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Numerical Computation of the Temporal Development of Currents in a Gas Discharge Tube*

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The behavior of electrical currents in a gas discharge tube including space charge effects is investigated by numerical integration of the governing nonlinear partial differential equations. Both stationary solutions and the temporal development, under the influence of space charge effects, are considered. It is found that the truncation error can be greatly reduced by comparison with formal solutions for constant fields. The discussion is essentially restricted to the more mathematical questions.

1. Introduction

The behavior of electron and ion currents in a gas discharge tube as a function of time and the applied voltage has been investigated by several authors [1, 2, 3, 4][1]. Most of these have omitted the effect of space charge, but have estimated when the effect appears. Space charge, however, results in a temporally growing distortion of the electrical field and, therefore, in a severe nonlinearity of the equations governing the behavior of electron and ion currents in the tube. A. L. Ward [5] suggested the numerical integration of the nonlinear equations on an electronic computer for an essentially "one-dimensional" tube, i.e., a tube whose electrodes are parallel plates of dimensions large compared to their separation.

This paper describes the mathematical treatment of the basic equations necessary for the applicability of numerical methods. For the sake of completeness, a short derivation of the basic equations (sec. 2) and the formal treatment of the case of constant field (sec. 5) is given, although many of these considerations can be found in other papers too. The stationary case is treated extensively in section 3. The difference schemes used for the time-dependent case are discussed in section 4. The discussion of the results in section 6 is restricted to the more mathematical questions like the influence of truncation errors, and certain other errors occurring during the computations. A discussion of the physical significance of the results is given by A. L. Ward [10, 11].

2. Basic Equations

We state the equations in an Eulerian coordinate system, denoting the space coordinate by x and the time by t. The cathode is located at $x=0$, the anode at $x=d$. Let n_+, n_-, j_+, j_-, v_+, v_- be the density of positive ions (number of particles per unit of volume), the density of electrons, the current density of ions (electric charge passing through a cross section of unit area per unit of time), the current density of electrons, the drift velocities of ions and electrons, respectively. The ion current density and the ion velocity are counted positive if the ions are moving toward smaller x, the electron current density and the electron velocity are counted positive if the electrons are moving toward larger x. We denote the intensity of the electrical field by E, counting it positive if directed from the anode to the cathode. Let $\alpha=\alpha(E)$ be

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[1] Figures in brackets indicate the literature references at the end of this paper.

the number of ionizations caused by each electron per unit length of its path, $q_+=en_+$ the charge density of positive ions, $q_-=en_-$ the charge density of electrons, counted positive all the time, e being the elementary charge. With ϵ_0 denoting the dielectric constant, the processes in the tube can be described by the following equations:

(a) Continuity (Townsend's equations)

$$\frac{\partial q_-}{\partial t}=\alpha j_- -\frac{\partial j_-}{\partial x} \tag{2.1}$$

$$\frac{\partial q_+}{\partial t}=\alpha j_- +\frac{\partial j_+}{\partial x}. \tag{2.2}$$

(b) The electrostatic Maxwell's equation (Poisson's equation)

$$\epsilon_0 \frac{\partial E}{\partial x}=q_- -q_+. \tag{2.3}$$

The outer circuit supplying the voltage for the tube can be described by a capacitance C parallel to the tube and a resistance R in series to both the tube and the capacitance. The outer voltage applied to this system as shown in figure 1 is called U. If V is the voltage across the gap of the tube and I the current to and from the tube, then the equation

$$U=R\left(I+C\frac{dV}{dt}\right)+V \tag{2.4}$$

describes the behavior of the outer circuit. The current I can be obtained from the mean current in the tube and the change in time of the voltage across the gap by the following considerations. Let us introduce the abbreviations $q=q_- -q_+$ and $j=j_- +j_+$. Furthermore let S denote the area of a cross section of the current in the tube, d the distance between cathode and anode, and Q the charge accumulated on a unit area of the cathode. Then the law of conservation of charge, applied to the cathode yields

$$I=S\left[j(0,t)-\frac{dQ}{dt}\right].$$

Now the charge Q is connected with the voltage across the gap by

$$Q=-\epsilon_0 E(0,t)$$

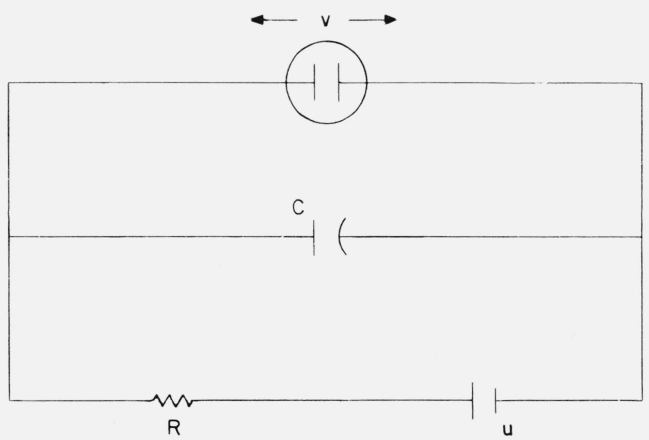

FIGURE 1. *Diagram of the electric circuit.*

and the equation

$$V=\int_0^d Edx=dE(0,t)+\epsilon_0^{-1}\int_0^d\int_0^x q(\xi,t)d\xi dx \tag{2.5}$$

following from eq (2.3) by integrating twice. If we differentiate eq (2.5) with respect to time and use $\frac{\partial q}{\partial t}=-\frac{\partial j}{\partial x}$, we obtain

$$\frac{dE(0,t)}{dt}=\frac{1}{d}\frac{dV}{dt}+\frac{1}{d\epsilon_0}\int_0^d j(x,t)dx-\frac{1}{\epsilon_0}j(0,t). \tag{2.6}$$

Now the current I can be expressed as

$$I=S\left(\frac{\epsilon_0}{d}\frac{dV}{dt}+\frac{1}{d}\int_0^d jdx\right).$$

Introducing this into eq (2.4) leads to the following ordinary differential equation for V:

$$R\left(C+\frac{S\epsilon_0}{d}\right)\frac{dV}{dt}=U-V-RSJ \tag{2.7}$$

where

$$J=\frac{1}{d}\int_0^d jdx$$

is the mean current density in the gap. We write: $J=qv$, where $v=v_+v_-/(v_-+v_+)$ is an average velocity. Instead of eliminating $\partial E(0,t)/\partial t$ one may eliminate dV/dt by using eq (2.6). Then, instead of eq (2.7), one obtains an ordinary differential equation in time for $E(0,t)$:

$$R(Cd+S\epsilon_0)\frac{dE(0,t)}{dt}=U-V+\frac{RCd}{\epsilon_0}J-R\left(\frac{Cd}{\epsilon_0}+S\right)j(0,t) \tag{2.8}$$

where V follows from eq (2.5).

The differential equation (2.7) or (2.8) furnishes the boundary condition corresponding to eq (2.3). For eqs (2.1) and (2.2) separate boundary conditions will have to be established. They describe the electron current at the cathode and the ion current at the anode respectively. The latter current is zero since there are no ions coming out of the anode:

$$j_+(d,t)=0. \tag{2.9}$$

The electron current at the cathode is given by

$$j_-(0,t)=j_p+\gamma_i j_+(0,t)+\gamma_p\int_0^d \sigma j_-dx \tag{2.10}$$

where j_p denotes the current density resulting from externally irradiating the cathode with photons. The second term describes electrons produced by ions hitting the cathode, γ_i being the probability that an incoming ion produces an outgoing electron. The last term comes from internally produced photons hitting the cathode. These "secondary" photons are assumed to be emitted from molecules which were excited by electron collisions. The number of excited molecules produced per unit of the path of a single electron is called $\sigma=\sigma(E)$, γ_p is the probability that a secondary photon produces an outgoing electron. It is assumed in eq (2.10) that the emission of photons occurs immediately after the excitation; however, the computer program contains a provision for an arbitrary time delay, simulating a delayed photoemission.

Equations (2.1), (2.2), (2.3), (2.5), (2.7), (2.9), and (2.10) together with initial distributions of q_+ and q_- and the initial voltage V determine the solution of the problem completely for times greater than the initial time up to infinity or to a certain time limit, provided the velocities v_+ and v_- and the ionization rates α and σ are given quantities.

Measurements show that v_+, v_-, and α can be approximated by the following types of functions of E and the pressure p in the tube:

$$v_+ = \begin{cases} \mu_+(1-B_1|E|/p)\dfrac{E}{p} & \text{if } |E|<W_1p \\ (B_3\sqrt{p/|E|}-B_2p^2/E^2)E/p & \text{if } |E|\geq W_1p \end{cases} \tag{2.11}$$

$$v_- = \mu_- E/p \tag{2.12}$$

$$\alpha = \begin{cases} C_1p \exp\left(-D_1p/|E|\right) & \text{if } |E|<W_2p \\ C_2p \exp\left(-D_2p/|E|\right) & \text{if } |E|\geq W_2p \end{cases} \Bigg\} \text{for molecular gases}$$
$$\Bigg\} \text{for atomic gases} \quad \begin{cases} C_3p \exp\left(-D_3\sqrt{p/|E|}\right) & \text{if } |E|<W_2p \\ C_4p \exp\left(-D_4\sqrt{p/|E|}\right) & \text{if } |E|\geq W_2p \end{cases} \tag{2.13}$$

where μ_+, μ_-, the B_i, C_i, D_i, and W_i are certain constants. The quantity σ is approximated by the same type of function as α with possibly different constants.

