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On the Geometrical Optics of Curved Surfaces With Periodic Impedance Properties [1]

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In a previous publication, a two-dimensional Green's function has been derived for a circular cylinder whose surface impedance around the periphery deviates from a constant value by a sinusoidal variation of small amplitude α. Here, this solution is evaluated asymptotically in the illuminated region under the assumption that the cylinder radius is large compared with the wavelength of the incident field. The asymptotic result is interpreted in terms of geometrical optics generalized to apply to cylindrically curved, convex reflection gratings, and comprises the first and higher-order diffracted rays associated with a plane grating, together with geometrical divergence coefficients accounting for the surface curvature. General properties of the spectrum of reflected rays are observed. The behavior of the first-order diffracted rays, and in particular their domain of existence as a function of surface periodicity, is discussed in detail.

1. Introduction

In the theory of diffraction at high frequencies, a lowest-order approximation to the scattered field solution can be constructed by the methods of geometrical optics. If a plane wave represented by a continuum of parallel rays falls on a perfectly conducting, gently curved convex object, the reflected geometrical optics field comprises rays which emerge from the obstacle surface at the angle of incidence; a reflected ray lies in the plane of incidence and has an amplitude along the ray direction governed by a divergence coefficient which accounts for the spreading, due to the surface curvature, of a narrow reflected ray bundle. If the surface properties are characterized by a constant surface impedance Z_s, the geometrical optics approximation assumes that the reflected ray amplitude is, in addition, determined by a reflection coefficient which is the same as for an infinite plane surface with impedance Z_s tangent to the object at the reflection point. To verify the validity of these approximations for a surface with constant curvature, one may examine the rigorous solution to the problem of diffraction by a constant impedance cylinder in the high-frequency limit [Wait, 1959]. If the surface properties of the scatterer are not constant but are characterized by a spatially varying surface impedance Z_s, geometrical optics predicts that the scattered field can be constructed as before provided that one now employs the appropriate impedance at the specular reflection point in calculating the ray reflection coefficient [Keller, 1956]. Implicit in all geometric-optical approximations is the requirement that any variations in surface characteristics—whether changes in the radii of curvature, or in the local impedance properties—are "small" within an interval of length equal to the wavelength of the incident field.

To gain a deeper insight into the simultaneous influence of surface curvature and variability of surface properties on the field diffracted by an object at high frequencies, the authors have studied the problem of diffraction by a large circular cylinder whose surface impedance around the periphery deviates from a constant value by a sinusoidal variation of small amplitude α (see (2)). Apart from the relative simplicity of the mathematical analysis leading to a solution of this problem, the sinusoidal variation is highly desirable since an appropriate choice of the periodicity permits the simulation of either slow monotonic changes, or of rapid fluctuations. In the former range, a conventional geometric-optical interpretation of the dominant portion of the high-frequency diffracted field is expected to apply. Its domain of validity as a function of both the surface curvature and the rapidity of impedance variation can be assessed by studying the high-frequency asymptotic solution for arbitrary values of periodicity. In this manner, one may arrive at estimates for the magnitudes of the "small" terms mentioned at the end of the preceding paragraph, which delimit the range of applicability of geometrical optics.

The above discussion suggests a division of the asymptotic analysis of the rigorous solution into two parts: (a) the evaluation of the high-frequency fields for arbitrary periodicity of the surface impedance, and (b) the transformation of this solution into the previously mentioned geometric-optical form when

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the periodicity is very slow. Part (a) of this program is carried out in the present paper wherein it is shown that the asymptotic solution of the scattered field is interpretable via the geometrical optics of a curved, convex reflection grating, and that it can be constructed in terms of the specular and diffracted rays familiar from the theory of the plane grating, modified by geometrical divergence coefficients to account for the surface curvature (sec. 2). An interesting feature is the dependence of the domain of existence of each diffracted ray on both the periodicity of the surface impedance and on the radius of curvature, as discussed in section 4. Phase (b) of the investigation, presented in the companion paper, is concerned with the case where the period of the surface impedance variation is very large compared to the incident wavelength, thereby obscuring the periodic aspect of the surface properties. As noted above, one then expects the reflected field to be derivable from geometric-optical considerations applied to surfaces with locally constant characteristics. The reducibility of the first-order asymptotic solution to the geometric-optical formula deduced from a "local reflection" argument can thus be employed to furnish some bounds on the range of applicability of conventional geometrical optics when used in connection with gently curved surfaces possessing a variable surface impedance.

