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Conversion of the Amplitude-Probability Distribution Function for Atmospheric Radio Noise From One Bandwidth to Another

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The amplitude-probability distribution function of atmospheric radio noise can be predicted with reasonable accuracy for a given bandwidth using only the first two moments of the noise measured at that bandwidth. This paper presents a method for predicting this distribution function for any specified bandwidth from the moments of the noise measured at a particular bandwidth.

1. Introduction

Atmospheric radio noise is often the limiting factor in radio communications at frequencies up to about 30 Mc/s. In the design of communications circuits, it is necessary to know the detailed statistical characteristics of interfering noise in order to calculate the channel capacity or error rate for a noisy channel. The cumulative amplitude-probability distribution (APD) has been found to be a useful tool in such analyses [Montgomery, 1954; Watt, et al., 1958; Crichlow and Disney, to be published].

Atmospheric radio noise is a nonstationary random process whose characteristics change not only with time but also with bandwidth. The APD is usually measured as a time sequence of several simultaneous levels, and the necessarily long integration times make difficult the obtaining of a continuous curve. The need to overcome this difficulty has led to the development of a method of predicting the APD from the three statistical moments of atmospheric radio noise measured by the ARN–2 noise recorder of the National Bureau of Standards [Crichlow et al., 1960a; Crichlow et al., 1960b; Fulton, 1961]. These moments are measured on a worldwide basis [Crichlow, 1957], and the data from these measurements are summarized and published quarterly [Crichlow, Disney, and Jenkins, 1957–1961].

The three moments are measured for a power bandwidth of about 200 c/s, so the APD derived from them is valid only for this bandwidth. For this reason, a method for converting the APD of atmospheric radio noise at a 200 c/s bandwidth to a range of other bandwidths was developed.

2. Parameter Definitions and Basic Assumptions

It has been shown that the distribution function for atmospheric radio noise can be determined from its following three statistical moments [Crichlow et al., 1960a]:

$$\text{rms voltage: } v_{\text{rms}} = \left(\int_{f_1}^{f_2}\int_{t_1}^{t_2} v^2 dp(f,t)\right)^{1/2}$$

$$\text{average voltage: } v_{\text{ave}} = \int_{f_1}^{f_2}\int_{t_1}^{t_2} v dp(f,t)$$

$$\text{log of the voltage: } \log v_{\log} = \int_{f_1}^{f_2}\int_{t_1}^{t_2} \log v dp(f,t)$$

where v is $1/\sqrt{2}$ times the instantaneous envelope voltage, p is the probability of v being exceeded, $f_2-f_1=b$ is the bandwidth, and t_2-t_1 is the time interval of measurement.

The three moments as measured are expressed, respectively as:

F_a=the effective noise figure
=the external noise power available from an equivalent, short, lossless, vertical antenna in decibels above ktb (the thermal noise power in a passive resistance at room temperature, t, in a bandwidth, b; where k is Boltzmans' constant).

$V_d=v_{\text{ave}}$ in decibels below v_{rms}.

$L_d=v_{\log}$ in decibles below v_{rms}.

F_a is independent of bandwidth for a uniform spectrum (a condition closely approximated in normal communications bandwidths), but v_{rms} varies as the square root of bandwidth. V_d and L_d are always positive, since for this type of distribution function, $v_{rms}>v_{\text{ave}}>v_{\log}$. Because the shape of these distribution curves is dependent only on V_d and L_d*, which have been normalized to F_a, it is possible to construct distribution curves of a given form factor for various combinations of V_d and L_d.

Measurements of V_d and L_d tend to show that L_d is a linear function of V_d. Figure 1 shows L_d versus V_d for various bandwidths, seasons, and time

*Unless otherwise stated, the moments used in the analysis are assumed to be the true moments of the distribution function.