3. Steady State Solutions

The steady state has been investigated by several authors. We report here on calculations of space charge distributions in cold cathode discharge tubes, which have been conducted for a number of years at the National Bureau of Standards. A. L. Ward, who suggested this program, reported on the results in several publications [4], [10], [11], and we shall confine ourselves here to stating the equations and the method to solve them.

3.1. Differential Equations and Boundary Conditions

We assume a state of equilibrium, which means

$$\frac{\partial q_-}{\partial t}=0 \text{ and } \frac{\partial q_+}{\partial t}=0$$

at all times, and eq (2.1) reduces to

$$\frac{dj_-}{dx}=+\alpha(E)j_-(x) \tag{3.1}$$

which is Townsend's steady state equation. In order to take account of collisions between electrons and metastable molecules inside the tube, we add an extra term which is proportional to $j_-^2(x)$:

$$\frac{dj_-}{dx}=+\alpha(E)j_-(x)+\beta(E)j_-^2(x). \tag{3.2}$$

Here $\alpha(E)$ is defined by eq (2.13) and $\beta(E)$ is assumed to follow the same law, with possibly different parameters C_i, D_i, and W_2.

For eq (2.2), which governs the ions, we assume that the total current density is constant:

$$j_-(x)+j_+(x)=j=\text{const.} \tag{3.3}$$

Equation (2.3) is then written as follows:

$$\epsilon_0 \frac{dE}{dx} = \frac{p}{E} \cdot \begin{cases} \dfrac{j(x) - j_-(x)}{\mu_+(1 - B_1|E|/p)} & \text{if } |E| < W_1 p \\ \dfrac{j(x) - j_-(x)}{B_3\sqrt{\dfrac{p}{|E|}} - B_2\left(\dfrac{p}{E}\right)^2} & \text{if } |E| \geq W_1 p. \end{cases} \tag{3.4}$$

Equations (3.2) and (3.4) are two ordinary first-order differential equations for $E(x)$ and $j_-(x)$, provided that we are given the total current density function j.

Physically accessible are the currents at the electrodes, leading to the following boundary conditions:

$$j_-(d) = j \text{ at the anode.} \tag{3.5}$$

$$j_-(0) = (j_p + \gamma j_+(d))/(1+\gamma) \text{ at the cathode.} \tag{3.6}$$

Here j_p is a contribution to the electron current density caused by external radiation and γ is a secondary ionization coefficient assumed to be a constant.

3.2. Integration of the System of Differential Equations

We distinguish here two cases:

(i) The current is so small that $\left|\frac{dE}{dx}\right| \ll \frac{E}{d}$ everywhere, i.e., we assume a constant field. $\alpha(E)$ and $\beta(E)$ are then constant also and eq (3.2) becomes a Bernoulli equation with constant coefficients, which can be integrated explicitly. The value of E is then determined from the second boundary condition.

(ii) Arbitrary large currents $j(x)$. The solution has to be found iteratively:

Starting at $x = d$ with $j_-(d) = j$, a value of $E(d)$ has to be assumed. (It seems most feasible to generate a whole family of solutions with increasing total current densities. The initial value $E(d)$ is then taken to be the solution of the previous case. For sufficiently small $j(x)$ assumption (i) holds and no difficulty arises in finding a starting value.)

Equations (3.2) and (3.4) are integrated simultaneously by means of a Runge-Kutta scheme. Iteration on $E(d)$ is performed until $j_-(0)$ is in sufficient agreement with the prescribed boundary condition.

As the total current densities increase, it becomes more and more difficult to find the proper $E(d)$. At $j_-(d) \approx 10^{-4}$ amp/cm^2 it was impossible to find a solution, even by the interval halving method.

Fortunately this limit covers most of the experimental data as far as the basic equations are valid. For the results we refer to the aforementioned publication of A. L. Ward.

4. Difference Equations

4.1. Difference Equations With Respect to Both Time and Space. Stability Considerations

In the most general case, it is not possible to give an exact solution of the system of equations described in section 2. Therefore, one tries to find an approximate numerical solution. The most convenient way to obtain a "numerical solution" is to introduce finite differences in time and space and to solve numerically the finite equations generated in this way for a certain set of values of the different parameters of the problem.

The way to transform the differential equations into difference equations is largely determined by requiring simplicity of the computational scheme and "stability" of the difference scheme. For the sake of simplicity we ask for explicit schemes as far as possible. For the same reason we use difference operators of the same order as the corresponding differential

operators. This ensures that the special computations at the boundaries are kept to a minimum. By requiring stability we exclude certain difference schemes which would lead to large amplification of any small errors (as rounding errors) when the time increases, at least in the limit of vanishing meshwidth. We use here the concept of stability as introduced by Lax and Richtmyer [8, 12].

Denoting by Δt and Δx the increments in time and space respectively, the difference equations corresponding to eqs (2.1) and (2.2) are

$$\frac{q_-(x, t+\Delta t)-q_-(x, t)}{\Delta t}=\alpha(x, t)j_-(x, t)-\frac{j_-(x, t)-j_-(x-\Delta x, t)}{\Delta x} \tag{4.1}$$

for $x=\Delta x, 2\Delta x, \ldots, M\Delta x=d$; $t=t_0, t_0+\Delta t, t_0+2\Delta t, \ldots$

$$\frac{q_+(x, t+\Delta t)-q_+(x, t)}{\Delta t}=\alpha(x, t)\cdot j_-(x, t)+\frac{j_+(x+\Delta x, t)-j_+(x, t)}{\Delta x} \tag{4.2}$$

for $x=0, \Delta x, 2\Delta x, \ldots, (M-1)\Delta x$; $t=t_0, t_0+\Delta t, t_0+2\Delta t, \ldots$.

The difference quotients with respect to x have been chosen unsymmetrically for the sake of stability, see [8, 12]. In order to obtain stability in the sense of Richtmyer for $\Delta t\to 0$, $\Delta x\to 0$, with $\frac{\Delta t}{\Delta x}=$const., we must require that

$$\Delta t<\frac{\Delta x}{\max\{v_-, v_+\}}, \tag{4.3}$$

and that both v_- and v_+ are nonnegative. Since v_- is much larger than v_+, this means that the timestep must not exceed the time an electron needs to go from one meshpoint in space to the next. This time is very short for many of the interesting developments in a gas discharge tube. Therefore, for many phenomena, it will be sufficient to assume that the electron density and current distribution will be in a quasi-equilibrium state, i.e., we replace the left hand side of eq (2.1) by zero. Since α is rather large in the interesting cases, we do not use the difference scheme obtained from eq (4.1) by putting the left hand side equal to zero. We rather integrate eq (2.1) formally:

$$j_-(x, t)=j_-(0, t)\cdot\exp\left\{\int_0^x \alpha(x', t)dx'\right\}$$

and replace the integral in the argument of the exponential function by a finite sum, using the trapezoidal rule:

$$j_-(m\cdot\Delta x, t)=j_-(0, t)\cdot\exp\left\{\sum_{k=0}^{m-1}\frac{1}{2}[\alpha(k\cdot\Delta x, t)+\alpha((k+1)\Delta x, t)]\Delta x\right\} \tag{4.4}$$

for $m=1, 2, \ldots M$.

The stability condition for the system eq (4.4), (4.2) is now

$$\Delta t\leq\frac{\Delta x}{v_+}, \tag{4.5}$$

and $v_+\geq 0$. Thus one can use much larger timesteps than for the original system, namely, the timestep must not exceed the time an ion needs to go from one meshpoint to the next.

Equation (2.3) does not contain time derivatives. It shows that E, and therefore also α, are obtained by integrating q. Hence, stability is not affected, and the question how to choose a proper difference scheme replacing this equation can be separated from the question how to choose the timestep.

Of course, the characteristic time of the outer circuit will provide another bound for the timestep. As long as the term containing J is unessential in eq (2.7), this characteristic time is apparently $[R(C+S\epsilon_0/d)]$. If one uses instead of eq (2.7), the difference equation

$$R\left(C+\frac{S\epsilon_0}{d}\right)\cdot\frac{V(t+\Delta t)-V(t)}{\Delta t}=U(t)-V(t)-RSJ(t), \tag{4.6}$$

where

$$J(t)=\frac{\Delta x}{d}\sum_{k=0}^{m-1}\frac{1}{2}\left[j(k\Delta x, t)+j((k+1)\Delta x, t)\right],$$

the stability condition for this scheme in the sense of Rutishauser [13] indeed is

$$\Delta t\leq 2[R(C+S\epsilon_0/d)],$$

as long as the dependence of J on V is small enough to be disregarded. As soon as this dependence becomes important, we can no longer consider the outer circuit separately from what happens inside. Then we will have to treat the system as a whole. Even in the case where E and α are independent of x, i.e., when space charge effects can be neglected, this leads to a nonlinear problem because of the product αj_- (α depends on V).