The formal solution to the diffraction problem mentioned above [Felsen and Marcinkowski, 1962, and Marcinkowski and Felsen, 1961. These references will henceforth be called I and II, respectively. A related study for a slightly noncircular cylinder has been carried out by Clemmow and Weston, 1961] has been obtained by a Green's function procedure wherein the problem of determining the amplitudes in a harmonic series expansion of the scattered field G_s is reduced to that of solving an inhomogeneous, second-order difference equation with variable coefficients. The solution of the latter has been found by assuming the expansibility of G_s as a power series in the small perturbation parameter α. While we discuss in detail only the first-order perturbation (sec. 3 and appendix), the form of the asymptotic solution is readily ascertained to any order in α and leads to the expressions in section 2. Stress is placed throughout on the physical interpretation, rather than the detailed derivation, of the mathematical results; the interested reader will find additional information in reference II.

The physical configuration and choice of coordinates are shown in figure 1. While the circular cylindrical coordinates (ρ, ϕ, z) are employed in the derivation of the formal solution and its asymptotic representation, the coordinates θ_i, θ_r, and s are convenient for the geometric-optical interpretation of the result. A plane wave of wavelength λ with a magnetic field H_z along the positive z axis is incident on the cylinder along the ray OQ identified by the angle ϕ'. The field reflected by the cylinder is observed in the (ρ, ϕ) plane at the point $P(\rho, \phi)$. On the surface of the cylinder $(\rho=a)$, the magnetic field $H_z(\rho, \phi, \phi')$ is required to satisfy the boundary

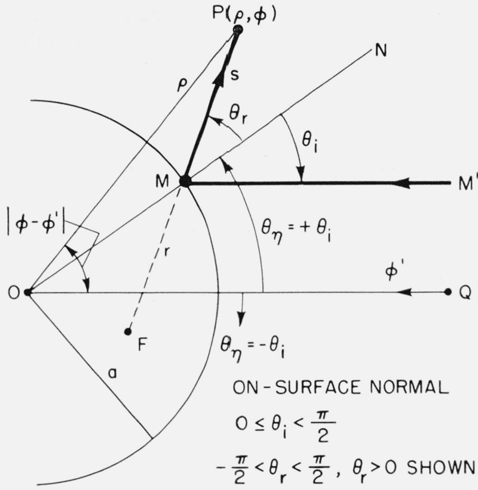

FIGURE 1. *The geometric optic parameters.*

condition

$$\left.\frac{\dfrac{1}{i\omega\epsilon}\dfrac{\partial H_z(\rho,\phi,\phi')}{\partial\rho}}{H_z(\rho,\phi,\phi')}\right|_{\rho=a}=-\overline{Z}(\phi), \tag{1}$$

with the variable surface impedance $\overline{Z}(\phi)$ given by

$$\overline{Z}(\phi)=Z\left[\frac{\mu}{\epsilon}\right]^{1/2}[1+\alpha\cos p(\phi-\phi_0)]. \tag{2}$$

ϵ and μ are the permittivity and permeability of the medium, and Z is an arbitrary complex number. For a passive surface impedance, the restriction Re $\overline{Z}(\phi)\geq 0$ is imposed. In (2), a small, sinusoidal variation with complex amplitude α $(0\leq|\alpha|<<1)$ is superimposed upon the constant impedance $Z(\mu/\epsilon)^{1/2}$. The periodicity of the surface properties is described by a finite, positive integer $p=2\pi a/L$, where L is the spatial period of the impedance variation. This provides the useful parameter $p/ka=\lambda/L$ involving the propagation constant $k=2\pi/\lambda$. For the sake of convenience the angle of observation ϕ and the arbitrary real phase angle ϕ_0 are restricted to the ranges $-\pi<(\phi-\phi')<\pi$ and $-\pi\leq(\phi_0-\phi')\leq\pi$. The units are rationalized MKS, and a time dependence $\exp(-i\omega t)$, where i is the imaginary unit, ω the angular frequency, and t the time, is suppressed. As mentioned previously, the formal solution of this boundary value problem has been presented in references I and II. We proceed now to its asymptotic evaluation in the illuminated region of the cylinder.

2. Spectrum of Reflected Grating Rays

The total axial magnetic field H_z in the illuminated region may be decomposed into incident and reflected contributions,

$$G(\rho, \phi, \phi')=G_i(\rho, \phi, \phi')+G_r(\rho, \phi, \phi'), \tag{3}$$

where the subscripts i and r identify the incident and reflected parts, respectively. The incident plane wave is given by

$$G_i(\rho, \phi, \phi') = e^{-ik\rho \cos(\phi-\phi')}, \; 0 \leq \theta_i < \pi/2. \tag{4}$$

A detailed asymptotic evaluation of G_r by the saddle point method has been carried out to $O(\alpha)$ and is summarized in the appendix and in section 3. However, the main properties of the asymptotic solution to any order in α can be inferred from the first-order results by repeated application of a recursion relation derived in references I and II. These more general relations are presented in this section. Their interpretation in geometric-optical terms is facilitated by the use of the geometric-optical coordinates θ_i, θ_r, s in figure 1.

