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Theory of Magneto-Telluric Fields

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This paper is a review of the present state of knowledge of magneto-telluric fields. The subject has to do with the combined analyses of the geomagnetic and the telluric (earth-current) fields on the surface of the earth. Usually, the objective of such investigations is to obtain information about the earth's crustal layers. However, for a sensible use of the method it is desirable to understand something about the source of the fields. In this paper, the various suggestions for the source mechanism are discussed. Then a fairly detailed review of previous work on the theory of the magneto-telluric interpretation is given. Included are a number of three-layer interpretation curves. The influence of earth curvature in magneto-telluric theory is treated in a mathematical appendix which is, in itself, a self-contained derivation of the various formulas.

1. Introduction

The temporal variations of the geomagnetic field have been investigated extensively for many years. However, only quite recently have the rapid magnetic variations been studied in conjunction with the variations of the telluric (earth-current) fields. In fact, only some twelve years ago, Tikhonov [1950] in the USSR, and Kato and Kikuchi [1950] in Japan, pointed out that the electrical characteristics of the deep strata of the earth's crust could be determined from a combined analysis of geomagnetic and telluric field variations. Since then, a large number of related investigations have been carried out, particularly in the USSR. It is the purpose of this paper to present a self-contained account of the theory of the phenomena. Sufficient curves are given to provide a basis for interpretation of experimental data.

2. Nature of the Sources

The actual mechanism which produces the short-period variations (i.e., frequencies of the order of 1 to 10^{-3} c/s) is not yet well understood. In recent years it has been suggested on numerous occasions that the phenomena are related to magneto-hydrodynamic (i.e., MHD) waves in the exosphere or the ionosphere of the earth.

Holmberg [1951], Lehnert [1956], and Maple [1959] have proposed that the sources of the observed micropulsations arise from intralayer MHD oscillations within the E and F layers of the ionosphere. The principal objection to such a hypothesis is the high attenuation of MHD waves in such regions as pointed out by Piddington [1959] and Ellis [1960].

Dungey [1955] has suggested that MHD waves form a standing wave pattern along the geomagnetic field lines. In the fundamental mode the period of oscillations would then be expected to increase with geomagnetic latitude. Some support for this view has been given by Obayashi and Jacobs [1958], who found that the period of some micropulsations did appear to increase with latitude. However, Ellis [1960], in Australia, found that there was no observable change in the period for three stations ranging from 28° S to 51° S geomagnetic latitude. Dungey [1955] has also suggested that instabilities of the Chapman-Ferraro outer boundary of the geomagnetic field could generate waves which propagate downward as MHD waves. It would be expected that such waves would tend to reach the ionosphere in auroral regions and consequently micropulsation from this source would not be observed at middle and equatorial latitudes. However, they might be propagated in horizontal MHD ducts as envisaged by Bomke et al. [1960].

Quite recently, Gallet [1959] and Ginzburg [1961] have suggested that solar corpuscular streams produce magneto-hydrodynamic waves. In particular, Ginzburg considers the radiation emitted by an isolated ion moving with velocity u along a magnetic field H in a magneto-active plasma. Two cases are distinguished, (1) slower-than-light motion, when the ion velocity u is less than the phase velocity v_{ph} of the emitted wave, and (2) faster-than-light motion, when the ion velocity is greater than the phase velocity of the emitted wave.

In the first case, mentioned above, the radiation results from the Larmor precession of the ion around the lines of force of the earth's steady magnetic field H. The angular frequency of the rotation is $\Omega_i = eH/Mc$, where e is the ion charge, M is the ion mass, and c is the velocity of light in vacuum. For a fixed observer in the plasma, the frequency of this radiation is

$$\omega' = \frac{\Omega_i}{1-(u/c)N\cos\theta} \tag{1}$$

where N is the refractive index of the plasma at frequency ω', and θ is the angle between the direction of wave propagation and the ion velocity u. This is really the normal Doppler effect and it results from the motion of the emitter which, in the present situation, is the gyrating ion.

