JOURNAL OF RESEARCH of the National Bureau of Standards—D. Radio Propagation
Vol. 66D, No. 6, November–December 1962

On the Limitations of Geometrical Optics Solutions for Curved Surfaces With Variable Impedance Properties [1]

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(Received May 14, 1962)

In the preceding paper, the authors have presented an asymptotic solution for the field in the illuminated region of a large circular cylinder whose surface impedance around the periphery deviates from a constant value by a sinusoidal variation of small amplitude α. To $O(\alpha)$, the reflected field comprises a specularly reflected ray and two first-order diffracted rays characteristic of a curved convex reflection grating. If the surface impedance varies "slowly," these three rays can be combined into a single specularly reflected ray having a reflection coefficient which depends solely on the local impedance at the reflection point. The "slowness" conditions necessary for the validity of this local reflection principle of geometrical optics are investigated and interpreted in physical terms. The results are presented in a manner which suggests their applicability to general, gently curved surfaces with slowly varying impedance properties.

1. Introduction

A plane electromagnetic wave incident on a large cylinder with a spatially periodic surface impedance around the periphery gives rise to a reflected field which may be interpreted as comprising a spectrum of rays appropriate to a curved convex reflection grating. These conclusions are presented in the preceding article wherein we have considered a circular cylinder whose surface impedance has a peripheral sinusoidal variation of small amplitude α superimposed upon a constant value [Marcinkowski and Felsen, 1962, henceforth referred to as III]. If the impedance varies slowly over a distance interval equal to the wavelength of the incident field, its periodic aspects in the vicinity of the specular reflection point lose their importance, and the specular and diffracted grating rays reaching a prescribed observation point are nearly parallel. From geometric-optical considerations it is reasonable to suppose that this ray bundle can be combined into a single specularly reflected ray having a reflection coefficient identical with that for a cylinder whose *constant* surface impedance is equal to the value of the variable surface impedance at the point of reflection. Such an assumption may be designated as the local reflection hypothesis of geometrical optics, and the conditions under which it obtains for the specular and the first order diffracted (grating) rays in III are discussed in this paper. These requirements are then phrased in a form which suggests their applicability also to impedance variations other than the one considered herein.

To avoid unnecessary repetition, all suitable definitions are to be found in III. The reflected rays considered here comprise a single, specularly reflected ray and two first order diffracted rays. Their analytical form is recalled in section 2, and the desired local reflection formula for a slowly varying surface impedance is presented. Section 3 contains, in outline, a derivation of the conditions required for the validity of the local reflection hypothesis. Some physical interpretations of these requirements are provided in section 4. While these conditions are derived from the periodic impedance function $\bar{Z}(\phi)$, the effects of periodicity in the neighborhood of the point of reflection play a minor role when $\bar{Z}(\phi)$ varies very slowly. Hence, a formulation as in section 5, which expresses the requirements for the validity of the local reflection hypothesis in a manner which makes no reference to the specific form of $\bar{Z}(\phi)$, may also be expected to apply to other slow impedance variations.

2. Geometric Optical Fields to Order α^1

To $O(\alpha)$ in the perturbation solution,[2] the total field at a point in the illuminated region can be characterized in terms of an incident ray (the magnetic field vector **H** is assumed to be parallel to the cylinder axis), a specularly reflected ray of $O(\alpha^0)$, and two grating rays of $O(\alpha)$. The axial magnetic field component is then given by

$$G(\rho,\phi,\phi')=G_i(\rho,\phi,\phi')+G_r(\rho,\phi,\phi'), \tag{1}$$

where the subscripts i and r identify the incident and reflected fields which are, to $O(\alpha)$,

$$G_i(\rho,\phi,\phi')=e^{-ik\rho\cos(\phi-\phi')} \tag{2a}$$

$$G_r(\rho,\phi,\phi')=G_r^0(\rho,\phi,\phi')+\alpha[G_r^1(\rho,\phi,\phi',p)+G_r^1(\rho,\phi,\phi',-p)]. \tag{2b}$$

[1] Sponsored by the Air Force Cambridge Research Laboratories under Contract No. AF-19(604)-4143. The work presented here is based on a dissertation submitted by one of the authors (C. J. M.) in partial fulfillment of the requirements for the Ph. D. degree.

