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Propagation of Spherical Waves Through an Ionosphere Containing Anisotropic Irregularities

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The propagation of a spherical wave through a medium containing anisotropic random irregularities is considered. The formulation follows closely that of Karavainikov. The mean square deviations and the correlation functions of the phase and of the logarithmic amplitude are derived by assuming that the autocorrelation function of the dielectric constant is Gaussian with ellipsoidal symmetry. This form of autocorrelation function is chosen because experimentally it has been found that irregularities in the ionosphere at heights of 300 kilometers are cigar-shaped with approximately Gaussian ellipsoidal symmetry. Methods of generalizing to other correlation functions are also indicated.

The theory is applied to a problem which is of particular interest in the investigation of ionospheric irregularities by means of radio transmissions from satellites. Specifically the dependence of scintillation (phase and logarithmic amplitude) on the height of the transmitter above a slab of irregularities and the dependence of the autocorrelation functions are investigated. The theory explains that the scintillation index should be relatively insensitive to the zenith angle of the satellite position for a temperate latitude station, in agreement with the present preliminary observations. The theory also suggests that the observation of phase scintillation may yield information concerning the physical location and the thickness of the region of irregularities.

1. Introduction

The study of wave propagation in a random medium is not only theoretically interesting, but also has wide applications in many branches of physics, including acoustics, ionospheric and atmospheric physics, and astronomy. The random irregularities would scatter waves and hence cause wave interference. As a result the amplitude and the phase of the wave fluctuate. The study of these fluctuations can yield certain statistical properties of the medium although, unfortunately, it usually cannot be done uniquely.

The early foundation of the theoretical work was laid by Bergman [1946], who used ray theory, and Pekeris [1947], who studied the problem of scattering. Since then, many papers have appeared in the literature and Keller [1960] has even classified the mathematical methods used into "honest" and "dishonest" methods. More recently, Silverman has translated the books of Chernov [1961] and Tatarski [1961], which largely represent the Russian contributions to the field. The present study is motivated by the observation of scintillation of radio signals received on the ground from an orbiting satellite-transmitter [Yeh and Swenson, 1959]. It is hoped that this paper will demonstrate that new information can be obtained by studying these scintillations and their relation to ionospheric irregularities.

The analysis in this report is carried out with the application to wave propagation through ionospheric irregularities in mind. The region of irregularities is found to be above about 300 km [Swenson and Yeh, 1961]. There is indication that the region may even be as high as 1,000 km in the auroral zone [Basler and DeWitt, 1962]. Its occurrence as evidenced by the radio-star scintillation studies is correlated with the spread F phenomenon observed in ionograms [Lawrence, Jespersen, and Lamb, 1961]. These irregularities mainly appear near midnight [Booker, 1958; Yeh and Swenson, 1959]. It should be pointed out that tropospheric irregularities have negligible effects in the present problem. (For an excellent review article on tropospheric scattering, see Wheelon, 1959.)

In this report it is assumed that the inductive permeability of the medium is that of free space, but the dielectric constant is a random variable of the position. The medium is characterized by the refractive index,

$$\vec{N(x)} = \langle n \rangle [1 + \epsilon \mu (\vec{x})], \tag{1}$$

where $\langle n \rangle$ is the smoothed average background value and is assumed to be a constant, ϵ is a small constant parameter and μ is a random variable of position. Accordingly, the average value of μ is zero. It shall be assumed that the correlation function of μ is only a function of the difference of coordinates. In a medium of weak random irregularities the percentage change of the refractive index from its average value is small. This means that ϵ is small and may be used as an expansion parameter.

The problem of interest is essentially that of the diffraction of a spherical wave by a slab of intervening medium consisting of random irregularities. These irregularities are assumed to be so weak that only first-order scattering is of importance (Born approximation). In such a problem many characteristic lengths are involved. A brief discussion on these lengths is carried out in section 2. The formulation of the problem in section 3 follows closely that of Karavainikov [1951]. We start from the scalar Helmholtz wave equation and apply the method of perturbation. The solution is expressible in terms of a series in which the zeroth order term is the solution in the absence of the irregularities (i.e., the familiar spherical wave solution). The first-order term shows the effect of the wave being scattered once, and the remaining higher order term shows the effect of multiple scattering. In a medium whose property varies sufficiently slowly over one wavelength these higher order Born solutions may be neglected.

For purposes of obtaining explicit expressions the autocorrelation function of the fluctuations of the dielectric constant must be assumed. In the ionosphere the exact form of this correlation function has not yet been accurately determined. Experiments have suggested that the irregularities have a prolate spheroidal shape [Spencer, 1955]. The ratio of the major to minor axis is approximately 5. The length of the minor axis is of the order of 1 km. Hence one of the most important points that should be emphasized here is that these irregularities are highly anisotropic. For convenience the autocorrelation function is assumed to be

$$\rho_\mu(\vec{x}) = \exp - (x^2/l_x^2 + y^2/l_y^2 + z^2/l_z^2), \tag{2}$$

in sections 4, 5, and 6, in which expressions for the mean square values, and the transverse and longitudinal correlations are obtained, respectively. When the correlation function departs from (2) some of these results are still useful. A method of generalization is presented in section 7.

Applications of the theory to ionospheric studies are considered in section 8. For the numerical values chosen we find the scintillation of the logarithmic amplitude increases monotonically as a function of the height of the transmitter above the region of irregularities while that of the phase increases sharply at first when the transmitter is in the region and reaches a maximum near the top of the region and decays slightly above the region. Sample representative curves for the autocorrelation functions are also plotted. The existing experimental results are still very sparse, but rough agreement between the theory and the experiments has been found.

2. Characteristic Lengths

The geometry of the problem is shown in figure 1 where a, b, and c respectively represent the distance between the transmitter and the top of the slab, the thickness of the slab, and the distance between the receiver and the bottom of the slab. The z-axis is in the direction AB. In case the transmitter is inside the slab we shall ignore the backscattering and hence the effective thickness is reduced accordingly. As a approaches infinity the problem reduces to that of scattering from a uniform plane wave.

It is convenient to normalize the lengths by the wave number k defined by,

$$k = \omega \langle n \rangle / c, \tag{3}$$

where ω is the angular frequency of the wave and c the velocity of light in free space. Unless otherwise specified it shall be understood that all lengths are measured in this dimensionless unit.