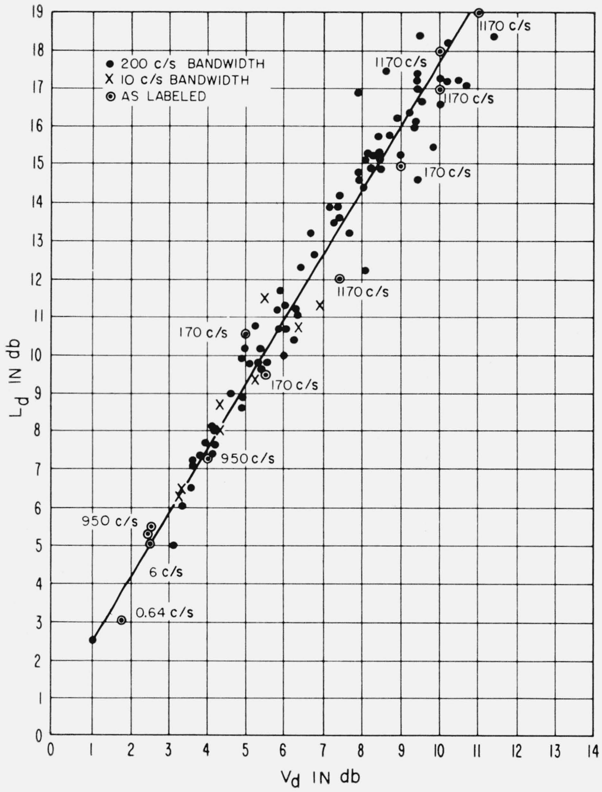

FIGURE 1. L_d *versus* V_d *from measured distributions.*

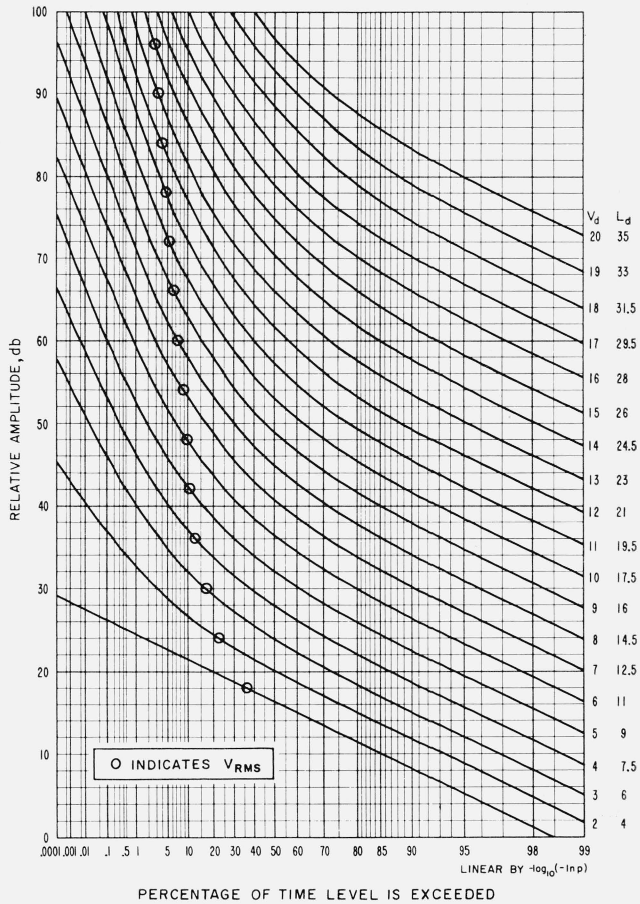

FIGURE 2. *Most probable amplitude-probability distribution functions for atmospheric radio noise.*

of day from measured distributions. The relation $L_d = 1.69\ V_d + 0.72$ seems to hold in gerenal. Because of the correlation between V_d and L_d, one can determine the most probable APD's of atmospheric radio noise for the range of V_d normally encountered. A family of these is presented as figures 2 and 3.

The APD of atmospheric radio noise can be closely represented by a three-section curve on Rayleigh graph paper, i.e., two intersecting straight lines joined tangentially by a circular arc, figure 4. This particular Rayleigh graph paper has scales chosen so that a Rayleigh distribution function plots as a straight line of slope $-\frac{1}{2}$. These scales are labeled noise level in decibles above v_{rms} versus the percentage of time that each level is exceeded.

The lower section of the curve (low levels exceeded with high probabilities) represents the part of the noise composed of many random overlapping pulses, and plots as a straight line Rayleigh distribution [Crichlow et al., 1960a]. The upper straight line section of slope less than $-\frac{1}{2}$ (high levels exceeded with low probabilities) represents the part of the noise composed of large, infrequent nonoverlapping pulses. If the exponent in the expression for the Rayleigh distribution function is raised to the power $-\frac{1}{2}s$, where s is the slope of this upper straight line, an expression of the upper portion of the APD will be obtained. For this reason, the upper portion is sometimes called a power Rayleigh.

Four parameters are used to define the distribution as shown in figure 4. A is the decibel difference between the v_{rms} level and the Rayleigh line at 0.5 probability, and determines the amplitude of the Rayleigh section of the distribution. C and X fix the amplitude and slope of the power Rayleigh. C is the decibel difference between the Rayleigh line and the power Rayleigh line at 0.01 probability and X is the ratio of the slope of the power Rayleigh relative to the slope of the Rayleigh, that is, $X = -2s$. The parameter B describes the circular arc and is defined as the decibel difference between the intersection of the two straight lines and the line tangent to the circular arc at its center. For atmospheric radio noise, an experimental correlation between B and X has been found, with $B = 1.5\ (X-1)$. These parameters have been determined as functions of V_d and L_d.

The following assumptions are used to determine the relationship between the APD's for a sample of atmospheric radio noise received through different bandwidths. Each of these will be discussed in detail as they arise.

1. The shape of the distribution for the probabilities of interest (10^{-6} and greater) will be of the above form for any bandwidth considered.

2. The rms value of the distribution will vary as the square root of the power bandwidth, increasing with increasing bandwidth.