[2] α is the amplitude of the surface impedance variation, and all fields are expressed as a power series in α [see III, and Felsen and Marcinkowski, 1962, henceforth referred to as I].

Via the geometrical optics interpretation in III, the specularly reflected ray of $O(\alpha^0)$ is the familiar one associated with a constant impedance cylinder

$$G_r^0(\rho,\phi,\phi')=A_0\, R_0\, D\, e^{iks}, \tag{3}$$

while a typical first order grating ray is

$$G_r^1(\rho,\phi,\phi',p)=A_0(p)R_1(p)D(p)e^{iks(p)}. \tag{4}$$

The A_0 term gives the phase of the incident plane wave at the point of reflection, R_0 and R_1 are the reflection coefficients, the D's are the divergence coefficients, and (iks) represents the phase along a reflected ray. We define the nonnegative integer $p=2\pi a/L$, where a is the cylinder radius and L is the spatial period of the sinusoidally varying impedance. From this definition it follows that $p/ka=\lambda/L$ where $k=2\pi/\lambda$ and λ is the wavelength of the incident plane wave. All of these quantities have been previously employed in III, eqs (6), (10), and (12).

If the surface impedance of the cylinder is slowly varying, a consideration of figure 1, and of the grating law (7) of III, shows that almost every point in the lit region $(0\leq\theta_i<\pi/2)$ is reached by the bundle of three reflected rays comprising G_r in (2b). In a typical situation, illustrated in figure 1, the three nearly parallel rays are reflected from the three different points E, F, and G on the cylinder to the same point of observation $P(\rho,\phi)$. Under the local reflection hypothesis, it should be possible to combine these three rays into a single specularly reflected ray $(\theta_r=\theta_i)$ emanating from the point of reflection F in figure 1. This specular ray should have a reflection coefficient associated with a cylinder having a constant impedance equal to $\overline{Z}(\hat{\phi})$ defined by (2) in reference III, where the angle $\hat{\phi}=\phi'\pm\theta_i$, $\theta_\eta\gtrless 0$, is the value of ϕ at the point of specular reflection, F. The value of $\overline{Z}(\hat{\phi})$ remains constant as the observation point P in figure 1 is moved along the ray FP, but changes for other locations of F on the cylinder. Under these conditions the local reflection coefficient $R_l(\theta_i)$ has exactly the same form as R_0 in (3) except for the substitution $Z\rightarrow Z[1+\alpha\cos p(\hat{\phi}-\phi_0)]$. Therefore, we seek to combine the three reflected rays in (2b) into the single, specular ray,

$$G_r(\rho,\phi,\phi')\simeq A_0\, R_l(\theta_i)\, D\, e^{iks}, \tag{5}$$

having a reflection coefficient

$$R_l(\theta_i)=\frac{\cos\theta_i-Z[1+\alpha\cos p(\hat{\phi}-\phi_0)]}{\cos\theta_i+Z[1+\alpha\cos p(\hat{\phi}-\phi_0)]} \tag{6}$$

Since (5) represents the result expected from conventional geometrical optics considerations applied to variable impedance surfaces [Keller, 1956], the conditions assuring its validity can be interpreted as requirements for conventional geometrical optics.

3. Requirements for the Validity of the Local Reflection Principle

Since a detailed derivation of the conditions for the validity of the local reflection hypothesis is somewhat lengthy, only an outline is presented in this section; the complete calculations are available elsewhere [Marcinkowski and Felsen, 1961, henceforth referred to as II]. In the analysis, the geometric optical parameters for the grating rays are expressed in terms of small deviations from the corresponding parameters for the specular ray; for example, the path length GP in figure 1 is given by $s(p)=s+\Delta s(p)$, with G falling below F for $p>0$. Thus, we define

$$\begin{aligned}\theta_i(p)&=\theta_i+\Delta\theta_i(p),\\ \theta_r(p)&=\theta_i+\Delta\theta_r(p),\\ s(p)&=s+\Delta s(p),\end{aligned} \tag{7}$$

and note that $\theta_r=\theta_i$ for the specularly reflected ray.