In the analysis of the scattering problem shown in figure 1 a few geometric parameters are of importance. Therefore, a brief discussion of them is in order.

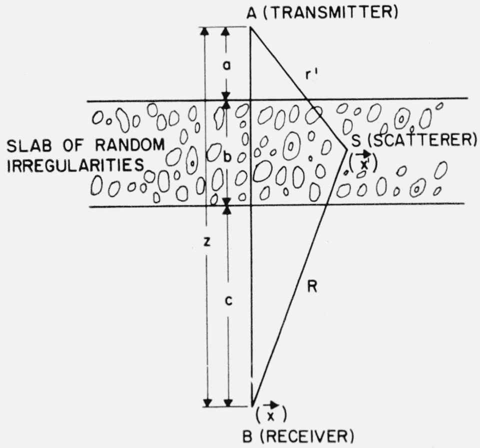

FIGURE 1. *Geometry of the problem.*

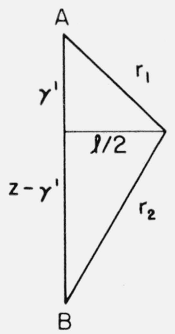

FIGURE 2. *Scattering from a single irregularity.*

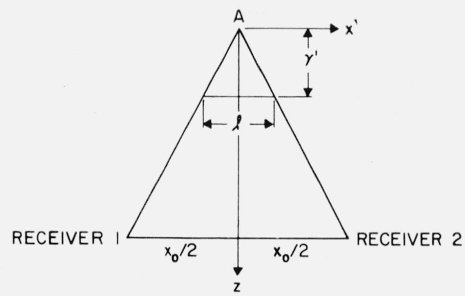

FIGURE 3. *Magnification parameter.*

Consider the case of scattering from a single irregularity of size l as illustrated in figure 2. When the condition,

$$r_1+r_2-z<<1/2, \tag{4}$$

is satisfied the phase change of the scattered waves at the receiver B introduced by the path difference is small. The scattered waves form almost a parallel beam and the fluctuation of the phase is predominately due to the change in refractive index along the irregularity. This case is usually referred to as Fresnel diffraction. On the other hand, when the converse of (4) is true it is then in the regime of Fraunhofer diffraction.

For convenience define a wave parameter,

$$D=4\gamma'(z-\gamma')/l^2z. \tag{5}$$

In terms of this parameter the following equivalent statements can be made:

$$\begin{array}{ll} D>>1 & \text{region of Fresnel diffraction, and} \\ D<<1 & \text{region of Fraunhofer diffraction.} \end{array} \tag{6}$$

For anisotropic irregularities, since the scale of the irregularity depends on the direction it is then necessary to define several wave parameters. These are discussed in a later section.

Another parameter that enters into the study of correlation functions can be seen as follows. In optics an object of size l has its shadow given by $x_0=zl/\gamma'$ (see fig. 3). For this reason one can define a "magnification" parameter,

$$M = x/x_0 = x\gamma'/zl. \tag{7}$$

Therefore, in the language of geometric optics, for a given γ' we would expect that the correlation function of the diffraction pattern on the ground has appreciable value only for values of x such that M is of the order of unity.

3. Formulation

In a slowly varying medium in which the property of the medium changes little in one wavelength the Helmholtz wave equation is approximately satisfied [Stratton, 1941]. In the normalized unit, it is written as (time dependence exp $(j\omega t)$ has been assumed),

$$\nabla^2\psi + [1+\epsilon\mu(x)]^2\psi = 0. \tag{8}$$

When the medium is homogeneous (i.e., $\epsilon=0$) a particular solution of (8) for an isotropic point source yields the well-known spherical wave function,

$$\psi_0 = (kA_0/r) \exp(-jr). \tag{9}$$

The problem is formulated by assuming the solution of the form,

$$\psi(\vec{x}) = \psi_0(\vec{x}) \exp -j[\epsilon\phi_1(\vec{x}) + \epsilon^2\phi_2(\vec{x}) + \ldots]. \tag{10}$$

Substitute (10) into the wave equation and equate coefficients of equal powers in ϵ. The result is a chain of equations. With suitable substitutions [Karavainikov, 1957] each of these equations can be transformed into an inhomogeneous wave equation and solved in the regular manner. When transformed back, the ϕ's in (10) are given by

$$\phi_1 = (jr/2\pi)\int_v \frac{\mu(\vec{x}')}{r'R} \exp[-j(r'+R-r)]d^3\vec{x}', \tag{11}$$

$$\phi_2 = (jr/4\pi)\int_v \frac{\mu(\vec{x}')^2 - [\nabla'\phi_1(\vec{x}')]^2}{r'R} \exp[-j(r'+R-r)]d^3\vec{x}', \tag{12}$$

where R is the distance connecting the source point (in this case the scatterer) and the observing field point (see fig. 1), i.e.,

$$R = |\vec{x} - \vec{x}'|. \tag{13}$$

The expressions given by (11) and (12) are physically reasonable. Each element of the transmitted spherical wave exp $(-jr')/r'$ is scattered at $\vec{x}'$ and reaches the receiver as another spherical wave exp $(-jR)/R$. Proper weighting functions have been introduced to take care of the amount of scattering. In (11) the weighting function is proportional to the percentage change of the refractive index. The weighting function in (12) has two terms; both are of higher order. Because of the appearance of the gradient term in (12), the expression for ϕ_2 is rather complex. From here on it is assumed that the irregularities are so weak that ϕ_2 can be neglected. Since, as shown in (14), ϕ_1 is related to the phase and logarithmic amplitude, the neglect of the gradient term in (12) is equivalent to the assumption that the change of these quantities be small in one wavelength.

It is convenient to rewrite the wave function (10) as,

$$\psi(\vec{x}) = (kA_0/r) \exp[-j(r+\epsilon\phi_1)] = [kA(\vec{x})/r] \exp -j[r+Q(\vec{x})]. \tag{14}$$

Note that when written this way A represents the amplitude of the wave and Q the phase departure. Substitute (11) into (14) and expand the exponential to the second order. For cases of scattering from a localized region the resulting expression can be shown to be identical to that derived by Booker and Gordon [1950], except for the factor that takes into account the polarization of the incident wave with respect to the direction of scattering. Since the present analysis is primarily concerned with the scattering in the forward direction this factor reduces to 1.

