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Effect of Error in Measurement of Elastic Constants on the Solutions of Problems in Classical Elasticity

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It is well known that a small error in the measurement of the elastic constants will result, for all physically interesting boundary value problems, in small errors in the computed values of the stresses and displacements. In this paper actual bounds are given for the error in both the first and second boundary value problems. In addition it is shown that as Poisson's ratio tends to ½ the results for compressible theory tend to those for the analogous problems in the classical incompressible theory.

1. Introduction

In the boundary value problems of classical elasticity the elastic constants of the material in question are assumed known. The solution of the problem is then expressed in terms of the "known" elastic constants. However, the values of these elastic constants are actually determined empirically and are therefore not known exactly. Rather the values of these constants may be determined to lie between certain limits.

It appears to be well understood by workers in classical elasticity that a small error in measurement of the elastic constants will not have much effect on the solution to the problem in question, and in problems where the solution can be determined explicitly for arbitrary elastic constants the effect of a small error can be easily determined. However, if the solution cannot be computed but instead pointwise bounds for the solution and its derivatives are sought, one must also take into account the error which might arise due to inaccuracies in the determination of the elastic constants.

In this paper we develop a priori inequalities appropriate to the first and second boundary value problems. These inequalities give upper and lower bounds for the pointwise error in approximation of the exact solution by an arbitrary function. The error in measurement of the elastic constants as well as the error in approximation of the data of the problem is taken into account in the bounds. Thus by choosing trial functions to closely approximate the data, a close approximation can be obtained, with a known bound on the error, provided the error in the constants is also known. The part of the error which results from the fact that the trial function does not satisfy exactly the data of the problem is quadratic. Thus, the Rayleigh-Ritz technique may be employed. This means that we choose from a class of trial functions the best function in the sense that the error term involving the data is minimized (see e.g., [3] [1]).

The above mentioned inequalities also are interesting in that they clearly display the fact that the solution to the first (or second) boundary value problem for an incompressible medium (Poisson's ratio, σ, having the value ½) as formulated in the literature, is actually the limit of the solution for values of $\sigma < ½$. We show in fact that at points in a given region the solution for $\sigma < ½$ converges with all its derivatives to the classical solution. To the authors' knowledge this fact has not been pointed out in the literature.

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[1] Figures in brackets indicate the literature references at the end of this paper.

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2. Notation

Let D be a bounded domain with boundary Γ in three dimensions. The displacement vector $u^\alpha(u_1^\alpha, u_2^\alpha, u_3^\alpha)$ corresponding to elastic constants μ_α and σ_α is assumed to satisfy in D the system of equations

$$\Delta u_i^\alpha = p^\alpha_{,i} - \mu_\alpha^{-1} F_i, \qquad i=1,2,3$$

$$u^\alpha_{j,j} = -c_\alpha p^\alpha. \tag{2.1}$$

Here Δ denotes the Laplace operator, the symbol, i, denotes partial differentiation with respect to x_i and the summation convention is employed for Latin but not Greek indices. The constant c_α is given by

$$c_\alpha = 1 - 2\sigma_\alpha, \tag{2.2}$$

and F_i denotes the body force per unit volume. In terms of u_i^α and the elastic constants the stress components τ_{ij}^α are expressed as

$$\tau_{ij}^\alpha = \mu_\alpha [u^\alpha_{i,j} + u^\alpha_{j,i} - (1-c_\alpha) p^\alpha \delta_{ij}]. \tag{2.3}$$

The physically interesting values of c_α lie in the interval

$$0 \leq c_\alpha < 3. \tag{2.4}$$

In this paper we shall, for simplicity, restrict our attention to the first and the second boundary value problems in classical elasticity. In particular, we assume that in D

$$\Delta u_i^\alpha - p^\alpha_{,i} = -\mu_\alpha^{-1} F_i; \qquad \Delta u_i^\beta - p^\beta_{,i} = -\mu_\beta^{-1} F_i$$

$$u^\alpha_{j,j} = -c_\alpha p^\alpha \qquad ; \qquad u^\beta_{j,j} = -c_\beta p^\beta. \tag{2.5}$$

We wish to compare the solutions u_i^α and u_i^β if one of the following sets of boundary conditions is satisfied:

$$u_i^\alpha = u_i^\beta = f_i \text{ on } \Gamma \tag{2.5A}$$

$$\mu_\alpha^{-1} \tau_{ij}^\alpha n_j = \mu_\beta^{-1} \tau_{ij}^\beta n_j = g_i \text{ on } \Gamma \tag{2.5B}$$

$$\tau_{ij}^\alpha n_j = \tau_{ij}^\beta n_j = h_i \text{ on } \Gamma. \tag{2.5C}$$

From the point of view of actual physical applicability, problem (2.5A) and (2.5C) are the two interesting cases.

In the subsequent sections we shall set

$$w_i = u_i^\alpha - u_i^\beta. \tag{2.6}$$

3. Pointwise Convergence

We demonstrate in this section that if p^β and F_i are square integrable in D and h_i is square integrable on Γ, then at interior points of D, u_i^α and all its derivatives tend to u_i^β and its corresponding derivatives as $c_\alpha \to c_\beta$, and $\mu_\alpha \to \mu_\beta$.