From the geometry of the problem and the requirements of the grating law (7) in III, it may be shown that

$$|\Delta\theta_i(p)|=O\left(\frac{p}{ka}\right)<<1$$

$$|\Delta\theta_r(p)|=O\left(\frac{p}{ka}\right) \tag{8}$$

uniformly in ρ and ϕ, provided that one imposes the restriction

$$\frac{p}{ka}<<\cos\theta_i\leq 1,\qquad 0\leq\theta_i<\pi/2. \tag{9}$$

Therefore, these approximations break down near the shadow-lit boundary where $\theta_i\rightarrow\pi/2$. (In this same transition region, the formula (2b) also becomes invalid, see III.) Upon utilizing (7), (8), and (9), one may show that to $O(p/ka)$,

$$\Delta\theta_i(p)\simeq\mp\frac{p}{ka\cos\theta_i}\,\frac{s}{2s+a\cos\theta_i},\qquad \theta_\eta\gtrless 0 \tag{10a}$$

$$\Delta\theta_r(p)\simeq\pm\frac{p}{ka\cos\theta_i}\,\frac{s+a\cos\theta_i}{2s+a\cos\theta_i},\qquad \theta_\eta\gtrless 0 \tag{10b}$$

$$\Delta s(p)\simeq-\frac{a\Delta\theta_i(p)\sin^2\theta_i}{\sin\theta_i+\Delta\theta_r(p)\cos\theta_i}, \tag{10c}$$

where $\theta_\eta=\pm\theta_i$ if $\phi-\phi'\gtrless 0$. Since $\Delta\theta_i(+p)\simeq-\Delta\theta_i(-p)$, it is evident that the ray bundle in figure 1 is located symmetrically about the specularly reflected ray $(EF\simeq FG)$. For an observation point $P(\rho,\phi)$ which approaches the surface of the cylinder $(s\rightarrow 0)$, (10) shows that $\Delta\theta_i(p)\rightarrow 0$ and $\Delta s(p)\rightarrow 0$, while $\Delta\theta_r(p)\rightarrow O(p/ka)\neq 0$. Consequently, even for the limiting condition of an observation point arbitrarily close to the surface, the bundle of three

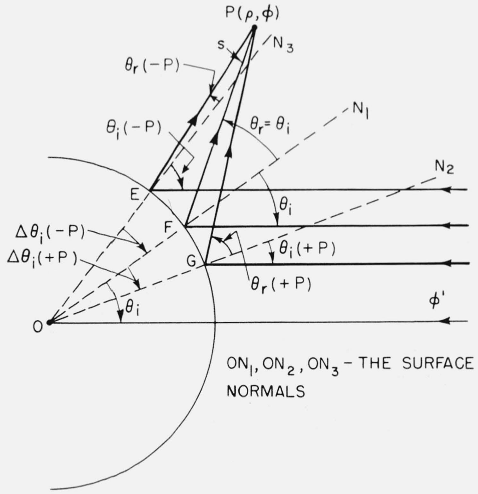

FIGURE 1. *Reflected rays to order α for a slowly varying surface impedance.*

rays has a finite, nonvanishing angular deviation of $O(p/ka)$. This result is a direct consequence of the grating law given by (7) in III.