4.2. Difference Equation for the Electrical Field E

The differential equation (2.3) is to be solved with the side condition (2.5) imposed on the integral over the unknown function E. The most reasonable way seems to be to replace eq (2.3) by

$$\frac{E(x+\Delta xt)-E(x, t)}{\Delta x}=\epsilon_0^{-1}\tfrac{1}{2}\left[q(x+\Delta x, t)+q(x, t)\right] \tag{4.7}$$

for $x=0, \Delta x, 2\Delta x, \ldots, (M-1)\Delta x$.

Using a trapezoidal rule on the left side of eq (2.5) would lead to a side condition

$$V(t)=\sum_{m=0}^{M-1}\left\{\frac{\Delta x}{2}\left(E((m+1)\Delta x, t)+E(m\Delta x, t)\right)\right\}. \tag{4.8}$$

Equations (4.7) and (4.8) form $M+1$ equations for the $M+1$ unknowns $E(m\Delta x, t)$, $m=0, 1, \ldots, M$.

But it turns out that these equations do not give the rigorous solutions even if $q(x)$ is a linear function of x. For, take $\epsilon_0^{-1}q(x)=x$ and $V=0$, then, according to eqs (2.3) and (2.5), $E=\frac{x^2}{2}-\frac{d^2}{6}$. But, eq (4.7) yields $E=\frac{x^2}{2}+c$, where c is a constant to be determined from eq (4.8):

$$0=\frac{d^3}{6}+\frac{d(\Delta x)^2}{12}+cd.$$

Hence, we obtain $E=\frac{x^2}{2}-\frac{d^2}{6}-\frac{(\Delta x)^2}{12}$, which is not in agreement with the rigorous solution.

By partial integration on the right hand side of eq (2.5) one gets

$$V=d\cdot E(0, t)+\epsilon_0^{-1}\int_0^d (d-x)q(x, t)dx.$$

If integrated by the trapezoidal rule this formula does not give the rigorous solution for linear q either. For the above example we obtain $E=\frac{x^2}{2}-\frac{d^2}{6}+\frac{(\Delta x)^2}{6}$. Therefore, we have to search for a different method of replacing eqs (2.3) and (2.5) by finite equations such that the results are correct at least for piecewise-linear functions q. For instance, one may expect that a weighted mean of the two formulas discussed might eliminate the term containing $(\Delta x)^2$. Indeed, this way is successful if the weight ratio is 2:1. The results are then correct even for piecewise-linear functions q, as is shown below.

Let us put $\epsilon_0^{-1}q=r$ for simplicity. A piecewise-linear function $r(x)$ is given by its values at the meshpoints $r_m=r(m\Delta x)$ for $m=0, 1, \ldots, M$:

$$r(x)=r_{m-1}\cdot\frac{m\Delta x-x}{\Delta x}+r_m\frac{x-(m-1)\Delta x}{\Delta x}$$

for $(m-1)\Delta x\leq x\leq m\Delta x \quad (m=1,2,\ldots,M)$.

Our task is to find a proper approximation for the integral

$$I=\int_0^d\int_0^x r(x')dx'dx=\int_0^d(d-x)r(x)dx. \tag{4.9}$$

For the sake of simplicity we take $d=1$. The value of I follows by summation

$$\int_{\frac{(m-1)}{M}}^{m/M}(1-x)[r_{m-1}(m-Mx)+r_m(Mx-(m-1))]dx$$

$$=r_{m-1}\left[\left(1-\frac{m-1}{M}\right)\frac{1}{2M}-\frac{1}{6M^2}\right]+r_m\left[\left(1-\frac{m}{M}\right)\frac{1}{2M}+\frac{1}{6M^2}\right],$$

namely

$$I=\sum_{m=0}^{M}\frac{r_m}{M}\left(1-\frac{m}{M}\right)-\frac{r_0}{2M}+\frac{r_M-r_0}{6M^2}.$$

The trapezoidal rule applied to the rightmost expression in eq (4.9) gives the approximation

$$I_{\mathrm{tr}}=\sum_{m=0}^{M}\frac{r_m}{M}\left(1-\frac{m}{M}\right)-\frac{r_0}{2M}.$$

Repeated application of the trapezoidal rule to the middle expression in eq (4.9) gives the different approximation

$$I_{\mathrm{rep}}=\sum_{m=0}^{M-1}\frac{1}{2M}\left[\sum_{k=0}^{m-1}\frac{1}{2M}(r_k+r_{k+1})+\sum_{k=0}^{m}\frac{1}{2M}(r_k+r_{k+1})\right]=\sum_{m=0}^{M}\frac{r_m}{M}\left(1-\frac{m}{M}\right)-\frac{r_0}{2M}+\frac{r_M-r_0}{4M^2}.$$

Hence

$$1/3I_{\mathrm{tr}}+2/3I_{\mathrm{rep}}=\left[\sum_{m=0}^{M}r_m\left(1-\frac{m}{M}\right)-\frac{r_0}{2}+\frac{r_M-r_0}{6M}\right]\frac{1}{M} \tag{4.10}$$

is an approximation to eq (4.9) and leads to rigorous results in the case of piecewise linear functions $r(x)$ with slopes changing at $x=0, \Delta x, 2\Delta x, \ldots, M\Delta x=d=1$.

Since the solution of eq (4.7) requires the same summation as the inner sum in I_{rep}, we actually compute the finite approximation by repeated application of the trapezoidal rule and by adding a certain correction:

$$\frac{1}{3}I_{\mathrm{tr}}+\frac{2}{3}I_{\mathrm{rep}}=I_{\mathrm{rep}}-\frac{r_M-r_0}{12M^2}.$$

For general d, we have to replace r by rd^2.

The computation of E from V and the given values of q at the meshpoints can therefore be done by the following formulas:

At first we compute the auxiliary quantities

$$\left.\begin{aligned}&E^*(m\Delta x,t)=\frac{\Delta x}{2\epsilon_0}\sum_{k=1}^{m}(q(k\Delta x,t)+q(k-1)\Delta x,t) \qquad \text{for } m=1,2,\ldots,M.\\&E^*(0,t)=0.\end{aligned}\right\} \tag{4.11}$$

Then we continue with

$$E(0,t)=\frac{1}{d}\left[V(t)-\frac{\Delta x}{2}\sum_{m=1}^{M}(E^*(m\Delta x,t)+E^*((m-1)\Delta x,t))+(\Delta x)^2\frac{q(M\Delta x,t)-q(0,t)}{12\epsilon_0}\right] \quad (4.12)$$

$$E(m\Delta x,t)=E^*(m\Delta x,t)+E(0,t) \qquad \text{for } m=1,2,\ldots,M. \quad (4.13)$$

One may ask whether using $E(0,t)$ instead of $V(t)$ as the parameter describing the outer circuit would not simplify the computation formulas. Indeed this is true. One may compute $E(0,t)$ from a difference approximation of eq (2.8), then compute $E(m\Delta x,t)$ for $m=1,2,\ldots,M$ from eq (4.7) successively, and finally one computes V from those values by the trapezoidal rule. But it turns out that the truncation error in V is much larger in this case than if we use eqs (4.6), (4.11), (4.12), and (4.13). After a certain number of integration steps, V may even exceed U which is physically impossible, when $U=$const. and C is uncharged at time $t=0$.

4.3. Method of Computation

If we use the trapezoidal rule in eq (2.10), this boundary condition takes the form

$$j_-(0,t)=j_p+\gamma_i\cdot j_+(0,t)+\gamma_p\sum_{m=1}^{M}\left[\frac{\Delta x}{2}\left[\sigma(m\Delta x,t)j_-(m\Delta x,t)+\sigma((m-1)\Delta x,t)j_-((m-1)\Delta x,t)\right]\right]. \quad (4.14)$$

The eqs (4.1), (4.2), (4.6), (4.11), (4.12), (4.13), (2.9), and (4.14) together with initial distributions of q_+ and q_- and the initial voltage V determine the finite problem completely, provided the velocities v_+ and v_- and the ionization rates α and σ are given functions of E. Instead of the eq (4.1) one may use eq (4.4) if the electron density is in a quasi-equilibrium state.

The above system of finite equations is not completely explicit. In the case where eq (4.1) is used, explicit formulas are achieved by taking some quantities of minor importance at an earlier time. Thus, in eq (4.14) the last sum has been taken at the time $t-\Delta t$ instead of t, and in eqs (4.11) and (4.12), $q_-(0,t)$ has been replaced by $q_-(0,t-\Delta t)$. These two changes were sufficient to achieve explicit formulas for all quantities.