Put (11) into (14) and equate separately the real and the imaginary parts,

$$Q(\vec{x})=(r\epsilon/2\pi)\int_v \frac{\mu(\vec{x}')}{r'R}\sin\,(r'+R-r)d^3\vec{x}', \tag{15}$$

$$\log\left(\frac{A}{A_0}\right)=(r\epsilon/2\pi)\int_v \frac{\mu(\vec{x}')}{r'R}\cos\,(r'+R-r)d^3\vec{x}'. \tag{16}$$

It is known that a wave cannot be used to reveal structures much less than one wavelength [Ratcliffe, 1956]. The case of general interest is that in which the characteristic scale of the irregularities is much larger than one wavelength. In the case of the ionosphere the physical size of l is roughly 1 km which corresponds to l (in the normalized unit) of the order of 400 at 20 Mc/s. In our normalized unit this is equivalent to the condition,

$$|l>>1. \tag{17}$$

Because of this condition the contribution to the integrals (15) and (16) comes predominately from the scatterers in the neighborhood of the straight line connecting the transmitter and the receiver. Therefore, certain approximations may be used to facilitate the evaluation of these integrals.

Define the normalized autocorrelation function by,

$$\rho_\mu(\vec{x})=\langle\mu(\vec{x}_1)\mu(\vec{x}_2)\rangle/\langle\mu^2\rangle. \tag{18}$$

Here the symbol $\langle\rangle$ is used to denote the spatial average and the $\vec{x}$ is the relative coordinates defined by,

$$\vec{x}=\vec{x}_2-\vec{x}_1. \tag{19}$$

It shall be assumed that the ergodic hypothesis is valid so that the ensemble average is equivalent to the spatial average. It is seen immediately from (15) and (16) that the mean value of the phase departure and the logarithmic amplitude vanish,

$$\langle Q\rangle=0, \tag{20}$$

$$\langle S\rangle=0, \tag{21}$$

where for simplicity S is written for $\log\,(A/A_0)$.

The correlation functions can be represented formally by,

$$\langle Q(\vec{x}_1)Q(\vec{x}_2)\rangle=(r_1r_2\epsilon^2\langle\mu^2\rangle/4\pi^2)\int_{v_1}\int_{v_2}\frac{\sin\,(r_1'+R_1-r_1)}{r_1'R_1}\cdot\frac{\sin\,(r_2'+R_2-r_2)}{r_2'R_2}\,\rho_\mu(\vec{x}')d^3\vec{x}_1'd^3\vec{x}_2', \tag{22}$$

and

$$\langle S(\vec{x}_1)S(\vec{x}_2)\rangle=(r_1r_2\epsilon^2\langle\mu^2\rangle/4\pi^2)\int_{v_1}\int_{v_2}\frac{\cos\,(r_1'+R_1-r_1)}{r_1'R_1}\cdot\frac{\cos\,(r_2'+R_2-r_2)}{r_2'R_2}\,\rho_\mu(\vec{x}')d^3\vec{x}_1'd^3\vec{x}_2'. \tag{23}$$

When $\vec{x}_2=\vec{x}_1$, the two receivers coincide and the above expressions reduce to mean square values.

4. Mean Square Values

For the purpose of computing the mean square values let $\vec{x}_2=\vec{x}_1=(0,0,z)$ in (22) and (23). It is convenient to make the following approximations. When the quantity appears in the phase let (see fig. 1),

$$R+r'-r\cong(x'^2+y'^2)z/2z'(z-z'), \tag{24}$$

and when the quantities appear in the denominator

$$r'\cong z',\, R\cong z-z'. \tag{25}$$

Define,

$$I_1=\pi(<Q^2>+<S^2>)/\epsilon^2<\mu^2>, \tag{26}$$

and

$$I_2=\pi(<Q^2>-<S^2>)/\epsilon^2<\mu^2>. \tag{27}$$

The expressions for these integrals can be found by the use of (22) and (23),

$$I_1=\frac{1}{\pi}\int\int\frac{\rho_\mu(\vec{x}')}{4\zeta_1'\zeta_2'}\cos\left(\frac{x_1'^2+y_1'^2}{2\zeta_1'}-\frac{x_2'^2+y_2'^2}{2\zeta_2'}\right)d^3\vec{x}_1'd^3\vec{x}_2', \tag{28}$$

$$I_2=-\frac{1}{\pi}\int\int\frac{\rho_\mu(\vec{x}')}{4\zeta_1'\zeta_2'}\cos\left(\frac{x_1'^2+y_1'^2}{2\zeta_1'}+\frac{x_2'^2+y_2'^2}{2\zeta_2'}\right)d^3\vec{x}_1'd^3\vec{x}_2', \tag{29}$$

where the new symbols are defined by

$$\zeta_1'=z_1'(z-z_1')/z, \text{ and } \zeta_2'=z_2'(z-z_2')/z. \tag{30}$$

In order to integrate (28) and (29) it is desirable to make the coordinate transformation by introducing the relative coordinates,

$$x'=x_2'-x_1', \qquad y'=y_2'-y_1', \qquad z'=z_2'-z_1', \tag{31}$$

and the center of mass coordinates

$$\alpha'=(x_1'+x_2')/2, \qquad \beta'=(y_1'+y_2')/2, \qquad \gamma'=(z_1'+z_2')/2. \tag{32}$$

Integrate (28) and (29) with respect to α' and β', resulting in,

$$I_1=\int_{\gamma'=a}^{a+b}\int_{z'=-b}^{b}\int_{y'=-\infty}^{\infty}\int_{x'=-\infty}^{\infty}\frac{\rho_\mu(\vec{x}')}{2(\zeta_1'-\zeta_2')}\sin\frac{x'^2+y'^2}{2(\zeta_1'-\zeta_2')}\,dx'dy'dz'd\gamma', \tag{33}$$

$$I_2=\int_{\gamma'=a}^{a+b}\int_{z'=-b}^{b}\int_{y'=-\infty}^{\infty}\int_{x'=-\infty}^{\infty}\frac{\rho_\mu(\vec{x}')}{2(\zeta_1'+\zeta_2')}\sin\frac{x'^2+y'^2}{2(\zeta_1'+\zeta_2')}\,dx'dy'dz'd\gamma'. \tag{34}$$