If the results of (10) are substituted into (6), (7), (12) and (23) in III, all the necessary grating ray parameters A_0, R_1, D, θ_i, θ_r, and s may be evaluated. The calculations show that the fractional changes of the geometric optic parameters, in passing from the specular to the grating rays, are $\Delta s(p)/s \sim \Delta r(p)/r \sim \Delta D(p)/D \sim O(p/ka)$. Use of these estimates and the results of (10) allows the transformation of (2b) into

$$G_r(\rho,\phi,\phi') \simeq A_0 R_0 D e^{iks} + \alpha A_0 R' D e^{iks} \cdot [e^{ip(\hat{\phi}-\phi_0)} e^{if(+p)} + e^{-ip(\hat{\phi}-\phi_0)} e^{if(-p)}] \quad (11)$$

where

$$f(p) \simeq k\Delta s(p) + ka\Delta\theta_i(p) \sin\theta_i \pm p\Delta\theta_i(p), \qquad \theta_\eta \gtrless 0 \quad (12a)$$

$$\simeq ka\Delta\theta_i(p)\left[\frac{\sin\theta_i \cos\theta_i}{\sin\theta_r(p)}\Delta\theta_r(p) \pm \frac{p}{ka}\right] \quad (12b)$$

$$R' = -\frac{Z\cos\theta_i}{(Z+\cos\theta_i)^2}. \quad (12c)$$

The restriction Re $(Z) \geq 0$, already previously implied in the requirement for time-average power flow into the cylinder, has been imposed here to ensure the validity of the power series expansion of the denominator of $R_1(p)$ in (12) of III. In deriving (11), the amplitude of the perturbation parameter α has been explicitly related to the relative periodicity $(p/ka) = (\lambda/L)$ by the neglect of all terms of $O(\alpha^2)$. This requirement implies that

$$\alpha\, O\left(\frac{p}{ka}\right) = O(\alpha^2)$$

or

$$\alpha \sim \frac{p}{ka}. \quad (13)$$

A similar estimate has been utilized by Hessel [1960] for an analogous study on an infinite plane surface with a sinusoidally varying surface impedance. If

$$|f(p)| \leq O\left(\frac{p}{ka}\right), \quad (14)$$

it is readily shown that (11) may be simplified to yield, to $O(\alpha)$, the local reflection formula in (5). We conclude, therefore, that the requirement prescribed by (14) is a necessary condition for the validity of the local reflection hypothesis.

4. Interpretation of Local Reflection Requirements

The contributions to $f(p)$ in (12a) arise from three differential effects which are not always of the same order of importance. The first two terms, of order unity, are much larger than the last term of $O(p/ka)$. The contribution described by the first term $k\Delta s(p)$ is due to the differential phase shift produced by the path difference $\Delta s(p)$ of the different rays reaching the same observation point $P(\rho, \phi)$ in figure 1. The second contribution, described by the $ka\Delta\theta_i(p)\sin\theta_i$ term in (12a), arises from the $A_0(p)$ term which defines the phase of the incident plane wave at the point of reflection. Because of the cylindrical geometry, the phase of the incident plane wave at the three points of reflection E, F, and G in figure 1 is different, thereby introducing a differential phase shift into the reflected rays. The third contribution $\pm p\Delta\theta_i(p)$ in (12a) is due to the angle $\hat{\phi}(p)$ in the exponential of (12) in III which provides the phase shift of the reflection coefficient $R_1(p)$.

A consideration of the geometric optical solution given by (4) and shown in figure 1 indicates that the differential phase shift produced by the path difference tends to cancel the differential phase shift produced by the incident plane wave. This may be verified by making use of (10) in (12a). As a result of this cancellation, the first two terms of order unity in (12a) combine to form the first term of $O(p/ka)$ in (12b). In the very narrow back-scattering region where $\theta_i < p/ka$, this first term in (12b) is of $O(\theta_i)$ and therefore tends to vanish as $\theta_i \rightarrow 0$. This agrees with physical expectations since the increments due to differences in the path lengths and in the phase of the incident plane wave both tend to be smaller in the vicinity of $\theta_i = 0$. (The apparent singularity produced by the vanishing of the denominator does not arise since $\sin\theta_r(p) \rightarrow \Delta\theta_r(p)$ as $\theta_i \rightarrow 0$.) In this region the dominant contribution to $f(p)$ is the phase shift term coming from the reflection coefficient $R_1(p)$. This contribution depends upon the angle $\hat{\phi}$ at the point of reflection but is independent of the asymmetry (with respect to ϕ') produced by the phase angle ϕ_0 in the variable impedance $\bar{Z}(\phi)$.