We assume that there exists a constant M_β such that

$$\int_D [p^\beta]^2 dv < M_\beta. \tag{3.1}$$

Let us assume now that we were able to obtain for either of problems (2.5A) to (2.5C), an L_2 bound for w_i in terms of L_2 integrals of the data of u_i^β. Suppose further that the coefficients of the integrals of data tend to zero as $c_\alpha \to c_\beta$ and $\mu_\alpha \to \mu_\beta$. Then if the data of u_i^β are in L_2 it follows that as $\alpha \to \beta$, $w_i \to 0$ in L_2. Pointwise convergence would then follow from the continuity of w_i. The required L_2 bounds for w_i are derived in appendix 1 and appendix 2. Since we are also interested in computing pointwise bounds, explicit constants will be obtained. In problem (2.5A) we have

$$\int_D w_i w_i dv \leq \frac{M_\beta}{2\lambda_1 c_\alpha}(c_\alpha - c_\beta)^2 + \frac{(\mu_\alpha^{-1} - \mu_\beta^{-1})^2}{\lambda_1^2}\int_D F_i F_i dv, \tag{3.2}$$

and in problem (2.5B)

$$\int_D w_i w_i dv \leq \frac{k_4}{2} F(c_\alpha) M_\beta (c_\alpha - c_\beta)^2 + (\mu_\alpha^{-1} - \mu_\beta^{-1})^2 k_4^2 \int_D F_i F_i dv. \tag{3.3}$$

These inequalities follow from (3.1) using (5.23) and (5.30) of appendix 1. For case (2.5C) we have instead from (3.1) and appendix 2,

$$\int_D w_i w_i dv \leq \frac{k_4(c_\alpha - c_\beta)^2}{(2-3\gamma)[1-(\gamma-1)c_\alpha]} M_\beta + k_4\left[\frac{k_1}{k_3}\frac{(\mu_\alpha^{-1} - \mu_\beta^{-1})^2}{(2-3\gamma)}\right]\int_\Gamma h_i h_i ds$$
$$+ \frac{8(\mu_\alpha^{-1} - \mu_\beta^{-1})}{(2-3\gamma)^2} k_4^2 \int_D F_i F_i dv, \tag{3.4}$$

where γ satisfies (5.8).

We wish to obtain from these expressions pointwise inequalities in terms of the appropriate sets of data. In order to do this we introduce the function $\Gamma_{ik}^\alpha(P, Q)$, the Kelvin fundamental solution of (2.1); i.e.,

$$\Gamma_{ik}^\alpha(P, Q) = \frac{1}{8\pi(1+c_\alpha)}\left\{\frac{(1+2c_\alpha)\delta_{ik}}{r_{PQ}} + \frac{[x_i(P) - x_i(Q)][x_k(P) - x_k(Q)]}{r_{PQ}^3}\right\} \tag{3.5}$$

where r_{PQ} is the distance from the point P to the point Q. Then a particular solution $\bar{u}_i^\alpha$ of (2.1) is given by

$$\bar{u}_i^\alpha(P) = \mu_\alpha^{-1}\int_D \Gamma_{ik}^\alpha(P, Q) F_k dv_Q. \tag{3.6}$$

Let u_i^α now be defined as

$$\hat{u}_i^\alpha = u_i^\alpha - \bar{u}_i^\alpha. \tag{3.7}$$

Clearly then $\hat{u}_i^\alpha$ satisfies the system

$$\Delta \hat{u}_i^\alpha + \frac{1}{c_\alpha}\hat{u}_{j,ji}^\alpha = 0 \text{ in } D. \tag{3.8}$$

We now define

$$w_i = \bar{w}_i + \hat{w}_i \tag{3.9}$$

where

$$\overline{w}_i=\overline{u}_i^\alpha-\overline{u}_i^\beta=\int_D [\mu_\alpha^{-1}\Gamma_{ik}^\alpha(P,Q)-\mu_\beta^{-1}\Gamma_{ik}^\beta(P,Q)]F_k dv \tag{3.10}$$

and

$$\hat{w}_i=\hat{u}_i^\alpha-\hat{u}_i^\beta. \tag{3.11}$$

Since $\hat{u}_i^\alpha$ and $\hat{u}_i^\beta$ are biharmonic functions, it follows that $\hat{W}_i$ is a biharmonic function. We may, therefore, make use of some mean value inequalities for biharmonic functions (see, e.g., Bramble and Payne [2]). That is, if R denotes the radius of the largest sphere contained in D with center at the point at which bounds are desired (we take this point to be the origin), then

$$\hat{w}_i(0)\hat{w}_i(0)\leq\frac{75}{16\pi R^2}\int_D \hat{w}_i\hat{w}_i dv. \tag{3.12}$$

Similarly, as is shown in [3], it is possible to obtain

$$D^\nu(\hat{w}_i)(0)D^\nu(\hat{w}_i)(0)\leq C^\nu\int_D \hat{w}_i\hat{w}_i dv. \tag{3.13}$$

where D^ν is any νth order partial derivative and C^ν is a constant depending on R and ν.

