}$$

For the case in which (4.11) can be used, the coupling between the mth and $m+1$th surfaces and the other surfaces is lost, and thus they behave independently in the formulas (3.1), (3.2), and (3.3) for the attenuation coefficient A.

5. Diffraction by Two Ridges

The special case of $n=3$ and $\xi_2, \xi_3 \ll 1$, for which (4.9) and (4.11) are valid, corresponds to the case of two-ridge diffraction, as illustrated in figure 26. A straight application gives the result

$$A = \tfrac{1}{4}[\mathscr{E}(\zeta_2 e^{i\pi/4}, \beta_2/\zeta_2) + \mathscr{E}(\zeta_3 e^{i\pi/4}, \beta_3/\zeta_3)]. \tag{5.1}$$

Here

$$\zeta_2 = \{(\theta_2+\theta_3)r_{21}+\theta_3 r_{32}\}\sqrt{k_1 r_{43}/2(r_{32}+r_{21})(r_{43}+r_{32}+r_{21})},$$

$$\zeta_3 = \{(\theta_2+\theta_3)r_{43}+\theta_2 r_{32}\}\sqrt{k_1 r_{21}/2(r_{43}+r_{32})(r_{43}+r_{32}+r_{21})},$$

$$\beta_2 = \theta_2\sqrt{k_1/2(1/r_{32}+1/r_{21})},\ \beta_3 = \theta_3\sqrt{k_1/2(1/r_{32}+1/r_{43})}. \tag{5.2}$$

Equation (5.1) corresponds to the Fresnel integral in the case of a single ridge. Indeed, letting $r_{32} \to 0$, it follows that

$$\zeta_3 = \zeta_2 = (\theta_3+\theta_2)\sqrt{k_1 r_{43} r_{21}/2(r_{43}+r_{21})},$$

and thus, according to the definition of $\mathscr{E}(z, n)$ in (4.10) and $\mathscr{E}(z)$ in (4.3a), eq (5.1) becomes

$$A = \tfrac{1}{2}\mathscr{E}(\zeta_2 e^{i\pi/4}). \tag{5.3}$$

Equation (5.1) is given in terms of the function $\mathscr{E}(a, b/a)$ whose precise analytical treatment has been studied in another paper [Furutsu, 1955] and also in the appendix of this paper, the results of which are briefly described for convenience of computation. It is a many-valued function and the following relations are conveniently available for (5.1):

$$\begin{aligned}\mathscr{E}(a, b/a) &= 2e^{b^2}\mathscr{E}(a) - \mathscr{E}(a, -b/a), \qquad a_R>0,\ b_R\leq 0,\\ &= -\mathscr{E}(-a, b/(-a)), \qquad a_R\leq 0,\ b_R>0.\end{aligned} \tag{5.4}$$

Here a_R and b_R are the real parts of a and b, respectively.

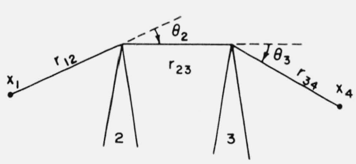


FIGURE 26. *Two ridges and the notations for eqs (5.1) and (5.2).*

Model experiments for two-ridge diffraction have been tried by Decker and his colleagues at the National Bureau of Standards. The results of comparison of the theory and the preliminary experiments are shown in figure 27. Here the theoretical values were obtained by the use of a high-speed computer. The agreement is surprisingly good.