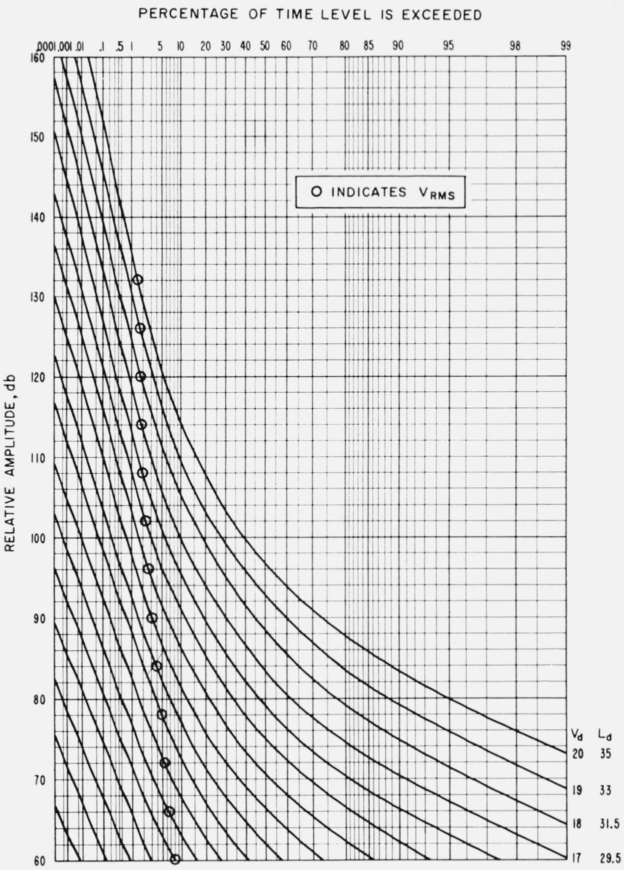

FIGURE 3. *Most probable amplitude-probability distribution functions for atmospheric radio noise.*

3. As the bandwidth decreases and becomes quite small, the APD approaches a Rayleigh distribution.

Since with the above assumptions the cumulative distribution is determined by its moments (V_d and L_d), the manner in which the moments vary with bandwidth needs to be determined. With V_{d_1} and L_{d_1} designating the moments of the original APD at bandwidth b_1, and V_{d_2} and L_{d_2} designating the moments of the desired distribution at bandwidth b_2:

1. X_2 and C_2 must be determined as functions of X_1, C_1, and w, where $w=b_2/b_1$ and,
2. V_{d_2} and L_{d_2} must be determined as functions of V_{d_1}, L_{d_1}, and w from (1).

3. Transformation of the Power Rayleigh Section of the Distribution

The high-amplitude, low-probability section of the APD represents a train of nonoverlapping pulses. If the response of a receiver with bandwidth b_1 to one of these pulses is a pulse of amplitude a and time duration t, and then the bandwidth is changed to b_2, the response will be a pulse of amplitude wa and time duration t/w. For this section of the APD Fulton [1961] has shown that every point (p, v) corresponding to a bandwidth b_1 transforms to the point $(p/w, wv)$ for a bandwidth b_2, as long as the

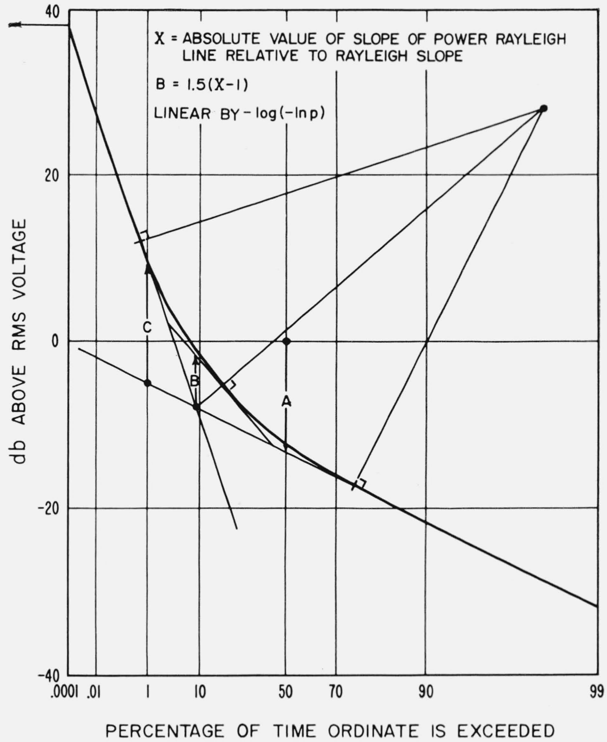

FIGURE 4. *Definition of parameters for amplitude-probability distribution for atmospheric radio noise.*