It is clear at this point that if in any of problems (2.5A to 2.5C) the body force F_i is identically zero, then $\overline{w}_i\equiv 0$ and $\hat{w}_i\equiv w_i$. By using the appropriate bound for $\int_D w_iw_i dv$ (given by (3.2), (3.3), or (3.4) we would obtain the desired pointwise bounds for $u_i^\alpha-u_i^\beta$. In fact, in any compact subdomain of D, we could obtain a uniform bound of the following type for any νth order partial derivative, $D^\nu(u_i^\alpha-u_i^\beta)$, of $u_i^\alpha-u_i^\beta$ (assuming that in (3.5), $\int_\Gamma h_ih_i ds$ is bounded):

$$D^\nu(u_i^\alpha-u_i^\beta)D^\nu(u_i^\alpha-u_i^\beta)\,|_{(0,\,0,\,0)}\leq k_\nu[c_\alpha-c_\beta]^2+K_\nu[\mu_\alpha^{-1}-\mu_\beta^{-1}]^2. \tag{3.14}$$

In either of problems (2.5A) or (2.5B), the constant K_ν would in fact be zero. We turn now to the case $F_i\not\equiv 0$.

Clearly

$$w_i(0)w_i(0)\leq 2[\overline{w}_i(0)\overline{w}_i(0)+\hat{w}_i(0)\hat{w}_i(0)]. \tag{3.15}$$

But from (3.12)

$$\hat{w}_i(0)\hat{w}_i(0)\leq\frac{75}{8\pi R^3}\left[\int_D w_iw_i dv+\int_D \overline{w}_i\overline{w}_i dv\right]. \tag{3.16}$$

Combining (3.15) and (3.16) we obtain

$$w_i(0)w_i(0)\leq\frac{75}{4\pi R^3}\int_D w_iw_i dv+2\,\overline{w}_i(0)\overline{w}_i(0)+\frac{75}{4\pi R^3}\int_D\overline{w}_i\overline{w}_i dv. \tag{3.17}$$

We seek now an appropriate bound for the last two terms of (3.17). An application of Schwarz's inequality to (3.10) gives

$$\overline{w}_i\overline{w}_i\leq\int_D[\mu_\alpha^{-1}\Gamma_{ik}^\alpha-\mu_\beta^{-1}\Gamma_{ik}^\beta][\mu_\alpha^{-1}\Gamma_{ik}^\alpha-\mu_\beta^{-1}\Gamma_{ik}^\beta]dv\int_D F_kF_k dv. \tag{3.18}$$

We now insert (3.5) into the first term on the right and make use of the arithmetic-geometric mean inequality to obtain

$$\int_D [\mu_\alpha^{-1}\Gamma^\alpha_{ik} - \mu_\beta^{-1}\Gamma^\beta_{ik}][\mu_\alpha^{-1}\Gamma^\alpha_{ik} - \mu^{-1}\Gamma^\beta_{ik}]dv \leqslant \frac{G(P)}{32\pi^2}\left\{2\,[\mu_\alpha^{-1} - \mu_\beta^{-1}]^2 + \left[\frac{(1+2c_\alpha)}{\mu_\alpha(1+c_\alpha)} - \frac{(1+2c_\beta)}{\mu_\beta(1+c_\beta)}\right]^2\right\} \tag{3.19}$$

where $G(P) = \int_D r_{PQ}^{-2} dv_Q$.

Combining (3.18) and (3.19) and integrating we have

$$\int_D \overline{w}_i\overline{w}_i dv \leqslant \frac{H}{32\pi^2}\left\{2\,[\mu_\alpha^{-1} - \mu_\beta^{-1}]^2 + \left[\frac{(1+2c_\alpha)}{\mu_\alpha(1+c_\alpha)} - \frac{(1+2c_\beta)}{\mu_\beta(1+c_\beta)}\right]^2\right\}\int_D F_kF_k dv, \tag{3.20}$$

where $H = \int_D\int_D r_{PQ}^{-2} dv_Q dv_P$.

It follows now from (3.18), (3.19), and (3.20) that

$$2\overline{w}_i(O)\overline{w}_i(O) + \frac{75}{4\pi R^3}\int_D \overline{w}_i\overline{w}_i dv \leqslant \frac{1}{16\pi^2}\left\{G(O) + \frac{75}{8\pi R^3}H\right\}$$

$$\cdot\left\{2[\mu_\alpha^{-1} - \mu_\beta^{-1}]^2 + \left[\frac{(1+2c_\alpha)}{\mu_\alpha(1+c_\alpha)} - \frac{(1+2c_\beta)}{\mu_\beta(1+c_\beta)}\right]^2\right\}\int_D F_kF_k dv. \tag{3.21}$$