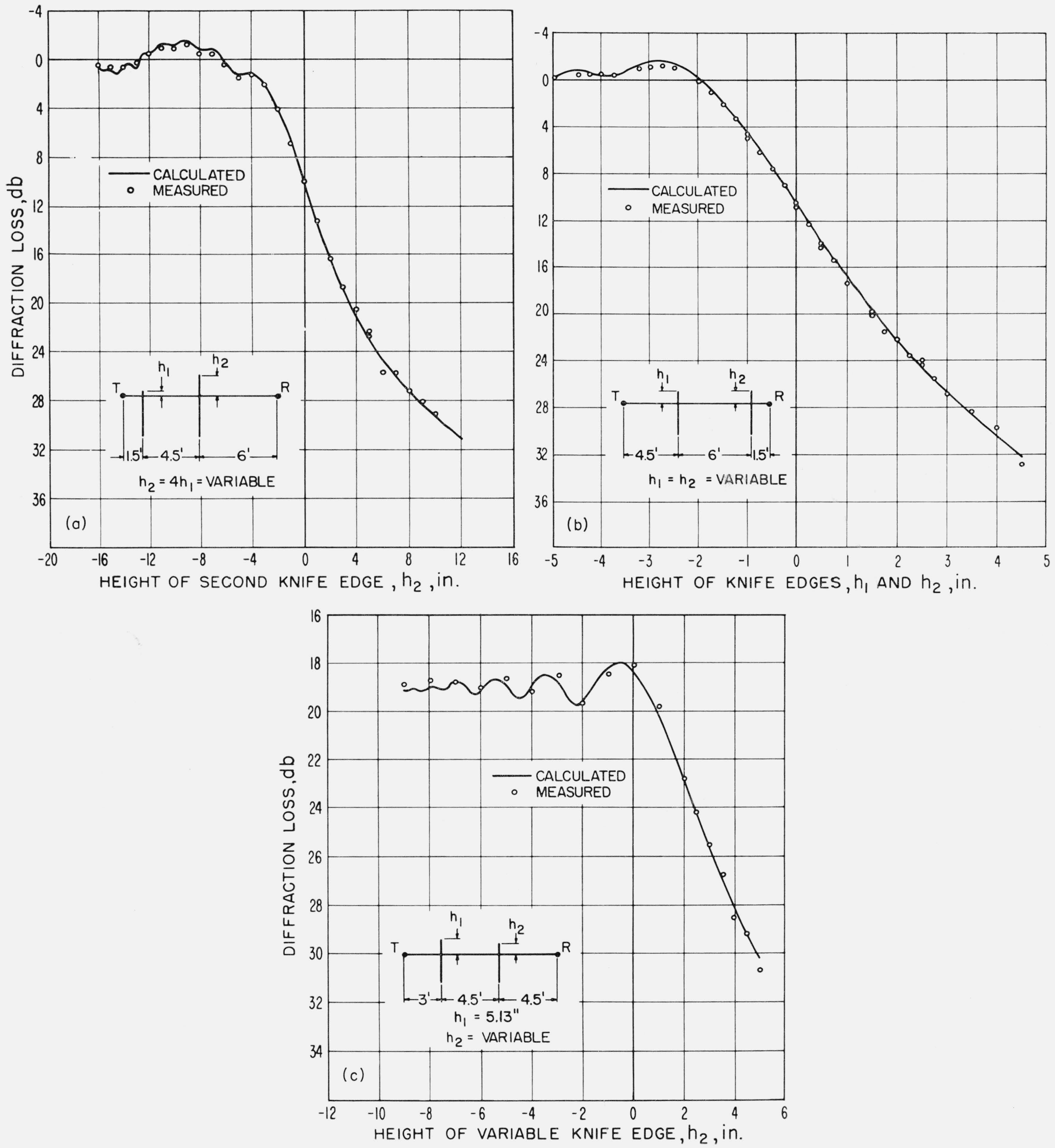


FIGURE 27. *A few comparisons of experimental and theoretical values for diffraction loss for a double knife-edge path.*

a. Knife edges at variable heights with transmitter and the two edges in line.
b. Knife edges at equal variable heights, frequency 24.21 Gc/s.
c. First knife edge fixed, second variable in height, frequency 24.14 Gc/s.

The result (5.1) was obtained in eq (12.2.15) of the reference [Furutsu, 1956]. Recently an equivalent formula was also derived by Millington et al. [1962], and the comparisons with the empirical methods of Bullington and Epstein-Peterson were discussed.