θ_i is the angle between the incident ray $M'M$ and the normal ON to the cylindrical surface at the point of reflection $M (0 \leq \theta_i < \pi/2)$. θ_r is the angle between the reflected ray MP and the same normal ON $(-\pi/2 < \theta_r < \pi/2)$. The coordinate s is the straight-line distance along a reflected ray MP from the point of reflection M to the point of observation P. To permit a simple interpretation of the asymptotic results, it is necessary to admit nonspecularly reflected rays so that the angle of reflection θ_r need not be equal to the angle of incidence θ_i; this generalization of conventional geometrical optics allows the inclusion of the diffracted rays associated with a periodic structure or grating. θ_r is considered positive when the reflected ray lies on the side of the normal ON opposite the incident ray as shown in figure 1; when both incident and reflected rays lie on the same side of ON then θ_r is considered negative. If $\theta_r < -\theta_i$ and s is sufficiently large, the observation point $P(\rho, \phi)$ may lie below the line OQ. Under these conditions the ray MP is reflected from a point M on the cylinder where $\phi-\phi'>0$ and is observed at a point P in space where $\phi-\phi'<0$.

With these definitions, the reflected field in the illuminated region may be shown to comprise a spectrum of reflected waves determined as follows:

$$G_r(\rho, \phi, \phi') = \sum_{n=-n_1}^{n_2} A_0(np) \mathscr{R}_n(p) D(np) e^{iks(np)} \tag{5}$$

where

$$A_0(np) = e^{-ika \cos \theta_i(np)}$$

$$\mathscr{R}_n(p) = \sum_{m=0}^{\infty} R(n, m, p)\, \alpha^{2m+|n|}, \; m=0, 1, 2, \ldots$$

$$D(np) = \left[1 + \frac{s(np)}{r(np)}\right]^{-1/2}$$

$$r(np) = a \frac{\cos \theta_r(np)}{1 + \dfrac{d\theta_r(np)}{d\theta_i(np)}}, \tag{6}$$

and

$$\sin \theta_r(np) = \sin \theta_i(np) \pm n \frac{\lambda}{L}, \quad \theta_\eta \gtrless 0, \tag{7}$$

with $0 \leq \theta_i(np) < \pi/2$ and $-\pi/2 < \theta_r(np) < \pi/2$.

Each reflected ray associated with each integer n will in general emerge from a different point M in figure 1 [with a different $\theta_i(np)$] to reach the same, prescribed observation point P. The exponent in the term $A_0(np)$ gives the phase of the incident plane wave at each reflection point, while $\mathscr{R}_n(p)$ gives the reflection coefficient associated with each reflected ray. It is well known that a parallel bundle of rays incident on a cylinder produces a divergent bundle of reflected rays [Bremmer, 1949 and Keller, 1956]. These rays appear to come from a point located by the focal length $r(np)$ which is measured along that part of the segment MP in figure 1 extended behind the reflection point M $(r(np)=MF)$. The resulting decrease in amplitude along a reflected ray is described by the divergence coefficient $D(np)$. $D(np)$ and $r(np)$ above are generalizations of analogous geometric optical parameters for the rays reflected specularly from a cylindrical surface with constant impedance. In the derivation of these parameters from purely geometrical considerations, the cylindrical surface and its tangent have been assumed to be continuous functions of ρ and ϕ, and $\theta_r(np)$ and its first derivative have both been assumed to be continuous functions of $\theta_i(np)$. The term $ks(np)$ in the exponential gives the phase of a reflected ray propagating from the point of reflection M to the observation point $P(\rho, \phi)$. The saddle point evaluations show that the angles of incidence and reflection for an nth order reflected ray are related by the well-known grating law in (7), familiar from the theory of a plane grating. The angle θ_η in (7) is the angle NOQ which the normal ON in figure 1 makes with the incident ray $\phi'(|\theta_\eta|=\theta_i(np),\ \theta_\eta>0$ in fig. 1). The explicit form of $A_0(np)$, $D(np)$ and $r(np)$ in (5) and (6) is obtained easily by comparing the integral expressions for the higher-order (in α) reflected contributions (see ref. II) with the $O(\alpha)$ expressions in the appendix. For the explicit evaluation of the reflection coefficients $\mathscr{R}_n(p)$, however, a detailed asymptotic evaluation of the higher order integral contributions is required and is not included here.