In the electron quasi-equilibrium case the equations are even more implicit. Therefore one uses an iterative scheme in order to obtain a solution starting from values of q_- at the previous timestep as initial approximations. The only quantity not treated iteratively is the last sum in eq (4.14). It is taken from the previous timestep throughout, i.e., instead of eq (4.14) one always uses:

$$j_-(0,t+\Delta t)=j_p+\gamma_i j_+(0,t+\Delta t)+\gamma_p\sum_{m=1}^{M}\frac{\Delta x}{2}\left[\sigma(m\Delta x,t)j_-(m\Delta x,t)+\sigma((m-1)\Delta x,t)j_-((m-1)\Delta x,t)\right]. \quad (4.15)$$

4.4. Convergence of the Iteration

It will be shown below that the proposed iteration scheme leads to convergent sequences for all quantities involved, if the electron density is small enough in the sense that it causes no distortions of the electric field comparable with the field itself.

For the sake of simplicity we consider the limit case $M\to\infty$ only. The quantities changing during the iteration are $q_-(x)$, $E(x)$, $v_-(x)$, $\alpha(x)$ and $j_-(x)$ for $0\leq x\leq d$, and furthermore, $v_+(0)$ and $j_+(0)$. All other quantities are fixed throughout the iteration. Since $q_-(x)$ is the only result of a previous iteration which enters the following iteration step, and since all other quantities depend on $q_-(x)$ continuously, it is sufficient to prove that the sequence of functions $q_-(x)$ converges.

According to [7], one has to consider the change $\delta q_-^{\text{new}}(x)$ of the result of one iteration step, caused by a certain change $\delta q_-^{\text{old}}(x)$ of the initial approximation. When measured by a

certain norm, the ratio of the changes has to be smaller than unity in a certain neighborhood of the true solution $q_-(x)$.

The following formulas describe the connection between $\delta q_-^{\text{new}}(x)$ and $\delta q_-^{\text{old}}(x)$. They are derived from the continuous analogs of the eqs (4.4), (4.11), (4.12), (4.13), from (4.15) and from the relations between v_-, v_+, α and E, and those between $q_\pm$, $v_\pm$ and $j_\pm$.

$$\delta q_-^{\text{new}}(x)=q_-^{\text{new}}(x)\left[\frac{\delta j_-(0)}{j_-(0)}-\frac{\delta v_-(x)}{v_-(x)}+\int_0^x \delta\alpha(u)du\right]$$

$$\frac{\delta j_-(0)}{j_-(0)}=\left(\frac{1}{v_+}\frac{dv_+}{dE}\right)\Bigg|_{x=0}\cdot\left[1-\frac{j_p+\Gamma}{j_-(0)}\right]\delta E(0)$$

and Γ is the last term on the right-hand side of eq (2.10).

$$\frac{\delta v_-(x)}{v_-(x)}=\frac{1}{v_-}\frac{dv_-}{dE}\,\delta E(x)$$

$$\delta\alpha(x)=\frac{d\alpha}{dE}\,dE(x)$$

$$\delta E(x)=\epsilon_0^{-1}\int_0^d K(x,u)\,\delta q_-^{\text{old}}(u)du$$

where

$$K(x,u)=\begin{cases} u/d & \text{if } u<x \\ u/d-1 & \text{if } u>x. \end{cases}$$

We restrict ourselves to the case for which not only v_- but also v_+ is proportional to E. Then

$$\frac{1}{v_+}\frac{dv_+}{dE}=\frac{1}{v_-}\frac{dv_-}{dE}=\frac{1}{E}.$$

Furthermore, we write

$$\frac{d\alpha}{dE}=\frac{\alpha}{E}\frac{d(\ln\alpha)}{d(\ln E)}$$

since the latter differential quotient varies more slowly in the interesting range of E. We introduce the norm

$$||f||=\int_0^d |f(x)|dx$$

for any function $f(x)$. Then, the following estimates can easily be derived:

$$|\delta E(x)|\leqq\epsilon_0^{-1}\max\left\{\frac{x}{d},1-\frac{x}{d}\right\}\cdot||\delta q_-^{\text{old}}||$$

$$||\delta q_-^{\text{new}}||\leqq K\frac{\epsilon_0^{-1}\int_0^d|q_-^{\text{new}}(x)|dx}{\min\limits_x|E(x)|}\cdot||\delta q_-^{\text{old}}||$$

where

$$K=1+\frac{\min\limits_x|E(x)|}{E(0)}+\frac{3}{4}\alpha d\max_x\left|\frac{d(\ln\alpha)}{d(\ln E)}\right|.$$

The condition for convergence is that

$$K\frac{\epsilon_0^{-1}\int_0^d|q_-^{\text{new}}(x)|dx}{\min\limits_x|E(x)|}=Q<1 \tag{4.16}$$

which can be interpreted the following way: The field distortions produced by the electrons and amplified by the numerical factor K must not exceed the minimum field strength in the gap.

From the rate of convergence given by eq (4.16) and the change of $q_-(x)$ between successive iterations, the corrections introduced by a further iterative step can be estimated. According to these considerations, the iteration may be stopped after a prescribed accuracy has been reached. A criterion of this kind has been used in the code.

5. Formal Solutions for Constant Field

5.1. General Considerations and Formulas

The nonlinearities contained in the equations of section 2 and in the corresponding difference equations disappear as soon as the quantities α, σ, v_+ v_- can be regarded as independent of the solution. If, moreover, these quantities are constant in space and time and if the voltage V across the gap is constant, the equations of section 2 and the corresponding difference equations can be solved explicitly. Then, of course, there is no room for eq (2.3), i.e., this treatment disregards space charge, and eq (2.7) cannot be taken into account, i.e., the reactions in the outer circuit are disregarded, or, in other words, the external resistance R is so small that it can be neglected by putting $R=0$, which leads to $U=V$. Work in this direction has been done by several authors [1, 2, 3, 9] as far as the differential equations are concerned. These papers discuss what happens in the tube. Here this special case will be considered again, but for a different purpose. We shall discuss the difference equations along with the differential equations in order to get insight into the effects of truncation errors. We hope that, to a certain extent, these effects carry over to the more general case, and therefore will allow us to correct the results obtained with finite steplengths Δt and Δx so that we obtain closer approximations to the case of infinitesimal steplengths.

We mainly deal with the difference equations. The results for the differential equations will be obtained by letting Δt, $\Delta x \to 0$. We restate the equations of section 4 for our special case putting $t_0=0$:

$$\frac{j_-(x, t+\Delta t)-j_-(x, t)}{v_-\Delta t}=j_-(x, t)-\frac{j_-(x, t)-j_-(x-\Delta x, t)}{\Delta x} \tag{5.1a}$$

for $x=\Delta x, 2\Delta x, \ldots M\Delta x=d$; $t=0, \Delta t, 2\Delta t, \ldots$

In the electron quasi-equilibrium case, we shall use instead of this:

$$j_-(x, t)=j_-(0, t) \cdot e^{\alpha x} \tag{5.1b}$$

for the same x and t as above. The ion currents behave according to

$$\frac{j_+(x, t+\Delta t)-j_+(x, t)}{v_+\Delta t}=j_-(x, t)+\frac{j_+(x+\Delta x, t)-j_+(x, t)}{\Delta x} \tag{5.2}$$

for $x=0, \Delta x, \ldots (M-1)\Delta x$; $t=0, \Delta t, 2\Delta t, \ldots$

There are two boundary conditions, namely

$$j_+(d, t)=0 \qquad \text{for } t=0, \Delta t, 2\Delta t, \ldots \tag{5.3}$$

and

$$j_-(0, t)=j_p+\gamma_i j_+(0, t)+\gamma_p \sum_{m=1}^{M} \frac{\Delta x}{2} \sigma \cdot [j_-(m\Delta x, t-\Delta t)+j_-((m-1)\Delta x, t-\Delta t)]$$

$$\text{for } t=\Delta t, 2\Delta t, \ldots \tag{5.4}$$

In order to make the solution unique, we have to introduce initial conditions, for instance by prescribing $j_-(x,0)$ and $j_+(x, 0)$ for $x=0, \Delta x, \ldots M\Delta x$. But we will be mainly concerned with

solutions which are proportional to $e^{\lambda t}$ with a suitable λ. For this type of solution, initial conditions do not have to be given.

