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Solutions of the Equation $\Psi_{xx}+\frac{1}{x}\Psi_x+Kx^n e^{\Psi}=0$

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The general solution of the equation $\Psi_{xx}+\frac{1}{x}\Psi_x+Kx^n e^{\Psi}=0$ is displayed in terms of simple tabulated functions. The existence and uniqueness of solutions of a simple boundary value problem are determined as a function of the parameter K.

For a class of problems in the steady flow of viscous fluids with a viscosity depending exponentially on temperature, an ordinary, nonlinear, second-order, differential equation arises [1.2][1]; viz,

$$\Psi_{xx}+\frac{1}{x}\Psi_x+Kx^n e^{\Psi}=0;\ x>0. \tag{1}$$

The same equation may arise in calculating the temperature distribution in a dielectric in an alternating field [3]. In this equation, Ψ is an unknown function of x, while K and n are arbitrary real constants. The equation is of considerable mathematical interest as an example of that exceedingly small class of strongly nonlinear equations whose general solution can be displayed in terms of simple tabulated functions.

The reader may verify by direct differentiation that the following array is indeed a collection of solutions of eq (1):

$$\Psi=\log\left\{\frac{2m^2}{Kx^{n+2}\cosh^2(m\log x+a)}\right\},\ K>0. \tag{2a}$$

$$\left.\begin{aligned}
&\Psi=\log\left\{\frac{-2m^2}{Kx^{n+2}\sinh^2(m\log x+a)}\right\}, && (2b')\\
&\Psi=\log\left\{\frac{-2m^2}{Kx^{n+2}\sin^2(m\log x+a)}\right\}, && (2b'')\\
&\Psi=\log\left\{\frac{-2}{Kx^{n+2}(\log x+a)^2}\right\}, && (2b''')
\end{aligned}\right\}K<0$$

$$\Psi=m\log x+a,\ K=0. \tag{2c}$$

In these equations, m and a are two arbitrary real constants which may be thought of as constants of integration.

It is important to show that this array contains all the real solutions of eq (1). One can get at this question through some well known properties of ordinary differential equations. Observe that eq (1) can be written in the form of a system of two first order equations:

$$\begin{aligned}\Psi_x&=\varphi=f_1(\varphi,\Psi,x)\\ \varphi_x&=-\frac{\varphi}{x}-Kx^n e^{\Psi}=f_2(\varphi,\Psi,x)\end{aligned} \tag{3}$$

It is immediately clear that the partial derivatives of f_1 and f_2 are continuous in the region $x>0$, and hence, by a well-known theorem [4] eq (1) satisfies a Lipschitz condition in some neighborhood of each point in the region. It follows immediately [5] that there is exactly one solution of eq (1) with a value and slope assigned at any given value of $x\neq 0$. We can thus be sure that the array of eq (2) contains all the solutions if we can find values of the constants of integration which determine a solution of preassigned value and preassigned derivative for arbitrary, nonnegative x. We notice that eq (1) may be transformed in the following way. Let $\Psi^*=\Psi+\Psi_0$ and $y=x/x_0$, then eq (1) becomes

$$\Psi^*_{yy}+\frac{1}{y}\Psi^*_y+Kx_0^{n+2}e^{-\Psi_0}y^n e^{\Psi^*}=0. \tag{4}$$

This equation is of the same form as eq (1) with K replaced by $Kx_0^{n+2}e^{-\Psi_0}$, while Ψ_0 and x_0 are unspecified constants. We see, therefore, that it is sufficient to show that among the solutions of eq (1) which vanish at $x=1$, there is a solution whose derivative at $x=1$ takes on any preassigned value.

Consider eq (2a) evaluated at $x=1$. By the vanishing of Ψ, we have that $m=\pm\sqrt{\frac{K}{2}}\cosh a$, and thus that

$$(\Psi_x)_{x=1}=-(n+2)\pm 2\sqrt{\frac{K}{2}}\sinh a.$$

It is clear that we may choose a so as to make $(\Psi_x)_{x=1}$ take on any real finite value.

[1] Figures in brackets indicate the literature references at the end of this paper.