The following bounds on $G(0)$ and H are easy to obtain:

$$G(0) \leqslant (4\pi d) \tag{3.22}$$

and

$$H \leqslant V(4\pi d) \tag{3.23}$$

where V is the volume of D and d is the diameter. Thus combining (3.17), and (3.21), (3.22), (3.23), and (3.26) we obtain

$$w_i(0)w_i(0) \leqslant \frac{75}{4\pi R^3}\int_D w_iw_i dv + \left[\frac{d}{4\pi}\left\{1 + \frac{75}{8\pi R^3}V\right\}\right].$$

$$\cdot\left\{2(\mu_\alpha^{-1} - \mu_\beta^{-1}]^2 + \left[\frac{(1+2c_\alpha)}{\mu_\alpha(1+c_\alpha)} - \frac{(1+2c_\beta)}{\mu_\alpha(1+c_\alpha)}\right]^2\right\}\int_D F_kF_k dv. \tag{3.24}$$

If F_k is square integrable then the insertion of (3.2), (3.3), or (3.5) into (3.24) gives in any compact subdomain D_1 of D a uniform bound of the following type.

$$[u_i^\alpha(P) - u_i^\beta(P)][u_i^\alpha(P) - u_i^\beta(P)] \leqslant B_1[\mu_\alpha^{-1} - \mu_\beta^{-1}]^2 + B_2[c_\alpha - c_\beta]^2 \tag{3.25}$$

where B_1 and B_2 are independent of the point P in D_1 and can be chosen independent of the elastic constants. This follows because of the fact that since $c_\alpha, c_\beta \geqslant 0$,

$$\left[\frac{(1+2c_a)}{\mu_a(1+c_a)} - \frac{(1+2c_\beta)}{\mu_\beta(1+c_\beta)}\right]^2 = \left\{(\mu_\alpha^{-1} - \mu_\beta^{-1})\frac{1+2c_\alpha}{1+c_\alpha} - \mu_\beta^{-1}\frac{(c_\alpha - c_\beta)}{(1+c_a)(1+c_\beta)}\right\}^2$$

$$\leqslant 2\,\{4(\mu_\alpha^{-1} - \mu_\beta^{-1})^2 + \mu_\beta^{-1}(c_\alpha - c_\beta)^2\}. \tag{3.26}$$

The inequality (3.25) clearly implies the uniform convergence of u_i^α to u_i^β in any compact subdomain of D as $\mu_\alpha \to \mu_\beta$ and $c_\alpha \to c_\beta$.

It is clear that if F_k has a sufficient number of square integrable derivatives then the convergence of the derivatives of u_i^α may be obtained in a similar way. To see this we note that

$$\frac{\partial \Gamma_{ik}^\alpha(P,Q)}{\partial x_P} \equiv -\frac{\partial \Gamma_{ik}^\alpha(P,Q)}{\partial x_Q}. \tag{3.27}$$

Thus (3.6) may be differentiated with respect to the variables of the point P, the order of differentiation and integration being interchanged on the right. Then (3.27) is used followed by an integration by parts which takes the differentiation off of Γ_{ik}^α. The bounds for $|u_{i,p}^\alpha(P)|$ are then obtained in a manner similar to that used in obtaining (3.25). To obtain the desired expressions for higher derivatives, we successively differentiate Γ_{ik}^α and use (3.27) to throw the differentiation off of Γ_{ik}^α in the volume integral on the right. Then if a sufficient number of derivatives of F_k are square integrable over D and over Γ we obtain a uniform bound similar to (3.25) for any desired derivative of u_i^α. This shows, then, that u_i^α and its derivatives converge to u_i^β and its derivatives.

We note, in particular, that if $c_\beta=0$ the u_i^β problem is the classical boundary value problem for an incompressible medium. It follows then that as $c_\alpha \rightarrow 0$ and $\mu_\alpha \rightarrow \mu_\beta$ the solution to the problem corresponding to α converges to the solution in the incompressible case. This fact, to the knowledge of the authors, has never been pointed out in the literature.

Another interesting fact is that F_k need only have the desired number of square integrable derivatives in a neighborhood of the point at which convergence is sought. The procedures used in this section could be applied to any neighborhood Ω of the origin. Then since

$$\int_\Omega w_i w_i dv \leq \int_D w_i w_i dv$$

it would follow that if F_k were square integrable over D and enough derivatives of F_k were square integrable over Ω and its boundary, uniform convergence in Ω of the derivative would follow.

4. Effect of Error in Measurement of Elastic Constants on Pointwise Bounds

We consider now problem (2.1) with either the displacement components or the surface tractions prescribed on Γ. Since c_α and μ_α must be determined empirically we shall assume that some error was made in making the measurements and that the values c_β and μ_β were actually obtained. If φ_i is a sufficiently smooth vector function chosen to approximate the solution u_i^α we may write

$$|u_i^\alpha-\varphi_i| \leq |u_i^\alpha-u_i^\beta| + |u_i^\beta-\varphi_i|. \tag{4.1}$$

We wish to obtain upper and lower bounds for $u_i^\alpha(0)$. If we can make each term on the right of (4.1) less than some prescribed ϵ we would then have

$$-2\epsilon < u_i^\alpha - \varphi_i < 2\epsilon \tag{4.2}$$

and φ_i would be a good pointwise approximation to u_i^α.