The above equations are linear and homogeneous with the single exception of (5.4) which contains the inhomogeneous term j_p. Since j_p is time independent and since time does not appear explicitly in any other term, there will exist, at least in general, a time independent solution. More general solutions will be obtained by superposition of any solutions of the corresponding homogeneous system with $j_p=0$. The homogeneous system allows for time separation in the form of a factor $e^{\lambda t}$. Then the stationary case results from putting $\lambda=0$ and from slight changes due to the inhomogeneous term. We introduce $j_\lambda^+(x)$ and $j_\lambda^-(x)$ by

$$\begin{aligned} j_-(x,t)&=j_\lambda^-(x)\cdot e^{\lambda t}\\ j_+(x,t)&=j_\lambda^+(x)\cdot e^{\lambda t} \end{aligned} \tag{5.5}$$

Then, eq (5.1a) can be solved by

$$j_\lambda^-(x)=j_\lambda^-(0)\cdot e^{\alpha^*\Delta x}, \tag{5.6}$$

where α^* is the solution of the transcendental equation

$$\frac{e^{\lambda\Delta t}-1}{v_-\cdot\Delta t}=\alpha-\frac{1-e^{-\alpha^*\Delta x}}{\Delta x} \tag{5.7}$$

In the electron quasi-equilibrium case, i.e., if we use eq (5.1b), all results will be correct in the sequel if we replace α^* by α, unless we distinguish explicitly between the two cases. Equation (5.2) now reduces to a single inhomogeneous equation for $j_\lambda^+(x)$, the inhomogeneous term being $\alpha j_\lambda^-(x)$. The solution satisfying the boundary condition (5.3) is easily found to be

$$j_\lambda^+(x)=Ae^{\beta x}[e^{(\alpha^*-\beta)d}-e^{(\alpha^*-\beta)x}], \tag{5.8}$$

where

$$A=\frac{j_\lambda^-(0)}{\dfrac{e^{\alpha^*\Delta x}-1}{\alpha\Delta x}-\dfrac{e^{\lambda\Delta t}-1}{v_+\alpha\Delta t}}$$

and β is determined by the equation

$$\frac{e^{\lambda\Delta t}-1}{v_+\Delta t}=\frac{e^{\beta\Delta x}-1}{\Delta x}. \tag{5.9}$$

The remaining unknowns are $j_\lambda^-(0)$ and λ. In the homogeneous case the quantity $j_\lambda^-(0)$ remains free, whereas λ follows from eq (5.4) with $j_p=0$, which transforms into the transcendental equation

$$1=\frac{\gamma_i[e^{(\alpha^*-\beta)d}-1]}{\dfrac{e^{\alpha^*\Delta x}-1}{\alpha\Delta x}-\dfrac{e^{\lambda\Delta t}-1}{v_+\alpha\Delta t}}+\gamma_p\sigma\Delta x e^{-\lambda\Delta t}\cdot\frac{1+e^{\alpha^*\Delta x}}{2}\cdot\frac{e^{\alpha^*d}-1}{e^{\alpha^*\Delta x}-1}.$$

5.2. Stationary Solution

For the stationary solution of the inhomogeneous eq (5.4), we put $\lambda=0$. Then (5.9) leads to $\beta=0$, and (5.4) shows that

$$j_0^-(0)=j_p\Big/\left\{1-\frac{\alpha\Delta x(e^{\alpha^*d}-1)}{e^{\alpha^*\Delta x}-1}\cdot\left(\gamma_i+\gamma_p\frac{\sigma}{\alpha}\frac{1+e^{\alpha^*\Delta x}}{2}\right)\right\}. \tag{5.11}$$

If we use (5.1a), we may simplify our equations further. From eq (5.7) we obtain α^* explicitly:

$$\alpha^*=\frac{1}{\Delta x}\ln\frac{1}{1-\alpha\Delta x}. \tag{5.12}$$

Equation (5.8) reduces to

$$j_0^+(x)=j_0^-(0)\cdot(1-\alpha\Delta x)(e^{\alpha^* d}-e^{\alpha^* x}), \tag{5.13}$$

and eq (5.11) reduces to

$$j_0^-(0)=j_p\Big/\left\{1-[e^{\alpha^* d}-1]\left[\gamma_i(1-\alpha\Delta x)+\gamma_p\frac{\sigma}{\alpha}\left(1-\frac{1}{2}\alpha\Delta x\right)\right]\right\}. \tag{5.14}$$

In the electron quasi-equilibrium case, we use eq (5.1b) and obtain

$$j_0^+(x)=j_0^-(0)\cdot\frac{\alpha\Delta x}{e^{\alpha\Delta x}-1}\,(e^{\alpha d}-e^{\alpha x}) \tag{5.15}$$

and

$$j_0^-(0)=j_p\Big/\left\{1-\frac{\alpha\Delta x}{e^{\alpha\Delta x}-1}\,(e^{\alpha d}-1)\left(\gamma_i+\gamma_p\frac{\sigma}{\alpha}\frac{1+e^{\alpha\Delta x}}{2}\right)\right\}. \tag{5.16}$$

By letting $\Delta x\to 0$ we obtain the corresponding relations for the differential equations; namely $\alpha^*=\alpha$ and, furthermore,

$$j_0^+(x)=j_0^-(0)\cdot(e^{\alpha d}-e^{\alpha x}), \tag{5.17}$$

$$j_0^-(0)=j_p\Big/\left\{1-(e^{\alpha d}-1)\left(\gamma_i+\gamma_p\frac{\sigma}{\alpha}\right)\right\}. \tag{5.18}$$

When the respective denominators in eqs (5.14), (5.16), (5.18), become zero, no stationary solution is possible. The voltage V for which this occurs is called the breakdown voltage. If we assume that σ/α is a given constant independent of V (it is often assumed that $\sigma=\alpha$) then the three respective denominators are equal to 1 for $\alpha=0$ and decrease monotonically in the interval $0\leq\alpha<\infty$. Therefore, there exists one and only one positive real root α, corresponding to the breakdown voltage. If the voltage V exceeds this value, the theoretical stationary solution shows negative values of j^-, which is physically impossible: No stable solutions exist beyond the breakdown voltage.

The physically interesting things happen if V is near the breakdown voltage. Therefore we have to look into the dependence of the breakdown voltage on the steplength Δx in order to establish a base for comparison of numerically obtained results and the true theoretical solutions, or the experimental measurements in this case.

In the infinitesimal case the value $\alpha=\alpha_{\mathrm{br}}$ corresponding to breakdown can be expressed explicitly from (5.18), if σ/α is given:

$$\alpha_{\mathrm{br}}=\frac{1}{d}\ln\left(\frac{1}{\gamma_i+\gamma_p(\sigma/\alpha)_{\mathrm{br}}}+1\right).$$

For the difference schemes, the corresponding transformation will lead to a form of the equation suitable for iteration. From eq (5.16), i.e., for the electron quasi-equilibrium case, we get

$$\alpha_{\mathrm{br}}=\frac{1}{d}\ln\left[1+\frac{\exp\,(\alpha_{\mathrm{br}}\Delta x)-1}{\alpha_{\mathrm{br}}\Delta x}\Big/\left(\gamma_i+\gamma_p\left(\frac{\sigma}{\alpha}\right)_{\mathrm{br}}\frac{1+\exp\,(\alpha_{\mathrm{br}}\,\Delta x)}{2}\right)\right].$$

In case we use eq (5.1a) instead, it follows from eqs (5.14) or (5.11) that

$$\alpha^*_{\mathrm{br}}=\frac{1}{d}\ln\left[1+\exp\,(\alpha^*_{\mathrm{br}}\,\Delta x)\Big/\left(\gamma_i+\gamma_p\left(\frac{\sigma}{\alpha}\right)_{\mathrm{br}}\frac{1+\exp\,(\alpha^*_{\mathrm{br}}\Delta x)}{2}\right)\right].$$

We shall see that these formulas are special cases of more general formulas for the homogeneous system.

5.3. Special Time-Dependent Solutions

We turn now to a discussion of the homogeneous system, especially the transcendental eq (5.10). Let us first consider the infinitesimal analog of this equation, obtained by letting $\Delta x \to 0$ and $\Delta t \to 0$. Equation (5.9) becomes then

$$\beta = \lambda / v_+, \tag{5.19}$$

and eq (5.7) transforms into

$$\alpha^* = \alpha - \lambda / v_-. \tag{5.20}$$

If we define v by $1/v = 1/v_+ + 1/v_-$, then eq (5.10) becomes

$$1 = \frac{\gamma_i \alpha}{\alpha - \lambda/v} \left[e^{(\alpha - \lambda/v)d} - 1 \right] + \frac{\gamma_p \sigma}{\alpha - \lambda/v_-} \left[e^{(\alpha - \lambda/v_-)d} - 1 \right], \tag{5.21}$$

in accordance with [9]. For the breakdown voltage, $\lambda = 0$ is a solution.