From (5) it is evident that a finite number of reflected rays reaches a prescribed point $P(\rho, \phi)$. The orders of the reflected rays range from $-n_1$ to $+n_2$, where n_1 and n_2 are nonnegative integers which, in general, are not equal. They are the largest values of n for which (7) can be satisfied subject to the indicated restrictions on $\theta_i(np)$ and $\theta_r(np)$. The perturbation procedure produces reflection coefficients $\mathscr{R}_n(p)$ which are expressed as power series in α involving terms of the form $\alpha^{2m+|n|}$. The coefficients of these series, and therefore the reflection coefficients $\mathscr{R}_n(p)$, are independent of the radius of the cylinder. Therefore, it is reasonable to expect that these reflection coefficients will be identical with those obtained for a plane surface with the same sinusoidally modulated surface impedance, where the plane surface is taken to be tangential to the cylinder at the point of reflection. This supposition has been verified to $O(\alpha^1)$ by comparing our reflection coefficient in (12) with that obtained by Hessel [1960] for the plane, sinusoidally modulated impedance surface. For an nth order reflected ray ($n \geq 0$), the leading term of the reflection coefficient $\mathscr{R}_n(p)$ is of $O(\alpha^n)$ in the perturbation parameter. For any prescribed $O(\alpha^n)$ there are a total of $n+1$ rays of $O(\alpha^n)$, a result which may be derived from the recursion relation appearing in the solution (ref. I).

It is significant to emphasize that the asymptotic reflected field representation in (5), expressed in terms of the geometric-optical coordinates (θ_i, θ_r, s), depends only on the local properties of the surface in the vicinity of the point of reflection of a given ray. Hence it appears reasonable to assume that the dominant contribution to the reflected field in the illuminated region of a general, gently curved cylindrical convex surface with periodic impedance properties can be constructed as in (5), provided that the constant cylinder radius "a", appearing in the focal length $r(np)$ and hence in the divergence coefficient $D(np)$, is replaced by the radius of curvature of the surface at the point of reflection of a given ray. For curved surfaces with nonperiodic, slowly varying properties, the above assumption was put forth by Keller [1956] and verified by Keller and others [Keller, Lewis, and Seckler, 1956] for various specific obstacle configurations, in which the reflected field comprises the usual specularly reflected rays only. The results herein suggest a direct extension to include as well the nonspecularly reflected rays associated with periodic surface properties.

3. Reflected Rays to Order α^1

To $O(\alpha^1)$ in the perturbation, the reflected fields in the illuminated region consist of three rays whose properties are described here:

$$G_r(\rho,\phi,\phi') = G_r^0(\rho,\phi,\phi') + \alpha[G_r^1(\rho,\phi,\phi',p) + G_r^1(\rho,\phi,\phi',-p)]. \qquad (8)$$

In terms of the geometric optic parameters, the reflected field to $O(\alpha^\circ)$ is given asymptotically by:

$$G_r^0(\rho,\phi,\phi') = A_0 R_0 D\, e^{iks}, \qquad 0 \leq \theta_i < \frac{\pi}{2}, \qquad (9)$$

with a reflection coefficient

$$R_0 = \frac{\cos\theta_i - Z}{\cos\theta_i + Z}. \qquad (10)$$

For this specularly reflected ray ($n=0$), the simpler notation $\theta_i(np) \to \theta_i$ is used, together with analogous simplifications for the other relevant parameters: $\theta_r = \theta_i$ and $d\theta_r/d\theta_i = 1$. Any point $P(\rho, \phi)$ in the lit region may be reached by a specularly reflected ray for which $0 \leq \theta_i < \pi/2$. These results are, of course, well-known from the theory of scattering by a constant impedance cylinder [Imai, 1954; Franz, 1957, 1954; Franz and Beckmann, 1956; Keller, 1956].