Further integration of I_1 and I_2 depends on the knowledge of the correlation function ρ_μ. In this paper ρ_μ is assumed to have a form given by (2). Substitute (2) in (33) and (34) and integrate with respect to x' and y',

$$I_1=\mathrm{Im}\int_{\gamma'=a}^{a+b}\int_{z'=-\infty}^{\infty}\frac{\pi\exp\,(-z'^2/l_z^2)dz'd\gamma'}{[-2z'(1-2\gamma'/z)/l_x^2-j]^{1/2}[-2z'(1-2\gamma'/z)/l_y^2-j]^{1/2}}, \tag{35}$$

$$I_2=\mathrm{Im}\int_{\gamma'=a}^{a+b}\int_{z'=-\infty}^{\infty}\frac{\pi\exp\,(-z'^2/l_z^2)dz'd\gamma'}{[D_x-z'^2/zl_x^2-j]^{1/2}[D_y-z'^2/zl_y^2-j]^{1/2}}. \tag{36}$$

Here the symbol Im stands for "the imaginary part of," and the wave parameters are defined by,

$$D_x = 4\gamma'(z-\gamma')/l_x^2 z, \qquad D_y = 4\gamma'(z-\gamma')/l_y^2 z, \tag{37}$$

and their significance was discussed in section 2. Since the thickness of the slab is much larger than the correlation distance of the irregulatities, contributions to the integrals I_1 and I_2 come predominately from the region $z' \lesssim l_z$. Therefore, (1) the limits of integration with respect to z' can be extended from $-\infty$ to $+\infty$ as in (35) and (36) without introducing appreciable errors, and (2) because of the additional condition (17) terms like $2z'(1-2\gamma'/z)/l^2$ in (35) and z'^2/zl^2 in (36) can be neglected as compared with unity. Introducing these approximations in (35) and (36), I_1 and I_2 are respectively given by,

$$I_1 = b\pi^{3/2} l_z, \tag{38}$$

$$I_2 = \int_{\gamma'=a}^{a+b} l_z \sqrt{\frac{\pi^3[(1+D_x^2)^{1/2}(1+D_y^2)^{1/2}+(1-D_x D_y)]}{2(1+D_x^2)(1+D_y^2)}}\, d\gamma'. \tag{39}$$

The exact integration of (39) is difficult to carry out since as seen in (37) D_x and D_y are functions of γ'. However, if the slab is thin or under conditions that D_x and D_y are almost constant and can be replaced by some convenient average values $\overline{D}_x$ and $\overline{D}_y$, then,

$$I_2 = b\pi^{3/2} l_z \left[\frac{\sqrt{(1+\overline{D}_x^2)(1+\overline{D}_y^2)}+(1-\overline{D}_x\overline{D}_y)}{2(1+\overline{D}_x^2)(1+\overline{D}_y^2)}\right]^{1/2}. \tag{40}$$

Since from (26) and (27),

$$<Q^2> = \epsilon^2 <\mu^2> (I_1+I_2)/2\pi, \tag{41}$$

and

$$<S^2> = \epsilon^2 <\mu^2> (I_1-I_2)/2\pi, \tag{42}$$

the mean square values are therefore given by,

$$<Q(0,0,z)^2> = \frac{\epsilon^2 <\mu^2>}{2}\pi^{1/2} l_z b \left\{1+\sqrt{\frac{\sqrt{(1+\overline{D}_x^2)(1+\overline{D}_y^2)}-(1-\overline{D}_x\overline{D}_y)}{2(1+\overline{D}_x^2)(1+\overline{D}_y^2)}}\right\} \tag{43}$$

$$<S(0,0,z)^2> = \frac{\epsilon^2 <\mu^2>}{2}\pi^{1/2} l_z b \left\{1-\sqrt{\frac{\sqrt{(1+\overline{D}_x^2)(1+\overline{D}_y^2)}-(1-\overline{D}_x\overline{D}_y)}{2(1+\overline{D}_x^2)(1+\overline{D}_y^2)}}\right\}. \tag{44}$$

Some special cases of interest will be considered in the following:

(1) $D>>1$. In this case it is found that the mean square values of the phase and the logarithmic amplitude are equal and independent of the dimensions of the irregularities transverse to the direction of propagation,

$$<Q^2> = <S^2> = \epsilon^2 <\mu^2> \pi^{1/2} l_z b/2. \tag{45}$$

This result reduces to that obtained by Chernov [1] [1961] when the irregularities have a spherically symmetric Gaussian correlation.

(2) $D<<1$. In this case D's are very small and (43) and (44) respectively become,

$$<Q^2> = \epsilon^2 <\mu^2> \pi^{1/2} l_z b, \tag{46}$$

and

$$<S^2> = \frac{\epsilon^2 <\mu^2> \pi^{1/2}}{16} l_z b(3\overline{D}_x^2 + 2\overline{D}_x\overline{D}_y + 3\overline{D}_y^2). \tag{47}$$

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[1] His eq (141) on page 75.

If, in addition $l_x = l_y = l_T$, hence $\overline{D}_x = \overline{D}_y = \overline{D}_T$,

$$<S^2> = \frac{\epsilon^2 <\mu^2> \pi^{1/2}}{2} l_z b \overline{D}_T^2. \tag{48}$$

It is interesting to note that for a thin slab with spherically symmetric irregularities ($l_x = l_y = l_z = l$),

$$\overline{D} \simeq 4ac/[l^2(a+c)]. \tag{49}$$

Put (49) into (48) it is seen that $(<S^2>)^{1/2}$ is halved when the transmitter moves from infinity ($a=\infty$) to a distance above the slab equal to that between the slab and the receiver ($a=c$).