The bound for the first term on the right of (4.1) is obtained for the first boundary value problem in elasticity from (3.24) and (5.22) of appendix 1.

$$(u_i^\alpha(0)-u_i^\beta(0))(u_i^\alpha(0)-u_i^\beta(0)) \leq \frac{75}{4\pi R^3}\left\{A_{\alpha\beta}\int_D F_iF_i dv + \frac{3(c_\alpha-c_\beta)^2\mu_\beta^{-1}E(u^\beta,u^\beta)}{\lambda_1 c_\alpha c_\beta(3-c_\beta)}\right\}, \tag{4.3}$$

where

$$A_{\alpha\beta}=(1/\lambda_1^2+2R^3d/15+5Vd/4\pi)(\mu_\alpha^{-1}-\mu_\beta^{-1})^2+d\mu_\beta^{-1}(2R^3/75+V/4\pi)(c_\alpha-c_\beta)^2.$$

A bound for $E(u^\beta, u^\beta)$ is obtained by the methods of Diaz and Greenberg [4], Fichera [6], Synge [10], or Bramble and Payne [3, eq (3.12)].

The second term on the right of (4.1) may be dealt with by the method of Bramble and Payne [3, eq (4.10)].

We look now at the stress components τ_{ij}^α. Using the triangle inequality we obtain

$$|\tau_{ij}^\alpha - \tau_{ij}| \leq |\tau_{ij}^\alpha - \tau_{ij}^\beta| + |\tau_{ij}^\beta - \tau_{ij}|, \tag{4.4}$$

where

$$\tau_{ij} = \mu_\beta \left[\varphi_{i,j} + \varphi_{j,i} + \frac{(1-c_\beta)}{c_\beta} \varphi_{j,j} \delta_{ij} \right] \tag{4.5}$$

is an approximation to τ_{ij}^α. Bounds for the first term on the right of (4.4) may be obtained using (3.13) following again the method outlined at the end of section 3. The second term on the right may be dealt with as described in [3].

For the second boundary value problem in elasticity we again use (4.1) and (4.4), where in this case u_i^α and u_i^β are solutions to (2.5C) and bounds for the second term on the right are given in [1].

5. Appendix 1

In appendix 1 we derive the inequalities (5.22), (5.23), (5.29), and (5.30) which are needed in the text. In order to do this we start with

$$\int_D w_{i,j}(w_{i,j} + w_{j,i})dv.$$

In particular, we bound this expression in terms of the solution of the problem with elastic constants μ_β, c_β. Consider then the Green's identity

$$\int_D w_{i,j}(w_{i,j} + w_{j,i})dv = \int_\Gamma w_i(w_{i,j} + w_{j,i})n_j ds - \int_D w_i(\Delta w_i + w_{j,ji})ds. \tag{5.1}$$

Making use of (2.5) and integrating by parts we obtain for w_i satisfying (2.6)

$$\int_D w_{i,j}(w_{i,j} + w_{j,i})dv = -\int_\Gamma w_i\{\mu_\alpha^{-1}\tau_{ij}^\alpha n_j - \mu_\beta^{-1}\tau_{ij}^\beta n_j\}ds + (u_\alpha^{-1} - \mu_\beta^{-1}) \int_D w_i F_i dv$$
$$+ (c_\alpha + c_\beta - 2c_\alpha c_\beta) \int_D p^\alpha p^\beta dv - c_\alpha(1-c_\alpha) \int_D (p^\alpha)^2 dv - c_\beta(1-c_\beta) \int_D (p^\beta)^2 dv. \tag{5.2}$$

In either of problems (2.5A) or (2.5B), the boundary integral on the right of (5.2) will vanish. In fact, this integral would vanish for various mixed boundary conditions which are encountered in physical situations. On the other hand, in problem (2.5C) the boundary term does not vanish. We shall thus consider first problems (2.5A) and (2.5B) and then treat (2.5C) at the end of the section. If $0 < c_\alpha \leq 3/4$ $(\sigma \geq 1/8)$ we rewrite (5.2) for (A) and (B) as

$$\int_D w_{i,j}(w_{i,j} + w_{j,i})dv = -c_\alpha(1-c_\alpha) \int_D \left[p^\alpha - \frac{(c_\alpha + c_\beta - 2c_\alpha c_\beta)}{2c_\alpha(1-c_\alpha)} p^\beta \right]^2 dv$$
$$+ (\mu_\alpha^{-1} - \mu_\alpha^{-1}) \int_D w_i F_i dv + \frac{(c_\alpha - c_\beta)^2}{4c_\alpha(1-c_\alpha)} \int_D (p^\beta)^2 dv. \tag{5.3}$$