For voltages near the breakdown voltage, there must be a solution λ near zero for continuity reasons. For this special root, the equation may be transformed into an iterative scheme similar to the one for the computation of breakdown voltage itself. We show this for the case where a solution is wanted for λ as a function of α.
We put

$$\alpha - \lambda / v = \alpha (1 - u)$$

and rewrite eq (5.21):

$$1 = \left[e^{\alpha (1-u) d} - 1 \right] \cdot \left[\frac{\gamma_i}{1-u} + \frac{\gamma_p (\sigma/\alpha)}{1 - \lambda/(\alpha v_-)} \cdot \frac{e^{\lambda d / v_+} - e^{-\alpha (1-u) d}}{1 - e^{-\alpha (1-u) d}} \right].$$

According to the assumptions made above, only the first factor on the right changes rapidly with α. Therefore we solve the equation for this factor, obtaining

$$\alpha = \frac{1}{(1-u)d} \ln \left\{ 1 + 1 \Big/ \left(\frac{\gamma_i}{1-u} + \frac{\gamma_p (\sigma/\alpha)}{1 - u \cdot v/v_-} \cdot \frac{e^{\lambda d / v_+} - e^{-\alpha (1-u) d}}{1 - e^{-\alpha (1-u) d}} \right) \right\}. \tag{5.22}$$

This form is suitable for iteration since the right hand side is only slowly varying with α, particularly, if we use u as an independent parameter. We describe the function $\lambda(\alpha)$ by the parametric representation $\lambda = \lambda(u)$, $\alpha = \alpha(u)$. From a first approximation of V and α we compute v_+, v_-, v, σ, and

$$\lambda = \alpha v u. \tag{5.23}$$

Since all variables on the right hand side of eq (5.22) are known, a new approximation for α can be computed from eq (5.22). Since the right hand side is slowly varying with α for sufficiently small u, we can hope for fast convergence of the iteration.

A similar but slightly more complicated approach is possible for solving eq (5.10), if eq (5.1a) is used for the computation of electron current. Here, in addition, the relation between α^* and α has to be established iteratively. Let us assume that we know a first approximation for α^* for a given value of the independent parameter

$$u = \frac{e^{\lambda \Delta t} - 1}{\alpha v \Delta t}.$$

Then α can be computed from (5.7) if we assume in addition that v/v_- is at least approximately known:

$$\alpha=\frac{1-e^{-\alpha^*\Delta x}}{\Delta x}\Big/\left(1-\frac{uv}{v_-}\right). \tag{5.24}$$

If necessary, from the α thus obtained one may compute a new value of v/v_- and repeat the computation of α iteratively.

From u and α one may compute

$$\lambda=\frac{1}{\Delta t}\ln\left(1+\alpha v u\Delta t\right) \tag{5.25}$$

and

$$\beta=\frac{1}{\Delta x}\ln\left(1+\alpha\frac{v}{v_+}u\Delta x\right). \tag{5.26}$$

Equation (5.10) may be written as

$$1=[e^{(\alpha^*-\beta)d}-1]\cdot\left[\frac{\gamma_i}{e^{\alpha^*\Delta x}\left(1-u\frac{v}{v_-}\right)-u\frac{v}{v_+}}+\frac{\gamma_p(\sigma/\alpha)e^{-\lambda\Delta t}}{e^{\alpha^*\Delta x}\left(1-u\frac{v}{v_-}\right)}\frac{1+e^{\alpha^*\Delta x}}{2}\cdot\frac{e^{\beta d}-e^{-(\alpha^*-\beta)d}}{1-e^{-(\alpha^*-\beta)d}}\right]$$

and is solved for the first factor:

$$\alpha^*=\frac{1}{(1-\beta/\alpha^*)d}\ln\left[1+\left(\frac{\gamma_i}{e^{\alpha^*\Delta x}\left(1-u\frac{v}{v_-}\right)-u\frac{v}{v_+}}+\frac{\gamma_p(\sigma/\alpha)e^{-\lambda\Delta t}}{e^{\alpha^*\Delta x}\left(1-u\frac{v}{v_-}\right)}\cdot\frac{1+e^{\alpha^*\Delta x}}{2}\cdot\frac{e^{\beta d}-e^{-(\alpha^*-\beta)d}}{1-e^{-(\alpha^*-\beta)d}}\right)^{-1}\right] \tag{5.27}$$

The computation of the breakdown voltage thus appears as a special case of these formulas, namely as the case $u=\lambda=0$.

In the equilibrium case the formula corresponding to eq (5.27) is

$$\alpha=\frac{1}{(1-\beta/\alpha)d}\ln\left[1+\left(\frac{\gamma_i}{(e^{\alpha\Delta x}-1)/(\alpha\Delta x)-u\frac{v}{v_+}}+\frac{\gamma_p(\sigma/\alpha)e^{\lambda\Delta t}}{(e^{\alpha\Delta x}-1)/(\alpha\Delta x)}\cdot\frac{1+e^{\alpha\Delta x}}{2}\frac{e^{\beta d}-e^{-(\alpha-\beta)d}}{1-e^{-(\alpha-\beta)d}}\right)^{-1}\right]. \tag{5.28}$$

This equation is used together with eqs (5.25) and (5.26) for solving eq (5.10) iteratively.

Because of the similarities between the eqs (5.22), (5.27), and (5.28) one can treat all three of these equations quite simply in a single computer program.

5.4 General Time-Dependent Solution for Constant Field

So far we have considered only a special root of eq (5.10). However, one can see that a transcendental equation of the type of eq (5.10) and its limit case eq (5.21) has more roots λ in general. For the case of finite Δt and Δx, the equation is rational in $e^{\lambda\Delta t}$. Therefore, there is a finite number of roots $e^{\lambda\Delta t}$, each of which corresponds to a infinite set of roots λ of the form $\lambda=\lambda_0+2\pi ik/\Delta t$ ($k=0, \pm1, \pm2, \ldots$). But all these roots describe the same function on the grid. Equation (5.21), however, has an infinity of roots λ, whose asymptotic distribution is shown in [6]. It turns out that at most a finite number of them can have a real part larger than the real root discussed before. The root with the largest real part will become dominant in any solution as time goes on. Therefore, one is mainly interested in the root with the largest real part. If we knew that this root was the real root discussed in section 5.3, we could confine our considerations to this root essentially. Unfortunately, a proof is not available at the time being. Therefore, it remains an open question whether or not the asymptotic behavior of the solution for larger t can be described by the formulas given in section 5.3.

For a complete formal solution of an arbitrary initial value problem, of course, one has to consider all of the solutions of eqs (5.10) and (5.21) and one has to develop the initial distribution into a series of the corresponding functions after substracting the stationary solution of the nonhomogeneous equation. No attempt has been made to go further in this direction, but see [1] for some results of this kind.

6. Results

The physical significance of the results obtained by the machine computations is discussed in [5, 11]. Therefore, we restrict our discussion to the more mathematical questions. The main question is how large the truncation error is, i.e., the error introduced by using finite differences instead of derivatives. No attempts to establish rigorous error bounds have been made. Instead of this, experiments with different step lengths have been carried out for the following set of parameters:

$j_p=10^{-12}$ amp/cm^2, $p=722$ torr, $d=1$ cm, $S=1$ cm^2,
$C=10^{-11}$ amp sec/volt, $R=10^6$ volt/amp, $U=V=25.6$ kv for $t\leq 0$.

U for $t>0$, γ_i, γ_p and Δx are different for different curves. The time step Δt is given by the formula $\Delta t=0.8\cdot\Delta x/\max\limits_x \{v_+\}$ in the electron quasi-equilibrium case, $\Delta t=0.8\cdot\Delta x/\max\limits_x \{v_-\}$ in the general case, unless something else is stated explicitly. The constants used to describe the functions α and σ are $C_1=C_2=8$ cm^{-1} torr^{-1}, $D_1=D_2=247$ volt cm^{-1} torr^{-1}, the mobilities of electrons and ions are constant: $\mu_-=4\times 10^5$ torr cm^2/(sec volt), $\mu_+=2\times 10^3$ torr cm^2/(sec volt), $B_1=0$, $W_1=+\infty$.

The calculations in section 5 show that in the most interesting area near the breakdown voltage the solution is very sensitive to changes in the voltage. On the other hand, a finite stepwidth of reasonable size, e.g., with 20 subintervals, introduces a change of the breakdown voltage of notable magnitude. Therefore, the differences between runs with different stepwidths are mainly due to the influence of the stepwidth on the breakdown voltage. It seems to be feasible to eliminate this influence by relating the applied voltages to the breakdown voltage, as computed according to section 5.3, i.e., neglecting space charge, for the stepwidth used in each case. This method turned out to give very satisfactory results for the electron quasi-equilibrium case, as can be seen from figures 2 and 3.

The reason why even the relatively large steplength $\Delta x=0.05$ cm ($\alpha\Delta x\approx 0.5$ to 0.6) gives a good approximation, can be seen from the figures 4, 5, and 6, where the coefficient of temporal growth λ, as computed from the formulas of section 5.3, i.e., without space charge effects, is plotted versus the voltage V across the gap and the overvoltage $V-V_{\text{br}}$. Figure 4 shows that the curves for the difference equation (quasi-equilibrium case, $\Delta x=0.05$ cm) and for the differential equation go almost parallel over a long range. Therefore after relating the voltages to the breakdown voltage, the curves almost coincide as can be seen from figure 5. This explains the good results obtained with that approximation, at least as long as space charge effects have small influence.

With the same value of $\gamma_i+\gamma_p$, but a portion of 10 and 20 percent γ_p, the temporal development goes faster than for $\gamma_p=0$, as one could expect from physical considerations.

According to figure 6, the difference scheme should give an approximation almost as good as for $\gamma_p=0$, even a better one in the 10 percent case, where the curves for the differential equation and the difference solution nearly coincide.