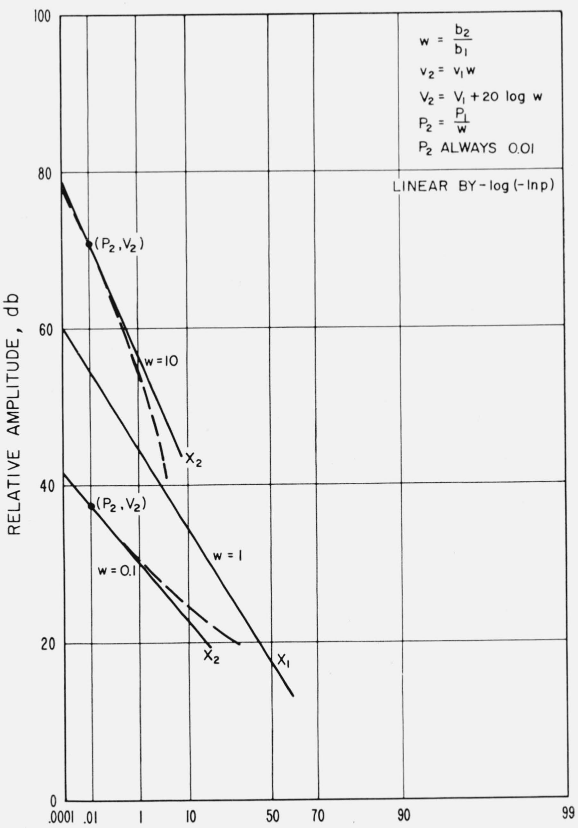

FIGURE 5. *Transformation of a power Rayleigh line.*

receiver response maintains the criterion of nonoverlapping pulses. However, if the transformation $(p, v)\rightarrow(p/w, wv)$ is made point by point, the results will not be another straight line, but a curve. As shown by the dashed lines in figure 5, the curvature is very slight in the probability range for which the power Rayleigh holds, because in the region in which the curvature becomes objectionable the criterion of nonoverlapping pulses is no longer met.

Since it is desired to portray the transformed distribution with the same form factor as the original, the curve will be approximated by an appropriate tangent to the curve. The slope and point of tangency of this line will be determined as a function of the bandwidth ratio w, so that a suitable approximation is obtained; and a reciprocal relationship results, i.e., transforming from b_1 to b_2, and then from b_2 to b_1, the original distribution is regained.

The equation of a cumulative Rayleigh distribution is:

$$p=\exp\left[-\frac{v^2}{\overline{v^2}}\right],$$

or

$$\log v=-\frac{1}{2}\left[-\log\left(-\ln p\right)\right]+\frac{1}{2}\log \overline{v^2},$$

where v is $1/\sqrt{2}$ times the instantaneous envelope voltage, p is the probability of v being exceeded, and $\overline{v^2}$ is the mean square voltage.

Since the slope of the power Rayleigh line is $-\frac{X}{2}$, the equation of the distribution function for the power Rayleigh will be

$$\log v=-\frac{X}{2}\left[-\log\left(-\ln p\right)\right]+\frac{1}{2}\log \overline{v_r^2},$$

where $\overline{v_r^2}$ is the mean square voltage of the corresponding Rayleigh distribution,

or

$$p=\exp\left[-\left(\frac{v^2}{\overline{v_r^2}}\right)^{\frac{1}{X}}\right].$$

Transforming the power Rayleigh by the bandwidth ratio w,

$$wp=\exp\left[-\left(\frac{v^2}{w^2\overline{v_r^2}}\right)^{\frac{1}{X}}\right],$$

or

$$\log v=\frac{X}{2}\log\left(-\ln wp\right)+\frac{1}{2}\log w^2\overline{v_r^2}.$$

Putting the above transformed curve in terms of our coordinates with

$$x=-\log\left(-\ln p\right)$$

or

$$p=\exp\left[-10^{-x}\right],$$

and

$$y=\log v,$$

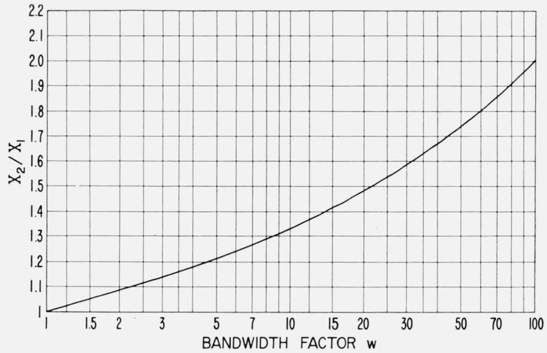

FIGURE 6. *Ratio of* X_2 *to* X_1 *versus bandwidth factor* w.

it is seen that

$$y=\frac{X}{2}\log\left(10^{-x}-\ln w\right)+\frac{1}{2}\log w^2\overline{v_r^2}.$$

$$\frac{dy}{dx}=\frac{X}{2}\left(\frac{-\ln p}{\ln p+\ln w}\right).$$

If $s_1(s_1=-X_1/2)$ is the slope of the original power Rayleigh and $s_2(s_2=dy/dx)$ is the slope of the resulting power Rayleigh,

$$\frac{X_2}{X_1}=\frac{\ln p}{\ln pw}.$$

The point for the best tangent approximation will change with each transformation w. The point of tangency must be chosen such that the above mentioned reciprocal relationship is maintained, and such that the transformed curve is well approximated by its tangent in the interval of interest. This tangent point for each transformation was determined to be $p\cong0.0001$ for $w>1$ and $b_1=200$ c/s. The above requirements will be met if the tangent point is chosen at $p=0.0001$ for $w>1$, and the appropriate value for $w<1$ to maintain the reciprocal relation.

In order to transform the power Rayleigh, then, the point $(0.01w\%, v)$ is translated to the point $(0.01\%, wv)$ and through this point the desired power Rayleigh with a slope given by figure 6 is drawn $(w>1)$. Figure 6 shows the ratio of X_2 and X_1 as a function of w.