The first term on the right is nonpositive. Thus

$$\int_D w_{i,j}(w_{i,j} + w_{j,i})dv \leq \frac{(c_\alpha - c_\beta)^2}{4c_\alpha(1-c_\alpha)} \int_D (p^\beta)^2 dv + (\mu_\alpha^{-1} - \mu_\beta^{-1}) \int_D w_i F_i dv. \tag{5.4}$$

The strain energy $E(u^\beta, u^\beta)$ corresponding to u^β is given by:

$$2\mu_\beta^{-1}E(u^\beta, u^\beta) = \int_D [u^\beta_{i,j}(u^\beta_{i,j}+u^\beta_{j,i}) + c_\beta(1-c_\beta)(p^\beta)^2]dv. \geqslant \frac{c_\beta}{3}(3-c_\beta)\int_D (p^\beta)^2 dv. \tag{5.5}$$

The last inequality results from the fact that

$$u^\beta_{i,j}(u^\beta_{i,j}+u^\beta_{j,i}) \leqslant 2[(u^\beta_{1,1})^2+(u^\beta_{2,2})^2+(u^\beta_{3,3})^2] \geqslant 2/3[u^\beta_{i,i}]^2. \tag{5.6}$$

Hence if $c_\beta \neq 0$ we have from (5.4) and (5.5)

$$\int_D w_{i,j}(w_{i,j}+w_{j,i})dv \geqslant \frac{3\mu_\beta^{-1}(c_\alpha-c_\beta)^2}{2c_\alpha c_\beta(1-c_\alpha)(3-c_\beta)}E(u^\beta, u^\beta) + (\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D w_iF_i dv. \tag{5.7}$$

Bounds for $E(u^\beta, u^\beta)$ in problems (2.5A) and (2.5B) in terms of the data of the problem have been given for instance by Diaz and Greenberg [4], Fichera [6], Prager and Synge [9], Synge [10], and Bramble and Payne [1, 3]. If $F_i \equiv 0$ such bounds would give then for $0 < c_\alpha \leqslant 3/4$ an upper bound for

$$\int_D w_{i,j}(w_{i,j}+w_{j,i})dv$$

in terms of the data of the u_i^β problem of (2.5A) or (2.5B).

For $c_\alpha > 3/4$ we choose a constant γ in the interval

$$1-c_\alpha^{-1} < \gamma < 2/3 \tag{5.8}$$

and consider the following identity derived from (5.2) by addition of the appropriate term to each side of the equation

$$\int_D w_{i,j}(w_{i,j}+w_{j,i})dv - \gamma\int_D w_{j,j}^2 dv = [c_\alpha+c_\beta+2(\gamma-1)c_\alpha c_\beta]\int_D p^\alpha p^\beta dv$$

$$-c_\alpha\{1+(\gamma-1)c_\alpha\}\int_D (p^\alpha)^2 dv - c_\beta\{1-(\gamma-1)c_\beta\}\int_D (p^\beta)^2 dv + (\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D w_iF_i dv. \tag{5.9}$$

Using (5.6) then we have

$$\left(1-\frac{3\gamma}{2}\right)\int_D w_{i,j}(w_{i,j}+w_{j,i})dv \leqslant -c_\alpha\{1+(\gamma-1)c_\alpha\}\int_D \left\{p^\alpha + \frac{[c_\alpha+c_\beta+2(\gamma-1)c_\alpha c_\beta]}{2c_\alpha[1+(\gamma-1)c_\alpha]}p^\beta\right\}^2 dv$$

$$+\frac{(c_\alpha-c_\beta)^2}{4c_\alpha[1+(\gamma-1)c_\alpha]}\int_D (p^\beta)^2 dv + (\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D w_iF_i dv. \tag{5.10}$$

Thus we obtain

$$\int_D w_{i,j}(w_{i,j}+w_{j,i})dv \leqslant \frac{(c_\alpha-c_\beta)^2}{2(2-3\gamma)c_\alpha[1+(\gamma-1)c_\alpha}\int_D [p^\beta]^2 dv + \frac{2(\mu_\alpha^{-1}-\mu_\beta^{-1})}{(2-3\gamma)}\int_D w_iF_i dv. \tag{5.11}$$

The choice $\gamma = 2c_\alpha/(3+2c_\alpha)$ yields

$$\int_D w_{i,j}(w_{i,j}+w_{j,i})dv \leqslant \frac{(3+2c_\alpha)^2(c_\alpha-c_\beta)^2}{4c_\alpha(3-c_\alpha)^2}\int_D (p^\beta)^2 dv + (\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D w_iF_i dv\cdot \tag{5.12}$$

Proceeding as before we obtain finally the inequality

$$\int_D w_{i,j}(w_{i,j}+w_{i,i})dv \leqslant F(c_\alpha)(c_\alpha-c_\beta)^2\int_D (p^\beta)^2 dv + (\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D F_iw_i dv, \tag{5.13}$$

where

$$F(c_\alpha)=\begin{cases}[4c_\alpha(1-c_\alpha)]^{-1}, 0<c_\alpha\leq 3/4\\ [(3+2c_\alpha)^2[4c_\alpha(3-c_\alpha)^2]^{-1}, c_\alpha>3/4.\end{cases} \tag{5.14}$$