The author expresses his cordial thanks to J. R. Wait for his useful discussions and also to M. T. Decker for kindly showing the data of his laboratory experiments on two-ridge diffraction.

Appendix

From the definition of $\mathscr{E}(z)$ in eq (4.3a),

$$\mathscr{E}(z)=e^{z^2}-\overline{\mathscr{E}}(z),\quad \overline{\mathscr{E}}(z)=\frac{2}{\sqrt{\pi}}\, e^{z^2}\int_0^z e^{-t^2}dt. \tag{A.1}$$

Here

$$\overline{\mathscr{E}}(z)=\sum_{n=1}^{\infty} C_n,$$

$$C_n=\frac{z^2}{n-\frac{1}{2}}\, C_{n-1},\quad C_1=\frac{2}{\sqrt{\pi}}\, z. \tag{A.2}$$

For $|z|\gg 1$ and $|\arg(z)|<3\pi/4$, $\mathscr{E}(z)$ takes the asymptotic form

$$\mathscr{E}(z)\sim\sum_{n=1}^{\infty} g_n,\quad |z|\gg 1,\ |\arg(z)|<3\pi/4, \tag{A.3}$$

where

$$g_n=-\frac{n-\frac{3}{2}}{z^2}\, g_{n-1},\quad g_1=\frac{1}{\sqrt{\pi}z}. \tag{A.4}$$

In the same way, from (4.10),

$$\mathscr{E}(a,\ b/a)=\frac{2}{\pi}\left(\tan^{-1}\frac{a}{b}\right) e^{a^2+b^2}-e^{b^2}\overline{\mathscr{E}}(a)+\overline{\mathscr{E}}(a,\ b/a). \tag{A.5}$$

Here $\overline{\mathscr{E}}(a)$ is the same function as in (A.1) and

$$\overline{\mathscr{E}}(a,\ b/a)=\left(\frac{2}{\sqrt{\pi}}\right)^2 e^{a^2+b^2}\int_0^a dt_2 e^{-t_2^2}\int_0^{(b/a)t_2} e^{-t_1^2}dt_1,$$

$$\mathscr{E}(0,\ b/a)=\frac{2}{\pi}\tan^{-1}\frac{a}{b}. \tag{A.6}$$

The function $\overline{\mathscr{E}}(a,\ b/a)$ can be expanded in the absolutely convergent series of

$$\overline{\mathscr{E}}(a,\ b/a)=\sum_{n=1}^{\infty} h_n. \tag{A.7}$$

Here

$$h_n=\frac{1}{n}\left\{(a^2+b^2)h_{n-1}+\alpha_n\right\},\quad \alpha_n=\frac{1}{n-\frac{1}{2}}\, b^2\alpha_{n-1},$$

$$h_1=\alpha_1=\frac{2}{\pi}\, ab. \tag{A.8}$$

For $|a^2+b^2|^{\frac{1}{2}}\gg 1$ and $|a/b|<1$ and $|\arg(a)|$, $|\arg(b)|<3\pi/4$, the function $\mathscr{E}(a,\ b/a)$ takes the asymptotic form

$$\mathscr{E}(a, b/a) \sim \frac{1}{\sqrt{\pi}}\left(\frac{a}{a^2+b^2}\right) \sum_{n=0}^{\infty} j_n, \quad |a^2+b^2|^{1/2} \gg 1, \quad |a/b| < 1, \quad |\arg(a)|, \quad |\arg(b)| < 3\pi/4. \qquad \text{(A.9)}$$