The corresponding solution of $O(\alpha^1)$ in (8) comprises two reflected waves for which $n = \pm 1$. The field for $n = +1$ is given asymptotically by (see appendix)

$$G_r^1(\rho,\phi,\phi',p) = A_0(p) R_1(p) D(p)\, e^{iks(p)}, \qquad (11)$$

where the reflection coefficient is found to be

$$R_1(p) = \frac{-Z\cos\theta_i(p)\, e^{ip(\hat{\phi}(p)-\phi_0)}}{[Z+\cos\theta_i(p)][Z+\cos\theta_r(p)]}. \qquad (12)$$

The angle at the point of reflection is $\hat{\phi}(p) = \phi' \pm \theta_i(p)$ for $\theta_\eta \gtrless 0$. Each of the two reflected rays of $O(\alpha^1)$ in (8) satisfies the grating law given by (7), with $n = \pm 1$. From the grating law it follows that the ray geometry of these two reflected rays is invariant to the simultaneous substitutions $p \to -p$ and $\theta_\eta \to -\theta_\eta$. As a result, the ray geometry is symmetrical about the incident ray ϕ' in figure 1 regardless of the phase angle ϕ_0 in (2) provided that a ray with the index $n=+1(-1)$ for $\theta_\eta > 0$ is compared with a ray of index $n=-1(+1)$ for $\theta_\eta < 0$, for the same angle of incidence $\theta_i(p) = |\theta_\eta|$. However, the symmetry of the reflected fields is, in general, different from the symmetry of the corresponding ray geometry due to the presence of the reflection coefficient $R_1(p)$, given to $O(\alpha^1)$ by (12). This coefficient depends upon the phase angle ϕ_0 through the term $\exp[ip(\hat{\phi}(p) - \phi_0)] = \exp[ip(\phi' - \phi_0 \pm \theta_i(p))]$ for $\theta_\eta \gtrless 0$. If ϕ_0 has the special values $p(\phi' - \phi_0) = \pm m\pi$, where $m=0$ or a positive finite integer, then both the ray geometry and the reflected fields are symmetrical about the incident ray ϕ'. This symmetry property follows directly from the symmetry of the variable surface impedance prescribed by (2). To $O(\alpha^1)$ in the perturbation and asymptotically in ka for $ka >> 1$, it is easy to verify that (8) satisfies the variable impedance boundary condition in (1).

If the periodicity parameter p is small or if the angle of incidence $\theta_i(p)$ is large, the angle of reflection for both grating rays in (7) will be positive ($\theta_r(\pm p) > 0$). With respect to the surface normal ON in figure 1, reflected rays of this kind lie on the side opposite the incident ray. This situation will occur simultaneously for both rays $n = \pm 1$ in (7). This is illustrated in Regions A and D of figure 2, where the three reflected rays to $O(\alpha^1)$ in the perturbation have been drawn in four different situations which can arise in the lit region. On the other hand, if the periodicity parameter p is large or if the angle of incidence $\theta_i(p)$ is small, the situation may arise where $\theta_r(-p) < 0$ for $\theta_\eta \geq 0$. This is illustrated in figure 2 for ray 3 in Region B and ray 2 in Region C. Let us consider the behavior of ray 3 as the surface normal ON passing through the point of reflection moves from Region A to B. The angle of reflection $\theta_r(-p)$ changes continuously from a positive value in Region A to a negative value in Region B. When the surface normal ON lies on the dashed line which bounds Regions A and B, then $\theta_r(-p) = 0$. Under this special condition the reflected ray lies along the surface normal ON. Similar conclusions hold for ray 2 in Regions C and D. From (7), the angle of incidence $\theta_i(p) = \theta_0$ appropriate to the special

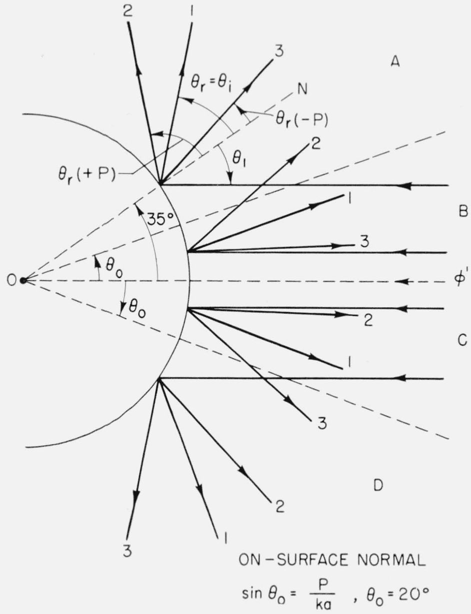

FIGURE 2. *Reflected rays to order α.*

limiting condition $\theta_r(\pm p)=0$ is given by the equation

$$\sin\theta_0=\frac{\lambda}{L}. \tag{13}$$

Figure 2 has been drawn to scale for $\theta_0=20°$.