4. Transformation of the Remaining Section of the Distribution

Having obtained X_2 as a function of X_1 and w, C_2 must be determined as a function of C_1, X_1, and w. This is obtained by choosing various X_1 and C_1 for a particular w, transforming the power Rayleigh portion according to the above, and then determining C_2 (that is, locate the Rayleigh portion of the desired distribution) such that the v_{rms} values of the original distribution and the desired distribution are related by the $\sqrt{w}$. An example of the procedure is given in figures 7 and 8. In figure 7, the distribution to be transformed is chosen ($X_1=6$, $C_1=10$ db), and from the known relationship between A, C, and X (fig. 9), A_1 is seen to be 9.9 db and the relative v_{rms} level is arbitrarily set at the 20 db level.

Transforming by the bandwidth factor 20, the point (P_1, V_1) is transformed to the point (P_2, V_2), where P_2 is 0.01 percent, P_1 is 20 (0.01%) or 0.2 percent, V_1 is 36.5 db, and V_2 is $V_1+20 \log 20$ or 62.5 db. X_2 is found to be 8.88 from figure 6 and the power Rayleigh section is transformed. The V_{rms} level of the desired distribution transforms as the $\sqrt{20}$ or 13 db, so V_{rms_2} will be 33 db. Using figure 9 and choosing various C_2, the correct C_2 is found to be 9.6 db (fig. 8). Figure 10 shows C_2 as a function of C_1 and X_1 for a bandwidth factor of 0.5.

From previous work [Crichlow et al., 1960a] and the above, the following functional relations are now known:

$$
\begin{aligned}
X &= f_1(V_d, L_d) \\
C &= f_2(V_d, L_d) \\
X_2 &= f_3(X_1, w) \\
C_2 &= f_4(X_1, C_1, w) \\
L_d &= f_5(V_d).
\end{aligned}
$$

These may be manipulated graphically to obtain V_{d_2} as a function of V_{d_1} and w. The results are shown in figures 11 and 12.

In order to determine the shape and amplitude of the APD for atmospheric radio noise at bandwidth b_2 from V_{rms_1} and V_{d_1} measured at bandwidth b_1, then, V_{d_2} is obtained from V_{d_1} and w from figures 11 or 12, and the shape of the APD for this V_{d_2} is found from figures 2 and 3. The amplitude of theAPD at bandwidth b_2 is determined from the relation $V_{\mathrm{rms}_2}=V_{\mathrm{rms}_1}+10 \log w$.

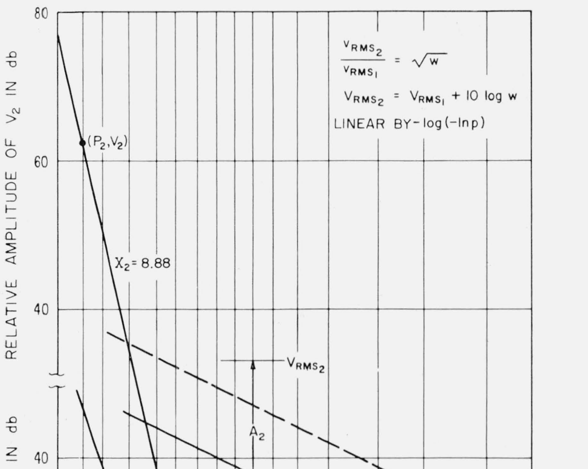

FIGURE 7. *The determination of* C_2 *as a function of* C_1, X_1, *and* w.

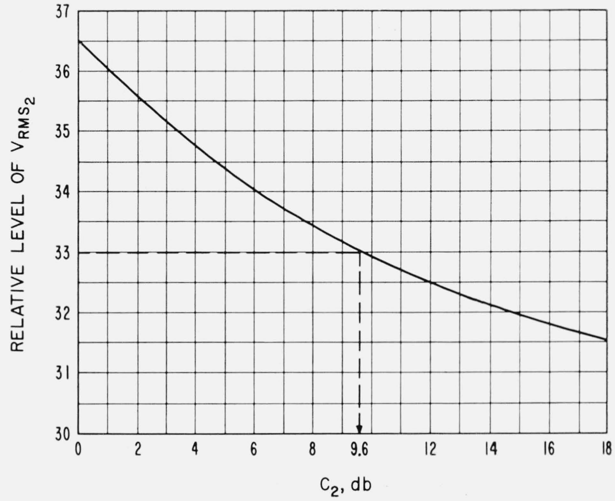

FIGURE 8. *Relative* V_{rms_2} *level versus* C_2 *for figure 5* ($X_2=8.88$).

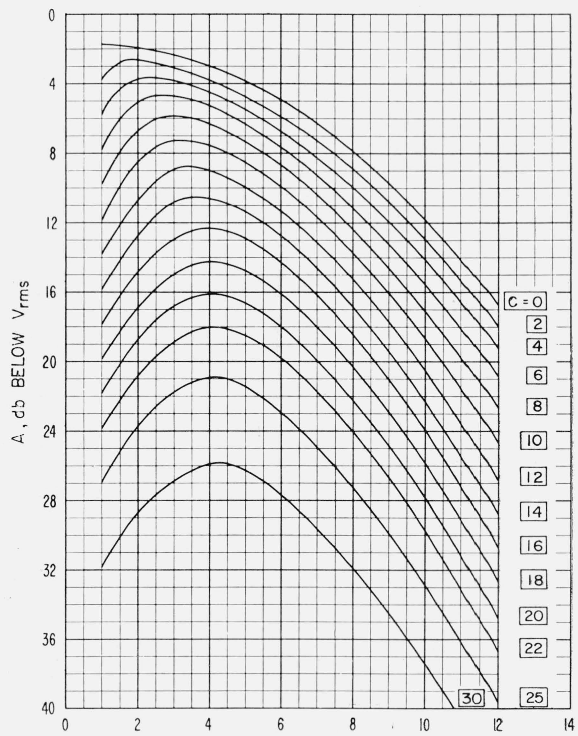

FIGURE 9. A *as a function of* C *and* X.