It follows then that for $c_\beta\neq 0$

$$\int_D w_{i,j}(w_{i,j}+w_{j,i})dv\leq\frac{6\mu_\beta^{-1}F(c_\alpha)}{c_\beta(3-c_\beta)}(c_\alpha-c_\beta)^2E(u^\beta,u^\beta)+(\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D w_iF_idv. \tag{5.15}$$

In problem (2.5A) we have the relation

$$\int_D w_{i,j}w_{i,j}dv=-\int_D w_i\Delta w_idv=-\int_D w_i(p^\alpha-p^\beta)dv$$

$$=-\int_D(c_\alpha p^\alpha-c_\beta p^\beta)(p^\alpha-p^\beta)dv+(\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D F_iw_idv. \tag{5.16}$$

or

$$\int_D w_{i,j}w_{i,j}dv=-c_\alpha\int_D\left[p^\alpha-\frac{(c_\alpha+c_\beta)}{2c_\beta}p^\beta\right]^2dv+(\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D F_iw_idv+\frac{(c_\alpha-c_\beta)^2}{4c_\alpha}\int_D(p^\beta)^2dv. \tag{5.17}$$

Thus, dropping the first term on the right,

$$\int_D w_{i,j}w_{i,j}dv\leq\frac{(c_\alpha-c_\beta)^2}{4c_\alpha}\int_D(p^\beta)^2dv+(\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D F_iw_idv \tag{5.18}$$

and if $c_\beta\neq 0$

$$\int_D w_{i,j}w_{i,j}dv\leq 3\mu_\beta^{-1}\frac{(c_\alpha-c_\beta)^2}{2c_\alpha c_\beta(3-c_\beta)}E(\mu^\beta,\mu^\beta)+(\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D F_iw_idv. \tag{5.19}$$

We have retained the term $(\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D F_iw_idv$ up to this point since in the important case $F_i\equiv 0$ this term will vanish and the inequality will not be altered. Let us assume now that $F_i\not\equiv 0$ and note that

$$\int_D w_iw_idv\leq\frac{1}{\lambda_1}\int_D w_{i,j}w_{i,j}dv\leq\frac{3(c_\alpha-c_\beta)^2\mu_\beta^{-1}E(u_\beta,u_\beta)}{2\lambda_1c_\alpha c_\beta(3-c_\beta)}+\frac{(\mu_\alpha^{-1}-\mu_\beta^{-1})}{\lambda_1}\int_D F_iw_idv. \tag{5.20}$$

Where λ_1 is the first eigenvalue in the three dimensional fixed membrane problem for D. A lower bound for λ_1 is given by the Faber-Krahn inequality [5, 7], i.e.,

$$\lambda_1\geq\pi^2\left[\frac{4\pi}{3V}\right]^{2/3}, \tag{5.21}$$

where V denotes the volume of D. From the arithmetic-geometric mean inequality applied to the last term of (5.20) it follows that

$$\int_D w_iw_idv\leq\frac{3(c_\alpha-c_\beta)^2\mu_\beta^{-1}}{\lambda_1c_\alpha c_\beta(3-c_\beta)}E(u_\beta,u_\beta)+\frac{(\mu_\alpha^{-1}-\mu_\beta^{-1})^2}{\lambda_1^2}\int_D F_iF_idv. \tag{5.22}$$

Similarly if we insert (5.18) into the first inequality in (5.20) and apply the arithmetic-geometric mean inequality to the last term we obtain

$$\int_D w_iw_idv\leq\frac{(c_\alpha-c_\beta)^2}{2\lambda_1c_\alpha}\int_D(p^\beta)^2dv+\frac{(\mu_\alpha^{-1}-\mu_\beta^{-1})^2}{\lambda_1^2}\int_D F_iF_idv. \tag{5.23}$$

For problem (3.5B) it was shown in [1, eq (3.14)] that if w_i is normalized so that $\int_\Gamma w_i ds=0$ then

$$\int_\Gamma w_iw_ids \leqslant \frac{k_1}{2k_3}\int_D w_{i,j}(w_{i,j}+w_{j,i})dv \tag{5.24}$$

where k_1 and k_3 are constants given in [1, eqs (2.9) and (2.19)], which are computable in terms of the geometry of D.