Here

$$\begin{cases} j_{2n+1} = Bj_{2n} - nAj_{2n-1} - k_n, \quad j_0 = \mathscr{E}(b), \\ j_{2n} = Bj_{2n-1} - (n - \tfrac{1}{2})Aj_{2n-2} \end{cases}$$

$$k_n = -(n-\tfrac{1}{2})(a^2+b^2)^{-1}k_{n-1}, \quad k_0 = \pi^{-1/2}b(a^2+b^2)^{-1},$$

$$A = a^2(a^2+b^2)^{-2}, \quad B = b^2(a^2+b^2)^{-1}. \qquad \text{(A.10)}$$

For the case of $|a/b| > 1$, the following relation is applicable:

$$\mathscr{E}(a, b/a) = \mathscr{E}(a)\ \mathscr{E}(b) - \mathscr{E}(b, a/b). \qquad \text{(A.11)}$$

References

Born, M., and E. Wolf, Principles of optics (Pergamon Press, Inc., New York 22, N.Y., 1959).

Fock, V. A., Diffraction of radio waves around the earth's surface (Academy of Sciences, U.S.S.R., 1946).

Furutsu, K., Propagation of e-m waves over a flat earth across two boundaries separating three different media, J. Radio Research Lab. (Japan) **2**, 239–279 (1955).

Furutsu, K., On the multiple diffraction of electromagnetic waves by spherical mountains, J. Radio Research Lab. (Japan) **3,** 331–390 (1956).

Furutsu, K., Wave propagation over an irregular terrain, Part I, J. Radio Research Lab. (Japan) **4,** 135–153 (1957a); Part II, **4,** 349–393 (1957b); Part III, **5,** 71–102 (1949a).

Furutsu, K., On the excitation of the waves of proper solutions, IRE Trans. **AP-7,** Special Supplement, p. 213 (1959b).

Logan, N. A., Fresnel diffraction by convex surfaces, Paper presented at Washington meeting of International Scientific Radio Union (May 4–7, 1959).

Millington, G., R. Hewitt, and F. S. Immirzi, The Fresnel surface integral, IRE Monograph 508E (Mar. 1962); *ibid.*, Double knife-edge diffraction in field strength predictions, IRE Monograph 507E (Mar. 1962).

Rice, S. O., Diffraction of plane radio waves by a parabolic cylinder-calculation of shadows behind hills, Bell System Tech. J. **33,** 417–504 (1954).

Wait, J. R., and A. M. Conda, Diffraction of electromagnetic waves by smooth obstacles for grazing angles, J. Research NBS **63D** (Radio Prop.), 181–197 (1959).

Other References

Bremmer, H., The extension of Sommerfeld's formula for the propagation of radio waves over a flat earth to different conductivities of the soil, Physica **XX,** 441 (1954).

Clemmow, P. C., Radio propagation over a flat earth across a boundary separating two different media, Trans. Roy. Soc. (London) **246,** 1 (1953).

Feinberg, E. L., On the propagation of radio waves along an imperfect surface, J. Phys. **IX,** No. 1 (1944); **X,** No. 5 (1946).

Feinberg, E. L., Propagation of radio waves along an inhomogeneous surface, Nuovo Cimento **11,** 60–91 (1959).

Furutsu, K., Propagation of electromagnetic waves over the spherical earth across boundaries separating different media, J. Radio Research Lab. (Japan) **2,** 345–398 (1955). See also J. Radio Research Lab. (Japan) **2,** 1–49 (1955).

Van der Pol, B., and H. Bremmer, Further note on the propagation of radio waves over a finitely conducting spherical earth, Phil. Mag. **29,** 261–275 (1939).

Wait, J. R., and J. E. Householder, Mixed ground wave propagation: 1. Short distance, J. Research NBS **57,** 1–15 (1956); 2. Larger distance; J. Research NBS **59,** 19–26 (1957).

Wait, J. R., and Anabeth Murphy, Influence of a ridge on the low-frequency ground wave, J. Research NBS **58,** 1–5 (1957).

Wait, J. R., On the theory of mixed-path ground-wave propagation on a spherical earth, J. Research NBS **65D** (Radio Prop.), 401–410 (1961).

(Paper 67D1–242)