5. Accuracy Considerations

As the bandwidth is increased indefinitely, the above analysis becomes inaccurate since the tangent approximation is no longer sufficient. However, since V_d increases with bandwidth, and for large V_d the integrals defining V_d are determined almost completely by the power Rayleigh portion of the cumulative distribution; the behavior of V_d for large bandwidth ratios may be determined by investigating the behavior of the power Rayleigh.

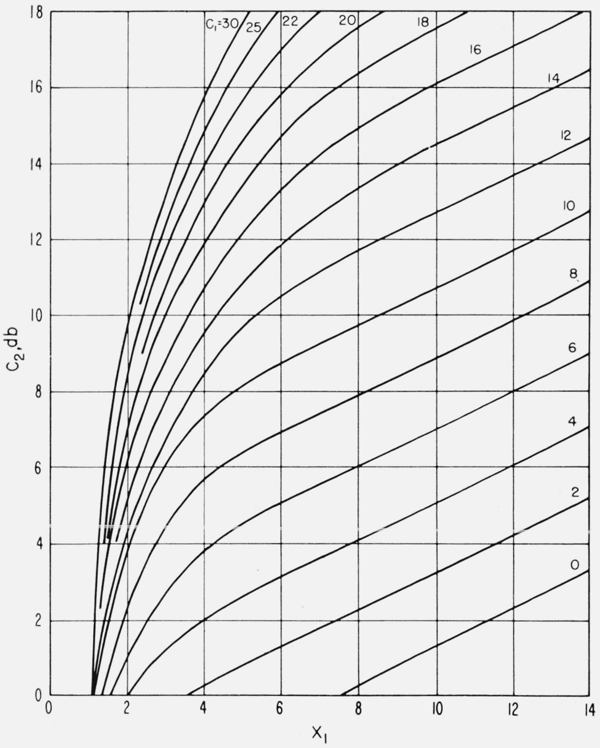

FIGURE 10. C_2 *as a function of* C_1 *and* X_1 *for* $w=0.5$.

Since the power Rayleigh is composed of nonoverlapping pulses, the character of the repsonse of a bandwidth-limited circuit to this train of pulses can be determined from the response to an individual pulse. Therefore, the average value is independent of bandwidth, and the rms value varies as the square root of the bandwidth ratio. V_d (the deviation in db between the average and rms values) will vary as the square root of the bandwidth ratio, or V_d will increase 10 db per decade of bandwidth ratio w. The above may also be shown by actual integration of the power Rayleigh and the transformed power Rayleigh.

The above also gives a good check on the accuracy of the previous analysis. The results must approach this 10 db per decade law as w increases and also as V_{d_1} increases. Figure 12 shows that this is indeed the case.

For a bandwidth of 200 c/s, it has been found experimentally that the distribution saturates at a probability of 10^{-6}, i.e., there will be almost no pulses with amplitude greater than that for which the probability of being exceeded is 10^{-6}. This is not true for other bandwidths. In general, the distribution will saturate at slightly lower probabilities for larger bandwidths and slightly higher probabilities for smaller bandwidths. By assuming the same form of the distribution for all bandwidths (saturation at $p=10^{-6}$), an error is introduced. This error is indicated by the shaded portions of figure 13.

An indication of the maximum error in V_d can be obtained by evaluating the average and rms values for a power Rayleigh in the probability ranges [0,1] (that is, assuming no saturation) and $[10^{-6},1]$. This will determine the maximum possible error in the average value and the rms value. Since these two errors will be in the same direction, the difference between them (in db) will be the error in V_d.