To obtain an overall picture of the behavior of $f(p)$, it is desirable to make the following simple estimates, valid for most θ_i,

$$\cos\theta_i = O(1)$$

$$\theta_r(p) = O(\theta_i) \tag{15}$$

$$\Delta\theta_r(p) = O\left(\frac{p}{ka}\right).$$

In view of the approximate nature of these estimates, the requirement on $f(p)$ prescribed by (14) seems overly precise and will be replaced by the simpler condition

$$|f(p)| \ll 1. \tag{16}$$

Use of (15) in (12b) permits (16) to be written as

$$L \gg 2\pi L_0, \quad L_0 \geq \lambda/2\pi \tag{17}$$

where we define

$$L_0 = 2a|\Delta\theta_i(p)|. \tag{18}$$

L_0 is the distance EFG in figure 1, measured along the surface of the cylinder, and can therefore be interpreted as the length of the cylindrical surface along which the variable impedance $\overline{Z}(\phi)$ is sampled by the three rays reflected from the points E, F, and G. The inequality (17) states that the local reflection phenomenon obtains if the variable impedance interval L_0 sampled by the diffracted rays is much smaller than the spatial period L. From (8) it is recalled that all the estimates in this section are subject to the basic restriction $p/ka = \lambda/L \ll 1$. Therefore, if the sampling distance $L_0 < \lambda/2\pi$, then (17) becomes ineffective and is replaced by the estimate $L \gg \lambda$. The relation (17) gives one equivalent of the restriction imposed by (16).

Another and perhaps more useful statement may be obtained by substituting the relations for $\Delta\theta_i(p)$ and $\Delta\theta_r(p)$ in (10a) and (10b), and the first two estimates in (15), into (12b). The inequality (16) is thereby transformed into a relation which explicitly involves the wavelength λ of the incident plane wave,

$$L \gg \lambda (ks)^{1/2} \frac{[a(3s+2a)]^{1/2}}{2s+a}. \tag{19}$$

This provides an estimation of the lower bound of the relative periodicity L/λ of the variable impedance in terms of the distance $s = FP$ in figure 1 and the radius of curvature "a" of the cylinder. The important behavior is given by the $(ks)^{1/2}$ term since the remaining terms are slowly varying. If (19) is evaluated for three different ranges of observation points s, one obtains the simpler relations

$$L \gg \lambda (2ks)^{1/2} \qquad \text{if } \frac{\lambda}{4\pi} \leq s \ll a \tag{20a}$$

$$L \gg \lambda \left(\frac{5}{9} ks\right)^{1/2} \simeq \lambda \left(\frac{5}{9} ka\right)^{1/2} \quad \text{if } s \simeq a \tag{20b}$$

$$L \gg \lambda \left(\frac{3}{4} ka\right)^{1/2} \qquad \text{if } s \gg a. \tag{20c}$$

It is seen that L/λ changes slowly from a dependence on the relative distance $(ks)^{1/2}$ in the near field $s \ll a$ to a dependence on the relative size of the cylinder $(ka)^{1/2}$ in the far field $s \geq a$. This behavior is in accord with physical expectations since only the relative distance ks would be expected to play a role near the surface, while the relative curvature $(ka)^{-1}$ would be expected to become important far from the cylinder. If the point of observation is so close to the cylinder that $s < \lambda/4\pi$ then the simple estimate $L \gg \lambda$ replaces the no longer effective estimate in (20a).

The relation (19) involves explicitly the radius of curvature a. In the limit $a \to \infty$, the cylindrical surface transforms into an infinite plane (19) then goes over exactly into (20a), thereby suggesting that this latter estimate should also predict the condition to be imposed on L/λ in order to have the local reflection hypothesis apply on a plane surface. For verification we note that (20a) agrees with a similar result obtained by Hessel [1960] who investigated an infinite plane with a sinusoidally varying surface impedance. This agreement highlights the negligible influence of the curvature of a large cylinder on the reflection phenomenon in the near field.