When $K<0$, any one of the three forms of the solution, 2b′, 2b″, and 2b‴ is possible. By the vanishing of Ψ at $x=1$ we have respectively, $m=\pm\sqrt{\frac{-K}{2}}\sinh a$, $m=\pm\sqrt{\frac{-K}{2}}\sin a$ and $a=\pm\sqrt{\frac{2}{-K}}$. The slope of Ψ at $x=1$ can then be seen to be,

$$(\Psi_x)_{x=1}=-(n+2)\pm 2\sqrt{\frac{-K}{2}}\cosh a, \qquad a\neq 0$$

$$(\Psi_x)_{x=1}=-(n+2)\pm 2\sqrt{\frac{-K}{2}}\cos a, \qquad a\neq 0$$

$$(\Psi_x)_{x=1}=-(n+2)\pm 2\sqrt{\frac{-K}{2}}$$

for the three forms 2b′, 2b″, and 2b‴, respectively. It is again clear that we may choose a in these equations so as to make $(\Psi_x)_{x=1}$ take on any real finite value. We see further that all three forms are necessary, indeed, that we must choose form 2b′, 2b″ or 2b‴ according to whether the absolute value of $(\Psi_x)_{x=1}+(n+2)$ is greater than, less than, or equal to $2\sqrt{\frac{-K}{2}}$.

For the case $K=0$, the solution must be of the form (2c), but since eq (1) is linear when $K=0$, the existence and uniqueness of solutions is well known and needs no further discussion.

We have thus shown that the array of forms of eq (2) contains all real solutions of eq (1) in the region $x>0$.

Of practical interest is the question of existence and uniqueness of solutions of boundary value problems. We shall consider a boundary value problem in which values of Ψ are given at two values of x and a solution of eq (1) is sought which is bounded and smooth in the interval. Assuming that two solutions exist, call them Ψ_1 and Ψ_2, we define the difference $\Psi_1-\Psi_2$ as a new function ω. We then perform the following integration by parts,

$$\int_{x_0}^{x_1}\omega\frac{\partial}{\partial x}(x\omega_x)dx=x\omega\omega_x\Big|_{x_0}^{x_1}-\int_{x_0}^{x_1}x\omega_x^2dx, \tag{5}$$

where x_0 and x_1 are the values of x at the boundaries of the interval. We replace the integral on the left of this equation by using the fact that both Ψ_1 and Ψ_2 satisfy eq (1) to get

$$K\int_{x_0}^{x_1}x^{n+1}e^{\Psi_2}\omega(1-e^{\omega})dx=x\omega\omega_x\Big|_{x_0}^{x_1}-\int_{x_0}^{x_1}x\omega_x^2dx. \tag{6}$$

Now it is very easy to see that the integrand on the left is always negative. Furthermore, if we restrict ourselves to solutions with bounded derivatives it is clear that the first term on the right vanishes (since ω vanishes on the boundaries) and, since the integral on the right is always positive, the right hand side of this equation is always negative. Hence, *if K is negative*, there is at most one solution to the boundary value problem. *If K is greater than zero* this uniqueness proof breaks down, and indeed we shall see that multiple solutions may be possible.

We will next consider the case when $K<0$, and show that there always exists a solution of the boundary value problem. We first use eq (4) to redefine K in terms of the given boundary data so that the vanishing of the scaled Ψ at $x=1$ corresponds to the boundary condition at the first boundary. When this condition is satisfied eq (2b) becomes

$$\left.\begin{aligned}
\Psi&=\log\left\{\frac{\sinh^2 a}{x^{n+2}\sinh^2\left(a\pm\sqrt{\frac{-K}{2}}\log x\sinh a\right)}\right\} &\text{(7a)}\\
\Psi&=\log\left\{\frac{\sin^2 a}{x^{n+2}\sin^2\left(a\pm\sqrt{\frac{-K}{2}}\log x\sin a\right)}\right\} &\text{(7b)}\\
\Psi&=\log\left\{\frac{1}{x^{n+2}\left(\sqrt{\frac{-K}{2}}\log x\pm 1\right)^2}\right\} &\text{(7c)}
\end{aligned}\right\}$$