Let us consider an angle of incidence such that all three reflected rays are in Region A. As the normal ON in figure 2 is varied continuously from Region A to D, all three rays change their position continuously. It is evident that the ray geometry is symmetric about the incident ray ϕ' providing the roles of rays 2 and 3 in figure 2 are reversed upon reflecting about ϕ'. If $p=0$, the three rays coalesce into one specularly reflected ray ($\theta_r=\theta_i$) and Regions B and C disappear. For this special condition, a single reflected ray is obtained from (8) by adding the three reflected rays into one specularly reflected ray. To $O(\alpha^1)$ it is easy to verify that the resulting reflected ray gives the expected asymptotic solution for $ka>>1$ for a constant impedance cylinder with the surface impedance $Z'=Z(\mu/\epsilon)^{1/2}(1+\alpha)$.

4. Domains of Existence of the Reflected Rays of Order α^1

The domains of existence of the reflected grating rays are those illuminated or "lit" regions wherein we find geometric-optical field contributions comprising real rays. For these regions, real angles of incidence and reflection exist which simultaneously satisfy the grating law given by (7), and the associated limitations on $\theta_i(np)$ and $\theta_r(np)$. In view of these limitations, the lit regions for the grating rays will depend upon λ/L and may be considerably smaller than the lit region for specularly reflected rays. For the two reflected rays of order α^1 this

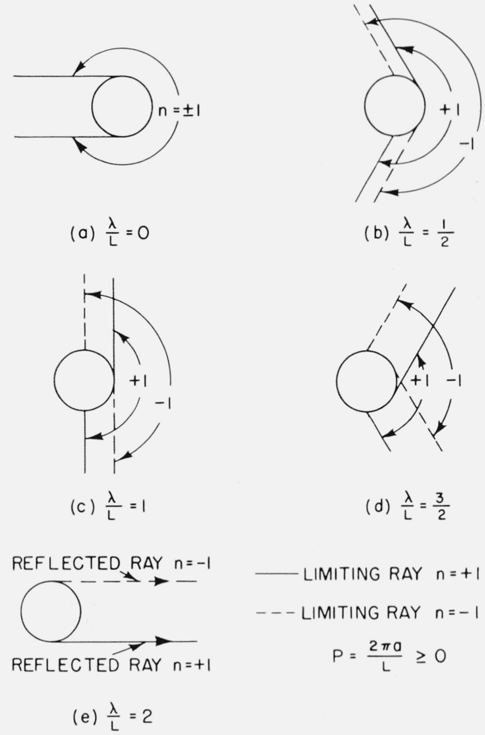

FIGURE 3. *The domains of existence of the real reflected grating rays* n=±1.

dependence is examined in this section as a function of λ/L, the relative periodicity of the variable surface impedance. The corresponding properties of the higher order grating rays may be obtained by appropriate generalizations of these simpler results.

Each of the two grating rays $n=\pm 1$ has its own lit region. As described previously in connection with figure 2, the region illuminated by the ray $n=+1$ is the mirror image of the region illuminated by the ray $n=-1$ reflected about the incident ray ϕ'. The ranges of the lit regions associated with the two reflected rays $n=\pm 1$ are shown in figure 3 for $\lambda/L=0$, 1/2, 3/2, and 2. (It should be noted that the saddle point evaluations, and therefore the asymptotic solutions herein, are not valid on the limiting rays or in the transition regions surrounding the limiting rays.)

For the constant impedance cylinder, $\lambda/L=0$. For this special condition, both rays $n=\pm 1$ have the same lit region as the specularly reflected ray $n=0$ (fig. 3a). As λ/L increases from $\lambda/L=0$ to $\lambda/L=2$, the extent of the lit regions associated with each ray progressively decreases as shown in figures 3a to 3e. When $\lambda/L=2$, each of the two lit regions degenerates into a single line as shown in figure 3e and the associated reflection coefficient for each ray vanishes from (12). If $\lambda/L>2$, the grating law cannot be satisfied for real angles, and the lit regions for the rays $n=\pm 1$ disappear.

For each ray family $n=\pm 1$ there are two characteristically different limiting rays which define the shadow-lit boundaries. For one limiting ray, the incident ray always arrives at grazing incidence, $\theta_i(\pm p)=\pi/2$. Since the angle of incidence is fixed, the corresponding reflected ray always emerges from the same point on the cylinder with an angle of reflection which varies with λ/L in the manner prescribed by the grating law. There is one limiting reflected ray of this type for each of the two ray families $n=\pm 1$ as shown in figure 3. For the other limiting ray, the roles of the angles of incidence and reflection are reversed and the reflected ray always leaves the cylinder surface at a grazing angle (grazing reflection). In this instance the angle of incidence, and therefore the point of reflection, varies with λ/L according to the grating law as shown in figure 3. There is one limiting ray of this type for each ray family $n=\pm 1$.