However, the restrictions for the plane and cylinder differ significantly in the far field ($s >> a$). For the infinite plane, the length L_0, along which the impedance is sampled by the three rays, becomes infinite as s becomes infinite. This follows from the geometry of the infinite plane and the requirements of the grating law. From the geometry it follows that the angle of incidence of each of the three incident rays is the same. From the grating law it follows that the angles of reflection differ by $O(\lambda/L)$. Therefore, as we move a distance s along the specularly reflected ray, the sampling distance L_0 must continually increase so that, as $s \to \infty$, $L_0 \to \infty$ for the infinite plane.

The corresponding limiting behavior for the cylinder is quite different. As a result of the cylindrical curvature, all of the angles of the different incident and reflected rays differ by $O(\lambda/L)$. As we move a distance s along a specularly reflected ray, it is only necessary that there be an angular deviation of the angles of incidence and reflection of $O(\lambda/L)$ to change the direction of a grating ray by the amount necessary to satisfy the grating law and pass through the point $P(\rho, \phi)$ in figure 1. This angular deviation is easily secured by changing the point of reflection by the angular amount $\Delta\theta_i(p) = O(\lambda/L)$ which results in only a small fractional change on the circumference of the cylinder. Therefore, as $s \to \infty$, the sampling distance for the cylinder $L_0 \to aO(\lambda/L)$ which is a finite number as verified upon substituting the expression for $\Delta\theta_i(p)$ from (10a) into (18). This difference in limiting behavior for the infinite plane and the cylinder is expressed mathematically by the different expressions (20a) and (20c). Evidently, if $s >> a$, the upper bound required by (20a) for the infinite plane is much larger than that required by (20c) for the cylinder. Therefore, far away from the surface, much more rapid changes in surface impedance are permissible for the cylinder than for the infinite plane, within the confines of the local reflection hypothesis. This is a direct result of the much smaller sampling distance L_0 for the cylinder than for the infinite plane.

It has been noted that as s varies from $s << a$ to $s >> a$, the lower bound on L/λ given by (20) changes from $(ks)^{1/2}$ to $(ka)^{1/2}$ in a transition region where $s \sim a$. It is interesting to obtain the value s_0 of s in this transition region in terms of parameters which do not involve the period L. Since (17) and (20) represent the same phenomenon, the lower bounds on L given by (17) and (20b) must be of the same order of magnitude. This provides the estimate

$$s_0 \sim 4\pi \frac{L_0^2}{\lambda}. \tag{21}$$

The expression for s_0 in terms of L_0 and λ has the same functional form as that for the distance from an aperture (having a diameter L_0) to the region wherein the radiation pattern undergoes a transition from the (near-field) Fresnel to the (far-field) Fraunhofer type (a standard estimate in this case is $s_0 \sim 2L_0^2/\lambda$).

If the sampling distance $L_0 < \lambda/2\pi$ or if the length of the reflected ray $s < \lambda/4\pi$ then L_0 and s can no longer provide significant estimates of the requirements for the validity of the local reflection phenomenon. Under these conditions L_0 and s are replaced by the wavelength λ in the single estimate $L >> \lambda$ wherein the wavelength itself provides the restriction on L because of (8) and (9).

The rays reflected from the points E, F, and G in figure 1 also have a certain resemblance to rays propagating through a random, inhomogeneous medium. If a geometric optical approximation is to be valid for such propagating rays it would be reasonable to expect that there should be certain restrictions on the extent of the inhomogeneities. In making a comparison of these restrictions with our results, the distance of propagation in the medium is analogous to s while the range of the inhomogeneities is analogous to L. Chernov [1960] has considered the conditions under which a geometric optical approximation is valid in a medium with spatially random inhomogeneities. It is interesting to observe that Chernov gives the conditions $\lambda << L$ and $\sqrt{\lambda s} << L$ as necessary conditions for the validity of such an approximation. The first requirement is precisely the basic approximation $p/ka << 1$ introduced in (9) and used in (14). The second requirement is given by (20a) if we set all numerical factors equal to unity.