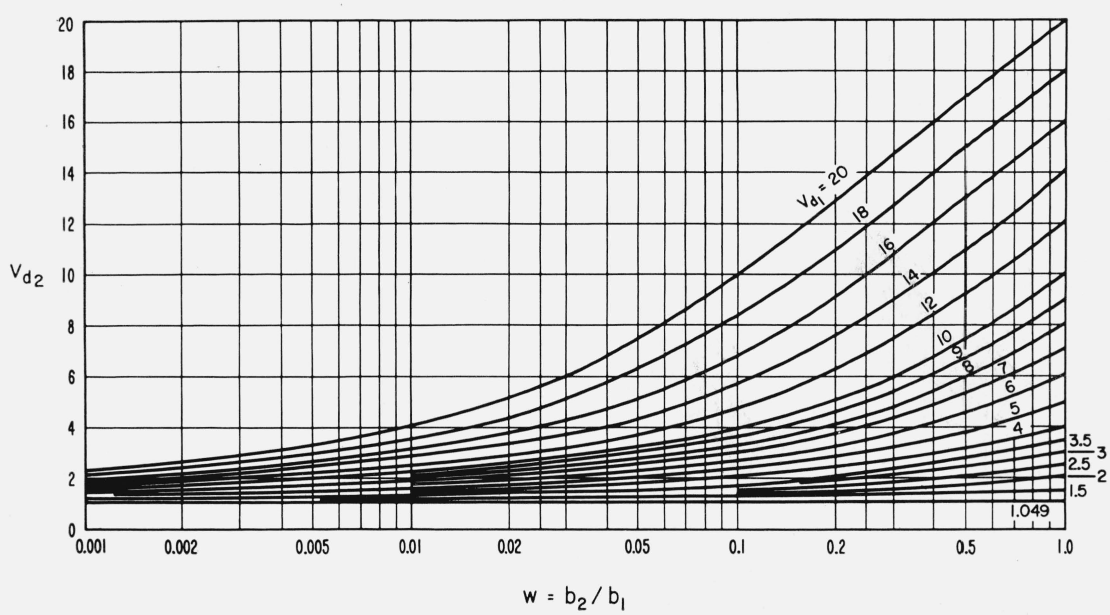

FIGURE 11. *Effect of bandwidth on* V_d.

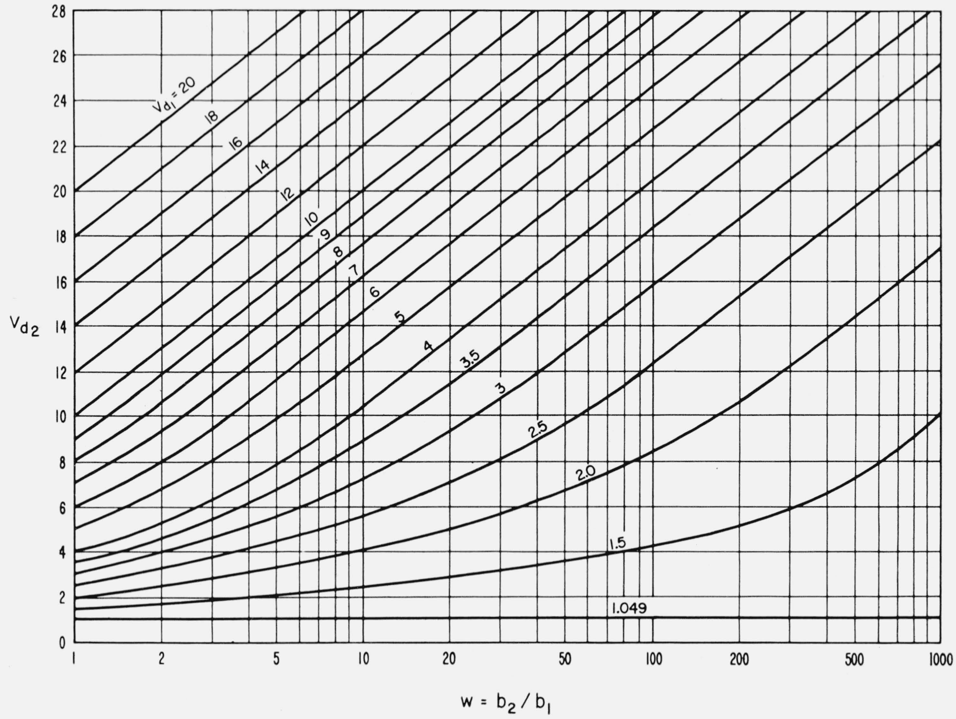

FIGURE 12. *Effect of bandwidth on* V_d.

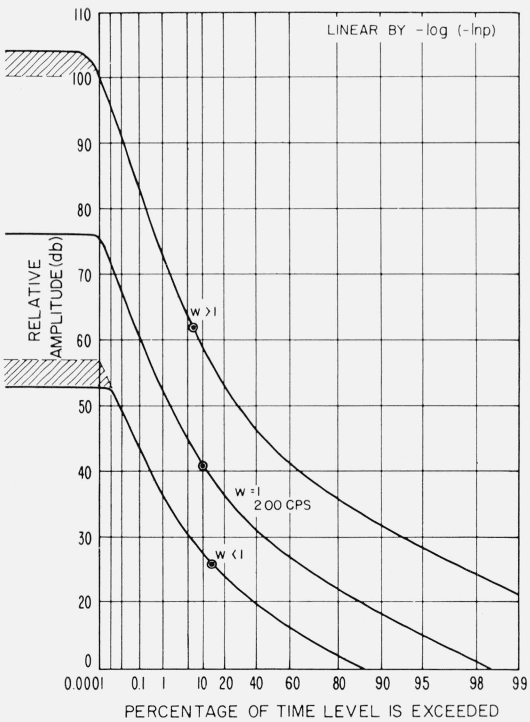

FIGURE 13. *Saturation of amplitude-probability distributions for various bandwidths.*

We must evaluate

$$v_{\text{ave}} = \int_{p_1}^{1} v\,dp \text{ and } v_{\text{rms}}^{2} = \int_{p_1}^{1} v^2 dp$$

where

$$p = \exp\left[-\left(\frac{v^2}{\overline{v_r^2}}\right)^{\frac{1}{X}}\right]$$

and p_1 assumes the two values 0 and 10^{-6}. Under the change of variable

$$Z = (v^2/\overline{v_r^2})^{\frac{1}{X}}$$

$$v_{\text{ave}} = \sqrt{\overline{v_r^2}} \int_0^{-\ln p_1} Z^{\frac{X}{2}} e^{-Z} dZ.$$