The domains of existence of the reflected rays $n=\pm 1$ described above can be understood from a consideration of the first-order diffracted rays on a plane reflection grating which are governed by the grating law in (7). In fact, the various regions depicted in figure 3 are precisely those which would be obtained for a plane grating of the same relative periodicity λ/L if the incident wave sweeps out the entire range from normal to grazing incidence. Consider, for example, the case $\lambda/L=3/2$ in figure 3d whence for the ray $n=-1$, with $\theta_\eta>0$, $\sin\theta_r=\sin\theta_i-3/2$. The diffracted ray does not emerge from the plane of the grating until $\sin\theta_i\geq 1/2$; for the maximum angle of incidence (i.e., grazing incidence) $\sin\theta_i=1$, whence $\sin\theta_r=-1/2$. Thus, the domain of existence of the first order diffracted ray is the angular region $\pi/2>-\theta_r>\pi/6$ (see fig. 4), with $-\theta_r=\pi/2$ and $-\theta_r=\pi/6$ corresponding to $\theta_i=\pi/6$ and $\theta_i=\pi/2$, respectively. This illuminated region for the plane grating is exactly equivalent to that shown for the ray $n=-1$ in figure 3d, if the curved periodic surface is approximated at the point of reflection by a plane surface. This is in accord with the previously noted quasi-optic expectation ($ka\gg 1$) that the field at any point in the illuminated region of a curved surface with periodic properties can be constructed from the geometric-optic results for a plane surface wherein diffracted rays arising due to the periodicity are included. The effect of surface curvature is accounted for by a divergence coefficient. This discussion of the domains of existence of the illuminated regions has been restricted to the two grating rays arising in the perturbation solution to $O(\alpha^1)$. Nevertheless, the conclusions obtained here may be easily extended to the higher order rays by making the substitution $\lambda/L\rightarrow n\lambda/L$ for an nth order ray. Consequently, if $\lambda/L=1/2$, then figure 3 may also be interpreted as representing the extent of the illuminated regions for the specular rays and for the first four orders of the grating rays ($n=1, 2, 3, 4$). In this case, no real diffracted rays exist for $n>4$. The regions outside those shown illuminated in figure 3 represent "grating shadow" regions whose properties remain to be investigated.

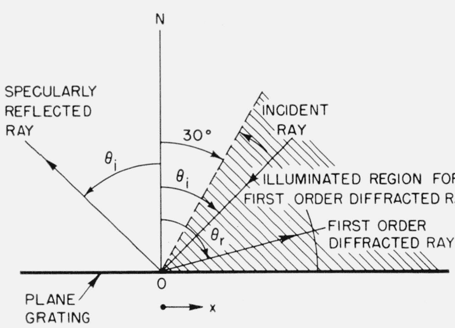

FIGURE 4. *The plane grating for* $\lambda/L=3/2$.

5. Appendix

This appendix contains a summary[2] of the derivation of the geometric-optical formula (11) from the rigorous integral expression for the reflected wave G_r^1, given by (see ref. II)

$$G_r^1(\rho,\phi,\phi',p)=\frac{Z}{\pi ka}e^{ip\left(\phi'-\phi_0-\frac{\pi}{2}\right)}\int_{-\infty}^{+\infty}\frac{e^{i\nu\left(\phi'-\phi-\frac{\pi}{2}\right)}H_\nu(k\rho)}{a(\nu)\cdot a(\nu+p)}\,d\nu \quad (14)$$

where

$$a(\nu)=H_\nu'(ka)+iZH_\nu(ka).$$

$H\nu$ indicates a Hankel function of the first kind and the prime denotes differentiation with respect to the argument. If the appropriate large argument asymptotic approximations are substituted for the Hankel functions, then this integral becomes

$$G_r^1(\rho,\phi,\phi',p)=\int_c Q(\nu,p)\,e^{ik\rho f(\nu,p)}\,d\nu \quad (15)$$

where

$$f(\nu,p)=\left(\phi'-\phi-\frac{\pi}{2}-\gamma\right)\cos\gamma+\sin\gamma$$
$$-\frac{a}{\rho}(\sin\gamma_1-\gamma_1\cos\gamma_1)-\frac{a}{\rho}(\sin\gamma_2-\gamma_2\cos\gamma_2) \quad (16)$$

$$Q(\nu,p)=\frac{-Ze^{ip\left(\phi'-\phi_0-\frac{\pi}{2}\right)+i\frac{\pi}{4}}}{(Z+\sin\gamma_1)(Z+\sin\gamma_2)}\left[\frac{\sin\gamma_1\sin\gamma_2}{2\pi k\rho\sin\gamma}\right]^{1/2} \quad (17)$$

$$\cos\gamma=\frac{\nu}{k\rho},\qquad \cos\gamma_2=\frac{\nu+p}{ka}$$

$$\cos\gamma_1=\frac{\nu}{ka},\qquad 0<\mathrm{Re}\,(\gamma,\gamma_1\text{ or }\gamma_2)<\pi. \quad (18)$$