The influence of the time lag in the term describing the production of electrons by secondary photons (last term in eq (5.10)) has been studied by introducing an artificial factor $\exp(-\lambda\Delta t)$ with γ_p into eq (5.22) for the continuous case. Δt was assigned a fixed value approximately equal to the ones used in the computation with $\Delta x=0.025$, namely $\Delta t=6.25\times 10^{-7}$ sec. The deviations due to that factor $\exp(-\lambda\Delta t)$ can be seen from table 1 below.

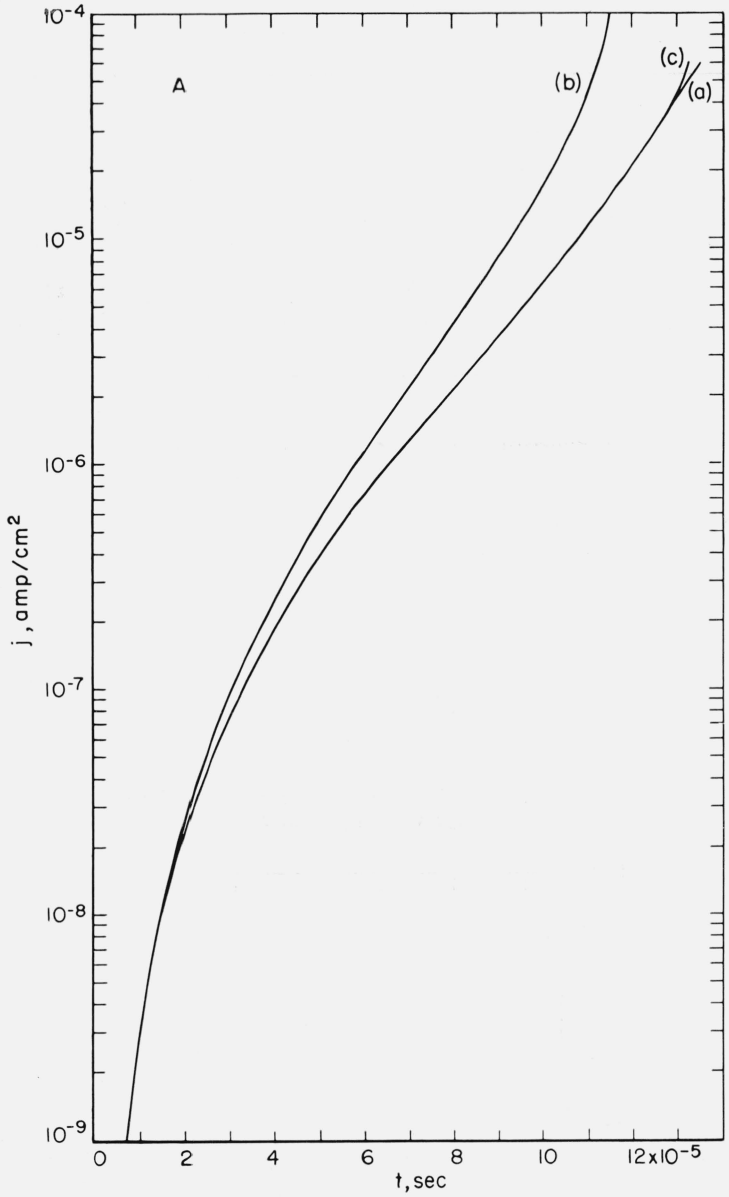

FIGURE 2A. *Mean current density* J *across the gap versus time* t *for* $\gamma_p=0$, $\gamma_i=1.5\ 10^{-5}$.

	Δx[cm]	U[kv]	$U-V_{br}$[kv]
(a)	0.05	28.870	0.233
(b)	.025	28.870	.295
(c)	.025	28.807	.233

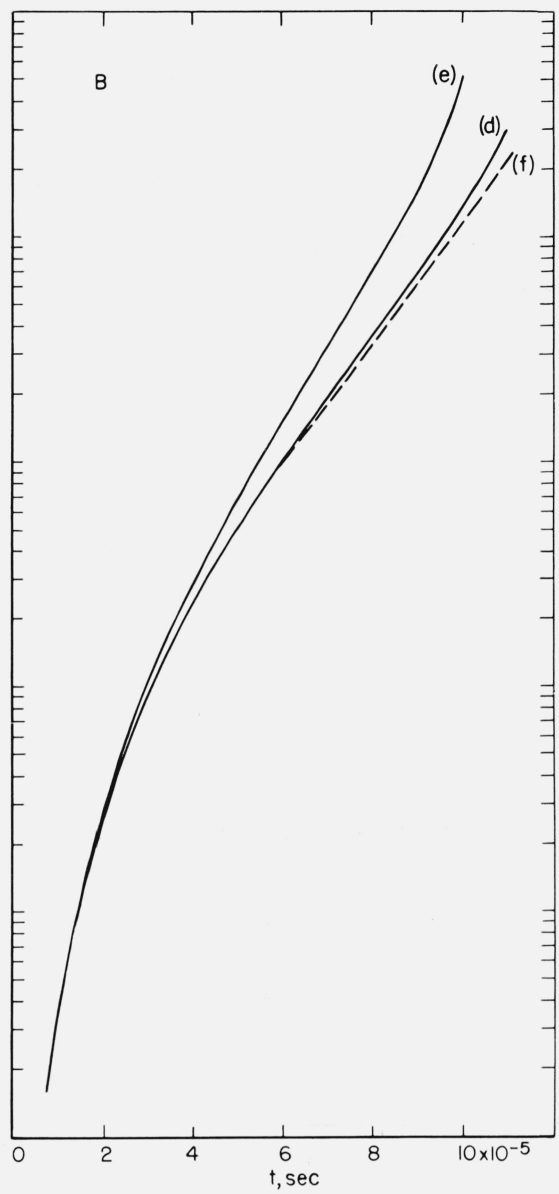

FIGURE 2B. *Mean current density* J *across the gap versus time* t *for* $\gamma_p=1.5\ 10^{-6}$, $\gamma_i=1.35\ 10^{-5}$.

	x[cm]	U[kv]	$U-V_{br}$[kv]
(d)	0.05	28.854	0.233
(e)	.025	28.854	.287
(f)	.025	28.800	.233

We conclude that the influence of that time lag is not very important, at least for the overvoltages and the small rates γ_p of production of secondary electrons considered here. We see that the time lag slows down the speed of development by a few percent at most, even in the worst case.

The electron quasi-equilibrium assumption $\partial q_-/\partial t=0$ in (2.1) is equivalent to letting $v_-\to\infty$ in this equation. Therefore, the influence of that assumption in the differential equation may be studied by replacing v_- by ∞ and v by v_+ in the eqs (5.22) and (5.23). If $\gamma_p=0$, the result is obvious. For fixed voltage, i.e., fixed α, the introduction of that assumption

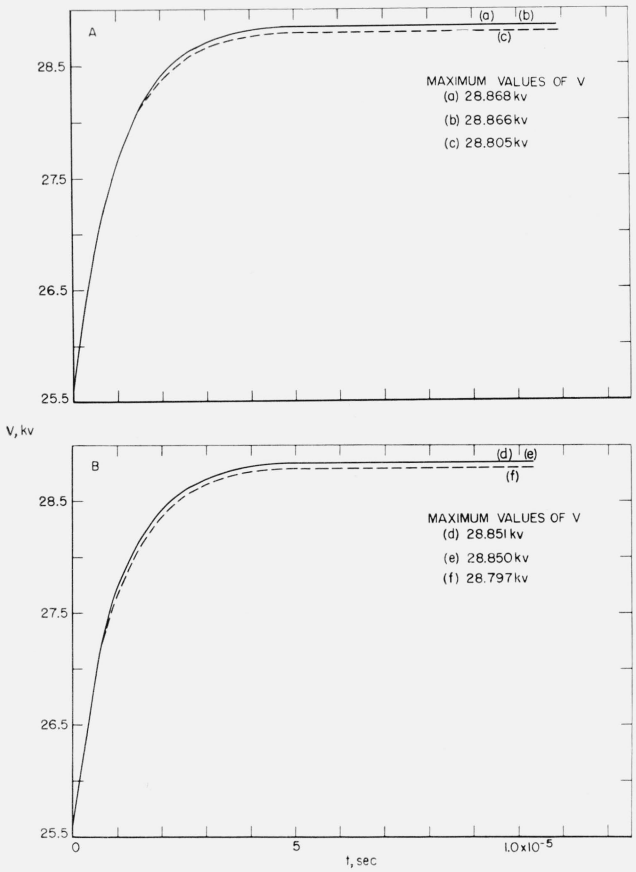

FIGURE 3A. *Voltage* V *across the gap versus time* t *for parameters of figure 2A.*

FIGURE 3B *Voltage* V *across the gap versus time* t *for parameters of figure 2B.*

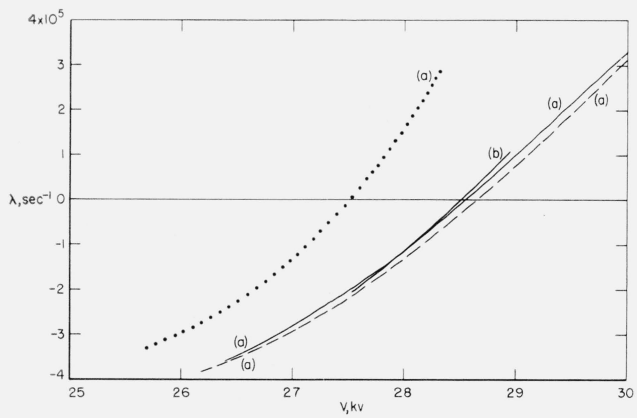

FIGURE 4. *Coefficient* λ *of temporal growth for constant field versus voltage* V *across the gap, for* Δx=0.05 *cm.*

(a) $\gamma_p=0$, $\gamma_i=1.5\ 10^{-5}$
(b) $\gamma_p=1.5\ 10^{-6}$, $\gamma_i=1.35\ 10^{-5}$ (continuous case only)
for electron quasi-equilibrium case (dashed)
general case (dotted)
continuous case (solid).