For $p_1 = 0$, then

$$v_{\text{ave}} = \sqrt{\overline{v_r^2}}\, \Gamma\left(\frac{X}{2}+1\right).$$

For $p_1 = 10^{-6}$, v_{ave} cannot be easily evaluated in closed form in terms of X. If X is chosen corresponding to the power Rayleigh with the largest slope likely to be encountered, v_{ave} can be evaluated. Choosing $X = 12$ or a slope of -6,

$$v_{\text{ave}} = \left[-\sqrt{\overline{v_r^2}}\, e^{-Z} \sum_{n=0}^{6} \frac{720}{n!} Z^n\right]_0^{13.82}$$

$$v_{\text{ave}} = 708.51 \sqrt{\overline{v_r^2}}.$$

For the probability interval [0,1], and $X=12$,

$$v_{\text{ave}}=\sqrt{\overline{v_r^2}}\,\Gamma(7)=720\sqrt{\overline{v_r^2}},$$

Therefore, the maximum relative error in v_{ave} is 0.13 db.

In similar fashion, for the probability range [0, 1],

$$v_{\text{rms}}^2=\overline{v_r^2}\,\Gamma(X+1)$$

$$v_{\text{rms}}=2.19\times10^4\sqrt{\overline{v_r^2}}.$$

For the probability range [10^{-6}, 1], and $X=12$

$$v_{\text{rms}}^2=\left[-\overline{v_r^2}e^{-Z}\sum_{n=0}^{12}\frac{12!}{n!}Z^n\right]_0^{13.82}$$

$$v_{\text{rms}}=1.726\times10^4\sqrt{\overline{v_r^2}}.$$

The maximum relative error in v_{rms} will, therefore, be about 2.07 db and the maximum possible error in V_d somewhat less than 2 db. The maximum error in L_d will be about the same, since the error in $v_{\log}$ will be even less than that in v_{ave}.

In practice, the error introduced will be much smaller than the 2 db, as it was assumed that the distribution never saturates in calculating the 2 db. Even so, an error of 2 db in V_d for large V_d will not noticeably change the shape of the cumulative distribution, but only the relative amplitude of the distribution.

Since the amplitude of the distribution (v_{rms}) is measured experimentally, we are only interested in the effect of the error on the shape of the distribution; therefore, the assumption of a uniform point of saturation for all bandwidths is a valid one, since the shape of the distribution for the probabilities of interest will not be changed.

In using the above, it should be remembered that the transformation depends on L_d being a linear function of V_d. Figure 1 shows this is generally correct, however, there is some variation of L_d for a given V_d, and this variation increases as V_d increases. That is, in any particular case, it is possible for the true L_{d_1} and the assumed true L_{d_1} (from fig. 1) to be significantly different, especially for large V_{d_1}. Also, V_{d_1} itself is subject to error in measurement, and this error will be propagated by the transformation. The possible error in V_{d_2} for a given V_{d_1} subject to error can be quickly determined from the transformation curves. For example, suppose V_{d1} was measured to be 5 db subject to a possible error of $\pm\frac{1}{2}$ db ($4.5<V_{d_1}<5.5$) and it is desired to transform by $w=100$. V_{d_1} of 5 transforms to V_{d_2} of 22.7, V_{d_1} of 4.5 goes to V_{d_2} of 21.7, and V_{d_1} of 5.5 to V_{d_2} of 23.7. In this case, then, the $\pm\frac{1}{2}$ db error in measurement has transformed to a possible ±1 db error in V_{d_2} ($21.7<V_{d_2}<23.7$). On the other hand, if we transform the above V_{d_1} by $w=0.01$, the $\pm\frac{1}{2}$ db error is seen to transform to a possible ±0.07 db error in V_{d_2}. For this reason, then, each transformation should be investigated as to the effect of measurement error in V_{d_1} on V_{d_2} and the effect of this possible error in V_{d_2} on the shape of the resulting APD.

8. Conclusions

Since measurements of the APD for atmospheric radio noise have shown the assumed form factor to hold over a wide range of bandwidths [Watt and Maxwell, 1957; Watt et al., 1958; Crichlow et al., 1960a], the above transformation will usually give good results. However, in using the above, it should be remembered that an error in measurement of V_{d_1} will generally propagate as a larger error in V_{d_2} as the bandwidth is increased and a smaller error in V_{d2} as the bandwidth is decreased. Also, the linear relation between V_d and L_d is only a good approximation and subject to possible error, especially for large V_d. If a V_{d_1} and L_{d_1} are encountered significantly different from the pairs corresponding to the APD's of figures 2 and 3, the APD for this pair of V_{d_1} and L_{d_1} may be obtained from NBS Monograph 23 [Crichlow et al., 1960b]. This APD may then be transformed by the method outlined above.

Experimental verification of this method of predicting the changes in V_d with bandwidth has been obtained, but with some doubts as to accuracy since the noise changed character in the period covering successive bandwidth measurements.

An improved method of measurement is currently under development using simultaneous magnetic tape recordings of the noise through different bandwidths. Results of the new measurements will be published upon completion.

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