When the slab is not thin the average value of the wave parameter squared in (48) must be found according to the integral (39), i.e.,

$$\begin{aligned} b\overline{D}_T^2 &= \int_a^{a+b} D_T^2 d\gamma' = \int_a^{a+b} \left[\frac{4\gamma'(z-\gamma')}{l_T^2 z}\right]^2 d\gamma' \\ &= \frac{16}{l_T^4 z^2} \left\{\frac{1}{3} z^2[(a+b)^3 - a^3] - \frac{1}{2} z[(a+b)^4 - a^4] + \frac{1}{5}[(a+b)^5 - a^5]\right\}. \end{aligned} \tag{50}$$

Equation (50) can be simplified if both the transmitter and the receiver are immersed in the irregularities. In this case $a=0$ and $b=z$, substitution of (50) into (48) yields,

$$<S^2> = \frac{4\pi^{1/2}}{15} \epsilon^2 <\mu^2> z^2 l_z / l_T^4. \tag{51}$$

It is seen that in this case the mean square values depend on the size of the irregularities along the direction of propagation as well as transverse to it. Equations (46) and (51) reduce to those obtained by Karavainikov [1957] when l's are all equal.

(3) D's$\simeq 1$. Fortunately or unfortunately, in most of the ionospheric applications the previous approximations are invalid and hence it is necessary to work with more complex expressions (43) and (44) of the original integral (39). Now it is seen that $<Q^2>$ and $<S^2>$ depend on D's in a different manner. If the scales of the irregularities are determined by a correlation method the height of the region of irregularities can be roughly estimated by making simultaneous measurements of $<Q^2>$ and $<S^2>$ as was done by Hewish [1952].

5. Transverse Correlation Functions

The general expressions for the correlation functions were derived in section 3. In order to evaluate these integrals, place one receiver at $(-x/2, 0, z)$ and one at $(x/2, 0, z)$. Since the spacing of these receivers is perpendicular to the direction of propagation the correlations of the field at these two receivers are called transverse correlation functions. Approximations similar to those given in the last section are,

$$(R_1 + r_1') - r_1 = [y_1'^2 + (x_1' - xz_1'/2z)^2]/2\zeta_1', \tag{52}$$

and

$$(R_2 + r_2') - r_2 = [y_2'^2 + (x_2' - xz_2'/2z)^2]/2\zeta_2'. \tag{53}$$

Define,

$$I_3 = (\pi/\epsilon^2 <\mu^2>)(<Q_1 Q_2>_T + <S_1 S_2>_T), \tag{54}$$

and

$$I_4 = (\pi/\epsilon^2 <\mu^2>)(<Q_1 Q_2>_T - <S_1 S_2>_T). \tag{55}$$

Putting (22) and (23) into above relations these integrals are found to be,

$$I_3=\frac{1}{\pi}\iint\frac{\rho_\mu(\vec{x}')}{4\zeta_1'\zeta_2'}\cos\left[\frac{y_1'^2+(x_1'-xz_1'/2z)^2}{2\zeta_1'}-\frac{y_2'^2+(x_2'+xz_2'/2z)^2}{2\zeta_2'}\right]d^3\vec{x}_1'd^3\vec{x}_2',$$

$$I_4=\frac{-1}{\pi}\iint\frac{\rho_\mu(\vec{x}')}{4\zeta_1'\zeta_2'}\cos\left[\frac{y_1'^2+(x_1'-xz_1'/2z)^2}{2\zeta_1'}+\frac{y_2'^2+(x_2'+xz_2'/2z)^2}{2\zeta_2'}\right]d^3\vec{x}_1'd^3\vec{x}_2'.$$

Transform the coordinates into a relative and center of mass system, and integrate with respect to α' and β' as was done in the last section,

$$I_3=\iiiint\frac{\rho_\mu(\vec{x}')}{2(\zeta_1'-\zeta_2')}\sin\frac{y'^2+(x'+x\gamma'/z)^2}{2(\zeta_1'-\zeta_2')}dx'dy'dz'd\gamma', \tag{56}$$

$$I_4=\iiiint\frac{\rho_\mu(\vec{x}')}{2(\zeta_1'+\zeta_2')}\sin\frac{y'^2+(x'+x\gamma'/z)^2}{2(\zeta_1'+\zeta_2')}dx'dy'dz'd\gamma'. \tag{57}$$

Here the limits of integration are identical to those given by (33) and (34). In fact I_3 and I_4 reduce to I_1 and I_2, respectively, when the spacing of the receivers x goes to zero as expected. For the correlation function of the form given by (2), the integral I_3 is given approximately by,

$$I_3=\frac{\pi^2 z l_x l_z}{2x}\left[\operatorname{erf}\left(\frac{(a+b)x}{l_x z}\right)-\operatorname{erf}\left(\frac{ax}{l_x z}\right)\right], \tag{58}$$

where it is seen that the arguments of these error functions are actually the magnification parameters (discussed in 2) at the bottom and top of the slab. Integrate (57) with respect to x', y', and z',

$$I_4=\pi^{3/2}l_z\ \mathrm{Im}\int_a^{a+b}\frac{\exp\left[-jx^2\gamma'^2/(j-D_x)z^2l_x^2\right]}{\sqrt{(D_x-j)(D_y-j)}}d\gamma'. \tag{59}$$

For cases in which D's are almost constant, (59) is approximately,

$$I_4=(\pi^2 l_z l_x z/2x)\ \mathrm{Im}\left\{j(1+jD_y)^{-1/2}\left[\operatorname{erf}\left(\frac{(a+b)x}{zl_x(1+jD_x)^{1/2}}\right)-\operatorname{erf}\left(\frac{ax}{zl_x(1+jD_x)^{1/2}}\right)\right]\right\}. \tag{60}$$

Define the normalized transverse autocorrelation functions by,

$$\rho_Q(x)=\langle Q_1Q_2\rangle_T/\langle Q^2\rangle,\ \text{and}\ \rho_S(x)=\langle S_1S_2\rangle_T/\langle S^2\rangle, \tag{61}$$

then

$$\rho_Q(x)=(I_3+I_4)/(I_1+I_2), \tag{62}$$

$$\rho_S(x)=(I_3-I_4)/(I_1-I_2). \tag{63}$$

For the special case in which the slab is very thin, (58) and (60) can be approximated. If, additionally, the D's are either large or small some simple results can be obtained. These are considered in the following:

(1) D's $>>1$. Keep only leading terms in (62) and (63) and consider the region of $x\lesssim l_xz/a$. The correlations of the phase and the logarithmic amplitude reduce to,

$$\rho_Q(x)=\rho_S(x)=\exp\left(-a^2x^2/l_x^2z^2\right). \tag{64}$$

Hence it is seen that both correlation functions are initially Gaussian and the "scale" of the irregular waves is a factor z/a times the "scale" of the fluctuations in the refractive index consistent with the idea of magnification discussed in section 2.