We now make use of the identity

$$5\int_D w_iw_idv=\int_\Gamma x^kw_i(2n_iw_k+n_kw_i)ds+2\int_D x^kw_i(w_{i,k}+w_{k,i})dv+2\int_D x^kw_kw_{i,i}dv. \tag{5.25}$$

An application of the arithmetic geometric mean inequality and (3.8) then yields (see e.g., [1, eqs (2.26) to (2.28)])

$$\int_D w_iw_idv \leqslant \frac{6r_M}{5}\int_\Gamma w_iw_ids+\frac{28r_M^2}{25}\int_D w_{i,j}(w_{i,j}+w_{j,i})dv \tag{5.26}$$

where r_M denotes the maximum distance from the origin to Γ. From (5.24) and (5.26) it follows then that

$$\int_D w_iw_idv \leqslant k_4\int_D w_{i,j}(w_{i,j}+w_{j,i})dv \tag{5.27}$$

where

$$k_4=\left[\frac{3}{5}r_Mk_1/k_3+\frac{28}{25}r_M^2\right]. \tag{5.28}$$

Thus if in (2.5B) $c_\beta\neq 0$,

$$\int_D w_iw_idv \leqslant \frac{12k_4\mu_\beta^{-1}F(c_\alpha-c_\beta)^2}{c_\beta(3-c_\beta)}E(u^\beta,u^\beta)+(\mu_\alpha^{-1}-\mu_\beta^{-1})^2k_4^2\int_D F_iF_idv. \tag{5.29}$$

Finally, after inserting (5.13) into (5.27) and using the arithmetic-geometric mean inequality on the last term we have

$$\int_D w_iw_idv \leqslant 2k_4F(c_\alpha)(c_\alpha-c_\beta)^2\int_D (p^\beta)^2dv+k_4^2(\mu_\alpha^{-1}-\mu_\beta^{-1})^2\int_D F_iF_idv. \tag{5.30}$$

6. Appendix 2

We derive here inequalities (6.3) and (6.4) which are needed to treat problem (2.5C).

Again we start with $\int_D w_{i,j}(w_{i,j}+w_{j,i})dv$ and instead of (5.2) we obtain

$$\int_D w_{i,j}(w_{i,j}+w_{j,i})dv=-\int_\Gamma w_ih_i[\mu_\alpha^{-1}-\mu_\beta^{-1}]ds+(c_\alpha+c_\beta-2c_\alpha c_\beta)\int_D p^\alpha p^\beta dv$$
$$-c_\alpha(1-c_\alpha)\int_D (p^\alpha)^2dv-c_\beta(1-c_\beta)\int_D (p^\beta)^2dv+(\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D w_iF_idv, \tag{6.1}$$

which follows from a calculation involving the use of the divergence theorem. We thus obtain as before

$$\int_D w_{i,j}(w_{i,j}+w_{j,i})dv+\frac{2}{(2-3\gamma)}[\mu_\alpha^{-1}-\mu_\beta^{-1}]\int_\Gamma w_ih_ids$$
$$-\frac{2}{(2-3\gamma)}(\mu_\alpha^{-1}-\mu_\beta^{-1})\int_D F_iw_idv\leqslant\frac{(c_\alpha-c_\beta)^2}{2(2-3\gamma)[1+(\gamma-1)c_\alpha]}\int_D (p^\beta)^2dv \tag{6.2}$$

where γ satisfies (5.8). It follows then from the arithmetic-geometric mean inequality, (5.24), and (5.27) that if w_i is normalized so that $\int_\Gamma w_i ds = 0$ then

$$\left[1-\frac{a}{2}-\frac{b}{2}\right]\int_D w_{i,j}(w_{i,j}+w_{j,i})dv \leq \frac{[\mu_\alpha^{-1}-\mu_\beta^{-1}]^2}{4(2-3\gamma)^2 a}(k_1/k_3)^2\int_\Gamma h_i h_i ds$$

$$+\frac{(c_\alpha-c_\beta)^2\int_D (p^\beta)^2 dv}{2(2-3\gamma)[1+(\gamma-1)c_\alpha]}+\frac{2(\mu_\alpha^{-1}-\mu_\beta^{-1})^2}{(2-3\gamma)^2 b}k^4\int_D F_i F_i dv \qquad (6.3)$$

for any positive numbers a and b. In particular for $a=\tfrac{1}{2}$, $b=\tfrac{1}{2}$ (6.2) becomes

$$\int_D w_{i,j}(w_{i,j}+w_{j,i})dv \leq \left[\frac{k_1}{k_3}\frac{(\mu_\alpha^{-1}-\mu_\beta^{-1})}{(2-3\gamma)}\right]^2\int_\Gamma h_i h_i ds$$

$$+\frac{8(\mu_\alpha^{-1}-\mu_\beta^{-1})^2}{(2-3\gamma)^2}k_4\int_D F_i F_i dv+\frac{(c_\alpha-c_\beta)^2}{(2-3\gamma)[1+(\gamma-1)c_\alpha]}\int_D (p^\beta)^2 dv. \qquad (6.4)$$

As before inserting (6.3) into (5.27) we have

$$\int_D w_i w_i dv \leq k_4\left[\frac{k_1}{k_3}\frac{(\mu_\alpha^{-1}-\mu_\beta^{-1})}{(2-3\gamma)}\right]^2\int_\Gamma h_i h_i ds+\frac{8(\mu_\alpha^{-1}-\mu_\beta^{-1})^2}{(2-3\gamma)^2}\int_D F_i F_i dv$$

$$+\frac{k_4(c_\alpha-c_\beta)^2}{(2-3\gamma)[1+(\gamma-1)c_\alpha]}\int_D (p^\beta)^2 dv. \qquad (6.5)$$

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