The contour c runs from $-\infty$ to $+\infty$ passing above the real ν axis for $\mathrm{Re}(\nu)<-ka$ and below for $\mathrm{Re}(\nu)>ka$. The integrand in (15) contains a rapidly varying exponential term and the slowly varying function $Q(\nu,p)$. Therefore, it is in a suitable form for an asymptotic evaluation by the saddle point method when $k\rho\geq ka>>1$ (for simplicity it

[2] Details of the evaluation can be found in reference II.

is assumed that $\arg(k)=0$). From the definitions given by (18) it follows that

$$\cos\gamma_2=\cos\gamma_1+\frac{p}{ka}. \tag{19}$$

The contour c in (15) may be deformed into a steepest descent path passing through the saddle point $\nu=\nu_0$ on the real ν axis. The equation for the location of this saddle point is

$$\phi'-\phi-\frac{\pi}{2}-\gamma_0+\gamma_{10}+\gamma_{20}=0 \tag{20}$$

where γ_0, γ_{10}, and γ_{20} are defined in (18) by letting $\nu=\nu_0$.

The asymptotic approximation to G_r^1 is thus given by

$$G_r^1\sim\left[\frac{-2\pi}{ik\rho\,\dfrac{\partial^2 f(\nu_0,p)}{\partial\nu_0^2}}\right]^{1/2} Q(\nu_0,p)\,e^{ik\rho f(\nu_0,\,p)} \tag{21}$$

which involves the angles γ_0, γ_{10} and γ_{20}. Appropriate transformations (whose details are omitted here) replace (18) by the simpler restrictions $0\leq\theta_i<\pi/2$, $-\pi/2<\theta_r<\pi/2$ defining the illuminated region, and furthermore

$$ik\rho f(\nu_0,p)=iks(p)-ika\cos\theta_i+ip\left(\frac{\pi}{2}\pm\theta_i\right), \qquad \theta_\eta\gtrless 0 \tag{22}$$

where

$$\begin{aligned} s(p)&=\rho\sin\gamma_0-a\sin\gamma_{10}\\ &=\rho\cos\left[\pm(\phi'-\phi)+\theta_i+\theta_r\right]\\ &\qquad -a\cos\theta_r, \qquad \theta_\eta\gtrless 0. \end{aligned} \tag{23}$$

When evaluated at the saddle point, the relation (19) transforms into the grating law given by (7). The focal length r previously defined is given here by the equation

$$r(p)=a\frac{\sin\gamma_{10}}{1+\dfrac{\sin\gamma_{20}}{\sin\gamma_{10}}}=a\frac{\cos\theta_r(p)}{1+\dfrac{d\theta_r(p)}{d\theta_i}}, \tag{24}$$

while the divergence coefficient D arises from

$$\begin{aligned} D(p)&=\frac{1}{k\rho}\left\{\frac{-\sin\gamma_{10}}{\sin\gamma_0\sin\gamma_{20}}\,\frac{1}{\left[\dfrac{\partial^2 f(\nu,p)}{\partial\nu^2}\right]_{\nu=\nu_0}}\right\}^{1/2}\\ &=\left[1+\frac{s(p)}{r(p)}\right]^{-1/2} \end{aligned} \tag{25}$$

where

$$\left[\frac{\partial^2 f(\nu,p)}{\partial\nu^2}\right]_{\nu=\nu_0}=\frac{1}{(k\rho)^2}\left(\frac{1}{\sin\gamma_0}-\frac{\rho}{a\sin\gamma_{10}}-\frac{\rho}{a\sin\gamma_{20}}\right). \tag{26}$$

By means of (16) to (26) the asymptotic result in (21) may be transformed into the simple geometric optical formula (11). The approximate expressions used in (15) for the Hankel functions break down in the transition regions where θ_i or $|\theta_r|\approx\pi/2$. To describe the transition phenomena in the narrow regions surrounding the limiting rays in figure 3, it is necessary to employ a more detailed analysis than that presented above.

6. References

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