(2) D's$\ll 1$. Again keeping only leading terms,

$$\rho_Q(x)=\exp\left(-a^2x^2/l_x^2z^2\right), \tag{65}$$

and

$$\rho_S(x)=\left[1-\frac{4a^2x^2\overline{D}_x\left(3D_x+\overline{D}_y-\frac{a^2x^2\overline{D}_x}{z^2l_x^2}\right)}{z^2l_x^2(3\overline{D}_x^2+3\overline{D}_g^2+2\overline{D}_x\overline{D}_y)}\right]\exp\left(-\frac{a^2x^2}{z^2l_x^2}\right). \tag{66}$$

The fluctuations in the phase is a factor z/a larger than the fluctuations in the refractive index. However, now the fluctuations in the amplitude is influenced by not only the fluctuations of the refractive index in the direction of the spacing of these receivers (x-direction) but also, though weakly, transverse to it (y-direction). If additionally $l_y=l_x$, then (66) reduces to,

$$\rho_S(x)=\left(1-\frac{2a^2x^2}{z^2l_x^2}+\frac{1}{2}\frac{a^4x^4}{z^4l_x^4}\right)\exp\left(-\frac{a^2x^2}{z^2l_x^2}\right). \tag{67}$$

Though (66) and (67) are not Gaussian they have a correlation distance of the order of l_xz/a and (66) approaches Gaussian in the limit $l_y \ll l_x$.

6. Longitudinal Correlation Functions

In this case the two receivers are placed along the direction of propagation at (o, o, z) and (o, o, $z+\Delta z$) with a separation $\Delta z=z_2-z_1$. Proceed in the same manner as the previous two sections by defining,

$$I_5=(\pi/\epsilon^2<\mu^2>)(<Q_1Q_2>_L+<S_1S_2>_L), \tag{68}$$

and

$$I_6=(\pi/\epsilon^2<\mu^2>)(<Q_1Q_2>_L-<S_1S_2>_L). \tag{69}$$

It can be shown that for the spacing Δz of the order of the correlation distance I_5 and I_6 reduces to I_1 and I_2 respectively. Therefore, the phase and the logarithmic amplitude are perfectly correlated in a distance $\Delta z \lesssim zl/(a+b)$.

7. Method of Generalization

In the foregoing sections expressions of mean square values and the autocorrelation functions are obtained by assuming that the fluctuating part of the dielectric constant has a Gaussian character given by (2). As mentioned before one of the most important features of the ionospheric irregularities is that they are strongly anisotropic. If ρ_μ departs from Gaussian but still possesses ellipsoidal symmetry some of the earlier derived expressions can be generalized. It is convenient to define

$$p=x^2/l_x^2+y^2/l_y^2+z^2/l_z^2. \tag{70}$$

Introduce $F(t)$ so that ρ_μ (p) and $F(t)$ are related through the Laplace transform, i.e.,

$$\rho_u(p)=\int_0^\infty F(t)\exp(-pt)dt. \tag{71}$$

The function $F(t)$ can be chosen to simulate almost any correlation function with ellipsoidal symmetry. The effect of introducing (71) in the integrals $I_i(l_x,l_y,l_z)$, $i=1$ to 6, is simply to replace them by

$$\int_0^\infty dtF(t)I_i\left(\frac{l_x}{\sqrt{t}},\frac{l_y}{\sqrt{t}},\frac{l_z}{\sqrt{t}}\right). \tag{72}$$

Similar generalizations can be applied to $<Q^2>$, $<S^2>$, ρ_S, and ρ_Q if they are evaluated explicitly. But now the averaging of D's is not permissible and there does not seem to be any simple way out.

By means of this device the generalization of (38), (45), and (46) is equivalent to multiplication of these respective equations by the factor,

$$\int_0^\infty \frac{F(t)}{\sqrt{t}}\,dt=\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{\rho_\mu(p)}{\sqrt{p}}\,dp, \tag{73}$$

and the factor that goes with (48) when (50) is substituted, and with (51) is

$$\int_0^\infty t^{\frac{3}{2}}F(t)dt=\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{p}}\frac{d^2\rho_u(p)}{dp^2}\,dp. \tag{74}$$

The generalization of the correlation functions can be done in an identical manner. If

$$M_a(x)=ax/zl_x, \tag{75}$$

the generalization of (64) and (65) can be shown to be

$$\frac{\displaystyle\int_0^\infty \rho_\mu(p+M_a^2)p^{-1/2}dp}{\displaystyle\int_0^\infty \rho_\mu(p)p^{-1/2}dp}. \tag{76}$$

The generalization of (67), when (49) is valid, is approximately

$$\rho_S(x)=\frac{1}{\displaystyle\int_0^\infty \frac{\partial^2\rho_\mu(p)}{\partial p^2}\,p^{-1/2}dp}\times\int_0^\infty\left[\frac{\partial^2\rho_\mu(p+M_a^2)}{\partial p^2}+2M_a^2\,\frac{\partial^3\rho_\mu(p+M_a^2)}{\partial p^3}+\frac{M_a^4}{2}\,\frac{\partial^4\rho_\mu(p+M_a^2)}{\partial p^4}\right]p^{-1/2}dp \tag{77}$$

In the general case it can still be proved that the longitudinal correlation functions are perfect for Δz of the order of $zl/(a+b)$.

8. Applications to the Ionosphere

The first clue to the existence of the irregularities in electron density was obtained by Hey, Parsons, and Phillips [1946]. They discovered that the intensity of the radiation from radio stars fluctuate on certain occasions in the VHF band. At first it was thought that this indicated the source was variable in its power output. Subsequently spaced-receiver experiments by Smith [1950] and by Little and Lovell [1950] gave convincing proof that the cause of these fluctuations is in the ionosphere. It is now generally believed that there exist in the ionosphere blobs with excesses or deficiencies of electrons which scatter waves irregularly. These blobs are elongated along the Earth's magnetic field lines with an axial ratio of five to one [Spencer, 1955]. Past studies of these related problems using radio stars and ground-based transmitters have been well summarized by Little et al. [1956], Ratcliffe [1956], and Booker [1958].