5. Extension to More General Types of Impedance Variation

Although the investigation herein is concerned with a cylinder having a periodic surface impedance variation, the effect of periodicity is obscured when $p/ka \ll 1$. Suppose that $p=1$ with $ka \gg 1$. In this instance the surface impedance as expressed by (2) in III goes through only one spatial period around the periphery of the cylinder. Viewed locally, one detects in any limited region of the cylinder surface an impedance variation devoid of any aspects of periodicity. It is reasonable to suppose, therefore, that the preceding restrictions delimiting the validity of the local reflection principle of geometrical optics, if properly rephrased, might apply as well to surface impedances with arbitrary, slow variations on a gently curved surface. The desired condition for local reflection should be phrased as a limitation on the allowed rate of impedance variation. Hence, we first determine the maximum relative rate of change of impedance occurring for the sinusoidal impedance variation $\overline{Z}(\phi)$, for which the geometrical optics approximation retains its validity subject to restriction (19).

Use of (2) in III for $\overline{Z}(\phi)$ and of the relation $\alpha = O(\lambda/L)$ in (13) yields

$$\left| \frac{\frac{1}{k}\frac{dz(x)}{dx}}{z(x)} \right|_m = O\left(\frac{\lambda}{L}\right)^2, \quad \frac{\lambda}{L} \ll 1, \tag{22}$$

where $z(x)$ is the variable surface impedance and $x = a\phi$ is the distance measured along the surface; the subscript m denotes the maximum value. We now seek to express (λ/L) in a manner which makes no reference to the sinusoidal impedance variation from which it has been derived. One such formulation utilizes the inequality in (19) which, when substituted into (22), leads to

$$\left| \frac{\lambda \frac{dz(x)}{dx}}{z(x)} \right| \ll \frac{\lambda}{s} \frac{(2s+a)^2}{a(3s+2a)}, \tag{23}$$

$$s \geq \frac{\lambda}{4\pi}.$$

The expression on the left hand side of this estimate, representing the relative impedance variation in an interval of one wavelength, is compared with quantities on the right hand side which contain the geometrical distance s along the reflected ray and the radius of curvature a at the point of reflection. Hence, this formulation of the restriction required for the validity of the conventional optics approximation is phrased in a manner independent of any specific functional variation of $z(x)$ and might therefore be expected to apply to an arbitrary, slowly changing surface impedance on a gently curved surface. The restriction $s \geq \lambda/4\pi$ follows from the same restriction in (20a). For $s<\lambda/4\pi$ (23) is superseded by (22) in which the replacement $O(\lambda/L)^2 \rightarrow << 1$ is made for reasons analogous to those discussed in the previous section. It should be emphasized that $z(x)$ is assumed finite in this expression and that the inequality becomes meaningless as $z \rightarrow 0, \infty$. As before it is of interest to examine (23) for various ranges of s/a. If we omit all numerical factors of order unity, then (23) becomes

$$\left| \frac{\lambda \frac{dz(x)}{dx}}{z(x)} \right| \ll \frac{\lambda}{s} \quad \text{if } \frac{\lambda}{4\pi} \leq s \ll a \tag{24a}$$

$$\ll \frac{\lambda}{a} \quad \text{if } \quad s \geq a \tag{24b}$$

Evidently the allowed impedance variation is more rapid for $s \ll a$ than for $s \geq a$. The reason for this behavior is connected with the negligible influence of surface curvature in the near field and its appreciable effect in the far field.

The limited applicability of the local reflection hypothesis of geometrical optics to variable impedance surfaces stems from the fact that the reflected field is influenced not only by the surface properties *at*, but also in the vicinity of, the specular reflection point. For periodic variations, the diffracted rays provide a means of "sampling" the nature of the surface in the neighborhood of the point of reflection. For a surface with monotonic impedance variation, it is perhaps suggestive to construct a quasi-periodic equivalent for the region under consideration and to use the associated diffracted rays to sample the surface properties near the specular reflection point.

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(Paper 66D6–230)