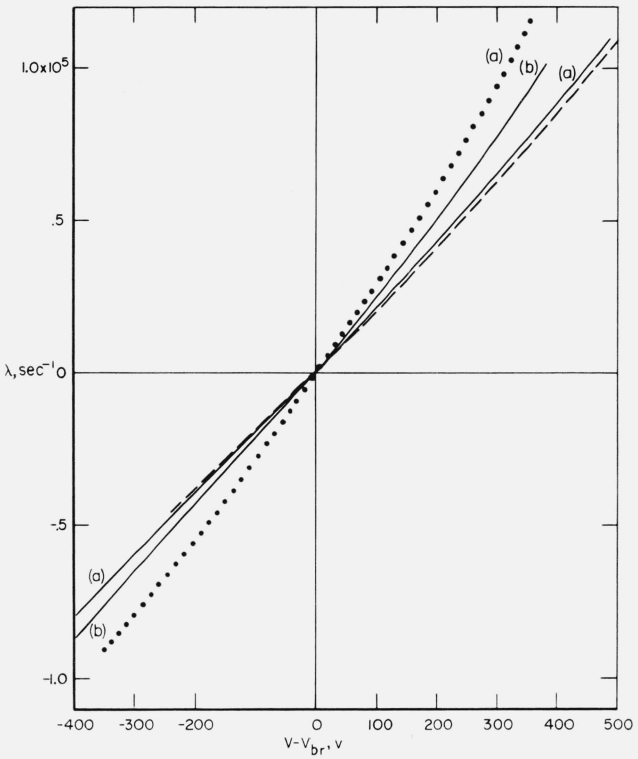

FIGURE 5. *Coefficient* λ *of temporal growth for constant field versus overvoltage* V—V_{br} *for same parameters as figure 4.*

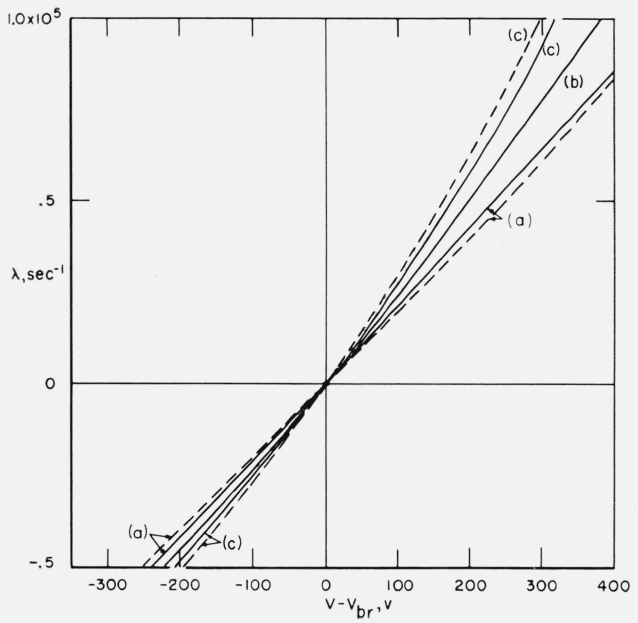

FIGURE 6. *Coefficient* λ *of temporal growth for constant field versus overvoltage* V—V_{br} *for* Δx=0.05 *cm.*

(a) $\gamma_p=0$, $\gamma_i=1.5\ 10^{-5}$
(b) $\gamma_p=1.5\ 10^{-6}$, $\gamma_i=1.35\ 10^{-5}$
(continuous and quasi-equilibrium case coincide)
(c) $\gamma_p=3.0\ 10^{-6}$, $\gamma_i=1.20\ 10^{-5}$
for electron quasi-equilibrium case (dashed)
continuous case (solid).

TABLE 1. *Influence of the time lag Δt for secondary photons on the coefficient λ of temporal growth*

$V-V_{br}$	$\frac{\gamma_p}{\gamma_i+\gamma_p}=0.1$		$\frac{\gamma_p}{\gamma_i+\gamma_p}=0.2$	
	$\Delta t=0$	$\Delta t=6.25\ 10^{-7}$	$\Delta t=0$	$t=6.25\ 10^{-7}$
Volt				
−400	−0.8643	−0.8624	−0.9291	−0.9248
−300	−.6595	−.6576	−.7147	−.7105
−200	−.4479	−.4463	−.4901	−.4864
−100	−.2285	−.2274	−.2532	−.2506
0	0	0	0	0
+100	+0.2396	+0.2377	+0.2746	+0.2696
+200	+.4930	+.4878	+.5802	+.5651
+300	+.7605	+.7538	+.9317	+.8978
+400	+1.0610	+1.0410	+1.3650	+1.2880

amounts to no change in u, and therefore to a relative increase of λ by about v_+/v_-. Hence the temporal development is speeded up by about 0.5 percent, which is small enough for the accuracy required in this problem.

A detailed analysis of the effect of that assumption in the case $\gamma_p \neq 0$, which is elementary, but too lengthy to be reproduced here, shows that this remains true as long as the relative influence of γ_p as compared to γ_i in eq (5.22) is small, i.e., roughly speaking, as long as $\gamma_p \cdot e^{\lambda d/v_+} \ll \gamma_i$. This is in agreement with physical considerations, since, if the influence of secondary photons is dominant, the development is dominated by a process with feedback, all of whose components go infinitely fast. Quantitatively, as long as $u \ll 1$, $v_+ \ll v_-$, $\alpha d \gg 1$, the change $\delta\lambda$ of λ effected by the equilibrium assumption is, with good approximation.

$$\frac{\delta\lambda}{\lambda} = \frac{v_+}{v_-} \cdot \left(1 + \frac{\alpha d - 1}{\alpha d - 1/(1-u)} \cdot \frac{\gamma_p (\sigma/\alpha)(1-u) e^{\lambda d/v_+}}{\gamma_i}\right).$$

The full difference equations, with eq (5.1a) included, have not been used[2], for the following reasons. For the same Δx and the same time interval to be covered, the stability conditions require 200 times as many time steps, if the electron-quasi-equilibrium condition is dropped. Furthermore, figures 3 and 4 show that, for $\Delta x = 0.05$ cm, the deviation from the limit case $\Delta x = 0$ is considerable, and cannot be diminished satisfactorily by a simple change in the outer voltage leading to the same overvoltage. It is estimated that a decrease of Δx to 0.01 cm is at least necessary in order to obtain results as good as the ones in the quasi-equilibrium case. Two trial runs were made, both leading to breakdown after a few ion transit times. The data for these runs were:

(a) $\Delta x = 0.05$, $U = 31.5$ kv, $V_{br} = 28.64$ kv, $\gamma_i = 1.5\ 10^{-5}$, $\gamma_p = 0$
infinite current after 2 ion transit times.
(b) $\Delta x = 0.5$, $U = 28.87$ kv, $V_{br} = 28.51$ kv, $\gamma_i = 0$, $\gamma_p = 1.5\ 10^{-5}$
infinite current after 3 ion transit times.

All the above considerations and the computations done so far are restricted to a domain where the temporal growth in one (natural) time-step (as indicated by the stability condition) is not too large. Near the actual breakdown, therefore, these arguments may not apply, and an investigation of the nature and quantitative features of the breakdown singularity is not given in this paper. Since the time required for the space change to become significant is very large compared to the remaining time until completion of the breakdown, and since the assumption of one-dimensionality is doubted for high current densities, there was no point in trying to describe the details of the breakdown more precisely.

The features discussed in this section have been tested only for a few sets of parameters. Up to this time it has not been proved that the conclusions are general. Since the number of

[2] Several successful computer runs have been made since by Dr. A. L. Ward, using this option of the program.

parameters is quite large, it has not been attempted to explore the limits of the region where the conclusions are valid. But the methods of computing and of guessing the effect of truncation errors as described here should be tried in other cases.

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