In the following three subsections the relation between fluctuations in refractive index and electron density, the scintillation indices and the autocorrelation functions are considered.

8.1. Fluctuations in Refractive Index and Electron Density

In case of ionospheric applications for a frequency of 20 Mc/s or higher the longitudinal approximation to the Appleton-Hartree formula [Appleton, 1932] is valid unless the ray is within a few degrees from the direction of exactly perpendicular to the Earth's magnetic field. Using the standard notations recommended by U.R.S.I. [Ratcliffe, 1959] the refractive index is given by,

$$n^2 = 1 - \frac{X + \Delta X(\vec{x})}{1 \pm Y}, \tag{78}$$

where $\Delta X\ (\vec{x})$ is the part due to electron density fluctuations and is a stochastic function of position. The upper sign in (78) is for the ordinary ray and the lower sign for the extraordinary ray. Taking the square root of (78) and keeping only first order terms, the refractive index is,

$$N = \langle n \rangle \left(1 - \frac{\Delta X(\vec{x})}{2(1 \pm Y - X)}\right). \tag{79}$$

Comparison of (79) with (1) shows that the expression for the mean square of the percent fluctuations in the refractive index as,

$$\epsilon^2 \langle \mu^2 \rangle = \frac{K}{f^4} \langle (\Delta N)^2 \rangle, \tag{80}$$

where

$$K = \frac{c^4 r_e^2}{4\pi^2 \langle n \rangle^4 (1 \pm Y)^2},$$

$$r_e = \frac{e^2}{4\pi\epsilon_0 mc^2} = 2.8178 \times 10^{-15}\ m \text{ (classical electron radius)},$$

ΔN = electron density fluctuations.

Since from (79) the fluctuations in the refractive index are proportional to the fluctuations in the electron density, the normalized autocorrelations of both must be identical. Therefore, the formulas derived in the previous sections can be applied by replacing the mean square electron density through the relation (80) and by noting that l's now represent the scale of ionization irregularities.

8.2. Scintillation Indices

With the advent of artificial satellites it is now possible to carry out a series of measurements with a transmitter at a variable height. The transmitter may be below, within, or above the slab of irregularities. Hence it is desirable to know how the scintillation indices of the amplitude and the phase (defined as $\langle Q^2 \rangle^{1/2}$ and $\langle S^2 \rangle^{1/2}$ respectively) vary with the height of the transmitting satellite. The result is shown in figure 4. Three cases have been considered: (1) The direction of propagation paralley to the major axis of the irregularities and; (2) two cases when the direction of propagation is transverse to it. The numerical values chosen are tabulated below:

$$b = \begin{cases} 2 \times 10^4 - |a| & \text{when} -2 \times 10^4 \leq a \leq 0. \\ 2 \times 10^4 & \text{when } a \geq 0, \end{cases}$$

In all cases:

$$c = 12 \times 10^4.$$

Case 1 Longitudinal case:

$$l_x = l_y = 4 \times 10^2, \qquad l_z = 2 \times 10^3.$$

Case 2 Transverse case 1:

$$l_x = 2 \times 10^3, \qquad l_y = l_z = 4 \times 10^2.$$

Case 3 Transverse case 2:

$$l_x = l_z = 4 \times 10^2, \qquad l_y = 2 \times 10^3.$$

In terms of physical units, for $k=400$/km (corresponding to a frequency of 19.1 Mc/s if the refractive index is 1) these values are given by,

$$b=\begin{cases} 50\text{km}-|a| \text{ when } -50 \text{ km} \leq a \leq 0, \\ 50 \text{ km when } a \geq 0, \end{cases}$$

$$c=300 \text{ km},$$

Case 1 Longitudinal case:

$$l_x=l_y=1 \text{ km}, \qquad l_z=5 \text{ km},$$

Case 2 Transverse case 1:

$$l_x=5 \text{ km}, \qquad l_y=l_z=1 \text{ km}.$$

Case 3 Transverse case 2:

$$l_x=l_z=1 \text{ km}, \qquad l_y=5 \text{ km}.$$

In calculation the approximate formulas (43) and (44) have been used. These formulas can be simplified considerably in case 1. Since $\overline{D}_x$ and $\overline{D}_y$ enter in a symmetrical manner, the corresponding scintillation indices in case 3 are identical to those in case 2. The actual plotted values are $(2<Q^2>/\epsilon^2<\mu^2> \pi^{1/2})^{1/2}\times 10^{-3}$ and $(2<S^2>/\epsilon^2<\mu^2> \cdot \pi^{1/2})^{1/2}\times 10^{-3}$. These naturally are proportional to the scintillation indices defined in this paper. As an approximation $\overline{D}$'s are calculated from (37) by assuming γ' at the midway of the slab. When the satellite is inside the slab the effective slab thickness is reduced accordingly.

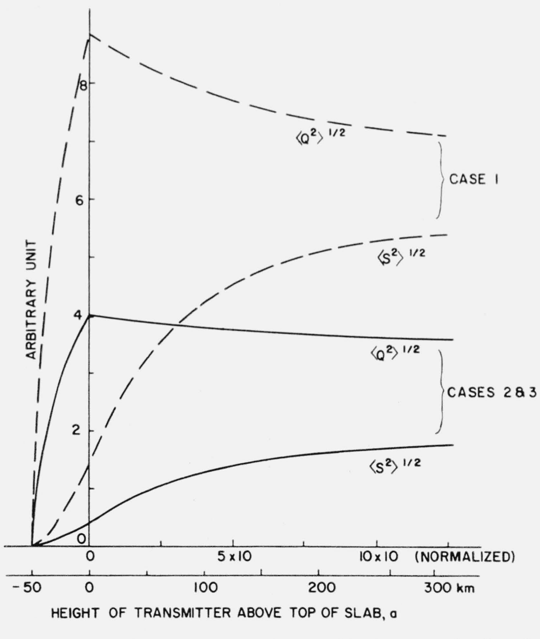

FIGURE 4. *Dependence of scintillation on the height of the transmitter.*

By careful examination of figure 4 the following interesting points are noted:

(1) The scintillation index of the amplitude varies monotonically as a function of the height while the scintillation index of the phase rises rapidly when the transmitter is within the slab and reaches a maximum value near the top of the slab and then decays slightly above the slab. Therefore, it appears that the measurement of phase scintillation may enable us

to determine the height of the bottom, the thickness, and the height of the top of the slab if it exists at all. The existing experimental data are rather meager. Some rough estimates using signals from satellite 1958Δ_2 on 20 Mc/s have been made [Swenson and Yeh, 1961]. The amplitude scintillation roughly agrees with figure 4. (See their figure 6, which is reproduced here as figure 5. Note that the scintillation index defined in that paper is identical to that defined here only in the limit of very weak amplitude fluctuations.) The data presented in figure 5 were collected in Baker Lake, Canada which is north of the auroral zone. There is evidence that the region of irregularities may vary in height by more than 500 km [Basler and DeWitt, 1962] in the auroral zone. This may account for the spread of these points.

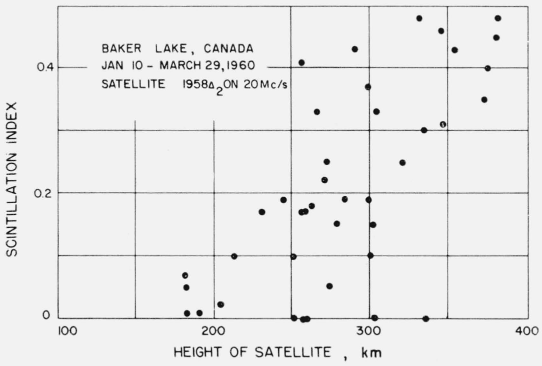

FIGURE 5. *Experimental results of scintillation dependence on the height of the satellite.*

(2) For a constant thickness of the slab figure 4 shows that there is more scintillation in the longitudinal case (case 1) than that in the transverse case (case 2 and case 3). The phase scintillation is 2 to 2.2 times more and the amplitude scintillation is 3.1 to 3.7 times more in the longitudinal case than the corresponding values in the transverse case. In the actual ionospheric observations the effective thickness varies according to the secant of the zenith angle of the direction of propagation. Since these irregularities are alined along the magnetic field lines and the dip angle at, for example, the University of Illinois is 71.3°, the effective thickness in the longitudinal case (case 1) is then a factor tan $71.3°=2.95$ smaller than that in the transverse case (cases 2 and 3). This will bring curves of cases 2 and 3 closer to the corresponding curves of case 1. Hence in actual ionospheric observations at temperate and high latitudes the scintillations should not change appreciably with the zenith angle of the transmitter and preliminary results of the observation of satellite signals at various stations operated by the University of Illinois seem to indicate that this is so. However, for a station near the magnetic equator we would expect that the effect of the effective thickness may enhance the differences in scintillations considered above and the signal from an orbiting satellite at low angles to the north or south of the station should exhibit stronger scintillations than when it is overhead.

8.3. Transverse Correlations

It has been shown in section 6 that the longitudinal correlations are perfect for a separation Δz of the order $zl/(a+b)$. Therefore only the transverse correlation will be considered here.

Examining the integral I_4 given by (59) or its approximate expression (60), it is seen that the correlation functions depend mainly on the dimension of the irregularities along the spacing of the two receivers and not on their dimension transverse to the spacing only if the wave parameter is small compared to 1. That this is intuitively true can perhaps be argued physically as follows. When $D<<1$ the irregularities are large compared with the first Fresnel zone. Since most of the contributions to the signal at the receiver come from the scattering in the first Fresnel zone it seems reasonable to expect that the correlations should not be sensitive to the dimension of the irregularities transverse to the spacing of the receivers. However, when $D>>1$ the irregularities are small compared with the first Fresnel zone and the scattering is all important toward the end of the irregularity. Therefore, in this case the transverse dimension of the irregularities affects the correlation function.

As specific numerical examples the three cases discussed in the last subsection are again considered. The numerical values are identical with those tabulated previously except that now a fixed value of $a=12\times 10^4$ is taken. For $k=400$/km this is equivalent to a height of 300 km above the top of the slab. The curves are shown in figure 6. The ordinate is the magnification parameter defined in (7) where γ' is assumed to be the distance from the transmitter to the center of the slab, i.e.,

$$\overline{M}_x = \frac{(a+b/2)\,x}{zl_x}. \tag{81}$$

In calculating these curves formulas (58) and (60) have been used.

Experimentally one usually obtains the fading records at a fixed station from a moving transmitting vehicle. For all practical purposes it is equivalent to the case considered in this paper. The autocorrelation function of the amplitude obtained experimentally (defined in a slightly different manner) agree reasonably well with those theoretical curves shown in figure 6 [Kent, 1959; Swenson and Yeh, 1961].

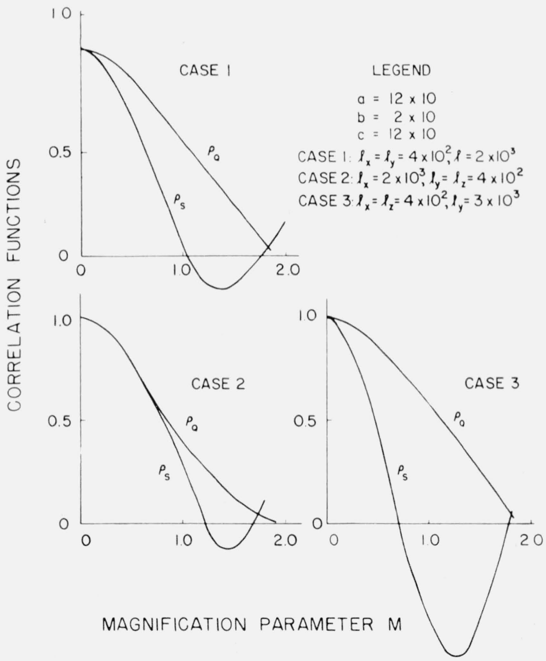

FIGURE 6. *Transverse correlation functions.*

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