If the second boundary condition is $\Psi=\Psi_2$ at $x=x_2$, $(x_2>1)$, we shall show that there is always one form of eq (7) which will satisfy the condition and which will define a *smooth bounded function in the interval.* For convenience, we define $\varphi=\exp\frac{1}{2}(\Psi_2+(n+2)\log x_2)$. Then φ is a number determined by the boundary data which can be expressed in terms of the constant of integration a. From eq (7) we see that there are five possible forms of φ. From (7a) we get two forms[2]

$$\varphi_1=\frac{\sinh a}{\sinh\left(a+\sqrt{\frac{-K}{2}}\log x_2\sinh a\right)}, \quad 0<a<\infty,$$

$$\varphi_2=\frac{\sinh a}{\sinh\left(a-\sqrt{\frac{-K}{2}}\log x_2\sinh a\right)}, \quad 0<a<\alpha,$$

from (7b) we get only one independent form[3]

$$\varphi_3=\frac{\sin a}{\sin\left(a+\sqrt{\frac{-K}{2}}\log x_2\sin a\right)}, \quad 0<a<\pi,$$

[2] The symbol α here represents the positive root of the equation

$$\frac{\sinh a}{a}=1/\sqrt{\frac{-K}{2}}\log x_2.$$

[3] The form with the negative sign is equivalent to this form with a replaced by $a+\pi$. On the other hand $\varphi_3(2\pi-a)=\varphi_3(a)$, hence it is only necessary to consider values of a between zero and π.

and from (7c) we get two more forms

$$\varphi_4 = \frac{1}{\sqrt{\frac{-K}{2}}\log x_2 + 1}$$

$$\varphi_5 = \frac{1}{\sqrt{\frac{-K}{2}}\log x_2 - 1}$$

We confine our considerations, at first, to the situation when $\sqrt{\frac{-K}{2}}\log x_2 < 1$. A straightforward study of the range of values of the various forms of φ reveals the following scheme:

$$0 < \varphi_1 < \frac{1}{1+\sqrt{\frac{-K}{2}}\log x_2} < \varphi_3 < \frac{1}{1-\sqrt{\frac{-K}{2}}\log x_2} < \varphi_2 < \infty$$

in which, for a suitable choice of a and of the form of φ, φ can take on any value between zero and infinity. It is clear, therefore, that there is always a solution to the boundary value problem.

When $\sqrt{\frac{-K}{2}}\log x_2 \geqslant 1$ some of the solutions represented by eq (7) are no longer suitable because they no longer correspond to smooth bounded functions in the interval. In particular, forms φ_2 and φ_5 correspond to inadmissible solutions, and in form φ_3, a must be restricted to the range $0 < a < \pi - \beta$, where β is the smallest positive root of the equation

$$\frac{\sin\beta}{\beta} = 1/\sqrt{\frac{-K}{2}}\log x_2.$$

We then find the following scheme for the possible values of the various forms of φ:

$$0 < \varphi_1 < \frac{1}{1+\sqrt{\frac{-K}{2}}\log x_2} < \varphi_3 < \infty$$

and again φ can take on any value between zero and infinity, so that there is always a solution to the boundary value problem.

When $K > 0$, we have seen that the uniqueness proof fails. In fact we cannot expect a unique solution of the boundary value problem. Upon fitting the boundary condition $\Psi = 0$ at $x = 1$, eq (2a) becomes

$$\Psi = \log\left\{\frac{\cosh^2 a}{x^{n+2}\cosh^2(a+\sqrt{\frac{-K}{2}}\log x \cosh a)}\right\}, -\infty < a < \infty$$

Again, we represent the second boundary condition by $\Psi = \Psi_2$ at $x = x_2$, ($x_2 > 1$). As before, we define $\varphi = \exp\frac{1}{2}(\Psi_2 + (n+2)\log x_2)$, and find

$$\varphi = \frac{\cosh a}{\cosh(a+\sqrt{\frac{K}{2}}\log x_2 \cosh a)}. \tag{8}$$

A study of this function shows it to be a smooth positive function of a, vanishing as a approaches plus or minus infinity and with a single maximum at $a = \alpha$, where α is the root of the equation

$$\tanh(\alpha + \sqrt{\frac{K}{2}}\log x_2 \cosh\alpha) = \frac{\tanh\alpha}{1+\sqrt{\frac{K}{2}}\log x_2 \sinh\alpha}. \tag{9}$$

In this case, therefore, there are two solutions to the boundary value problem when $\varphi < \varphi_{max}$, one solution when $\varphi = \varphi_{max}$ and no solutions when $\varphi > \varphi_{max}$. This non-existence of solutions of the boundary-value problem can be associated with an instability phenomenon in physical problems of viscous heating [6].

References

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