In the second case, denoted (2) above, the motion is faster-than-light and the frequency of the emitted radiation is given by

$$\omega' = \frac{\Omega_i}{\frac{u}{c}N\cos\theta - 1}. \tag{2}$$

This has been called the anomalous Doppler effect [Frank, 1942]. Contrary to the normal Doppler effect, in the latter case ω' may be less than Ω_i if u is sufficiently great.

By using known expressions for the refractive index of a plasma, and specifying the direction of wave propagation, the frequencies ω' emitted by the ion can be calculated. To simplify the discussion the waves are assumed to propagate along the earth's magnetic field (i.e., longitudinal propagation). Then, on the further assumption that the collision frequencies are negligible [Fejer, 1960; Wait, 1961],

$$N^2 \cong 1 - \frac{\omega_{0,i}^2}{\omega(\omega \pm \Omega_i)} - \frac{\omega_{0,e}^2}{\omega(\omega \pm \Omega_e)} \tag{3}$$

where $\omega_{0,i}$ and $\omega_{0,e}$ are the ion and electron plasma frequencies, respectively.

In the earth's exosphere and for $\omega << \Omega_e$, the magnitude of N is large. Then

$$N^2 \simeq \frac{K_0}{\left(1 \pm \frac{\omega}{\Omega_i}\right)\left(1 \mp \alpha \frac{\omega}{\Omega_i}\right)} \text{ where } K_0 = \frac{\omega_{0,e}^2}{\Omega_e \Omega_i} \tag{4}$$

and where $\alpha = \frac{\text{ratio of electron mass}}{\text{ratio of ion mass}} \simeq \frac{1}{1836}$ for hydrogen ions. The fast magneto-sonic wave corresponds to the adoption of the upper sign in the above expression. The Alfvén wave is obtained when the lower signs are used. It may be noted that for zero frequency the velocity v_A of the Alfvén wave is given by

$$v_A = \frac{c}{\sqrt{K_0}} = H\sqrt{\frac{\mu_0}{nM}}. \tag{5}$$

Inserting the equation for N into eqs (1) or (2), one easily finds that

$$\frac{u}{v_A} = \left(1 \mp \frac{\Omega_i}{\omega}\right)\left[\left(1 + \frac{\omega}{\Omega_i}\right)\left(1 - \alpha\frac{\omega}{\Omega_i}\right)\right]^{\frac{1}{2}} \tag{6}$$

where the upper sign is to be chosen for slower-than-light motion and the lower sign for faster-than-light motion.

Ginzburg [1961] has shown that the intensity of the cyclotron radiation is identically zero for the slower-than-light case when the propagation is longitudinal (i.e., $\theta=0$). Thus, the only case of physical interest is when the lower or + sign is used in the above equation.

Actually, a charge moving along a helical path also radiates harmonics of the Larmor frequency Ω_i. Thus, on replacing Ω_i in eq (6) by $p\Omega_i$ where p is an integer, we find that the equation, for faster-than-light motion, becomes

$$\frac{u}{v_A}=\left(1+\frac{p\Omega_i}{\omega}\right)\left[\left(1+\frac{\omega}{\Omega_i}\right)\left(1-\alpha\frac{\omega}{\Omega_i}\right)\right]^{\frac{1}{2}}. \tag{7}$$

This equation may be used to obtain the frequencies of emission from the specified value of u/v_A.

A specific example is now considered. At a distance of several earth radii H is of the order of 5×10^{-3} gauss * and if $u\simeq 7\times 10^6$ m/sec, $n\simeq 10^2/\text{cm}^3$, then the Alfvén velocity $v_A\simeq 10^6$ m/sec and $\Omega_i\simeq 50$ radians/sec. Solution of eq (7), when $p=1$, leads to the values $\omega/2\pi=0.77$, 777, and 13,000 c/s. It is only the lowest frequency that would be of interest in magnetotelluric investigations.

In general, when the conditions for longitudinal propagation no longer hold, the situation is much more complicated. In this case Ginzburg [1961] has shown that the radiation spectrum extends over a much wider range. He estimates that the excitation zone extends from about 13 earth radii down to about 1.5 earth radii. The lower limit depends on collision damping, noncollision cyclotron absorption, and the increase of the Alfvén velocity to unfavorably large values.

Magneto-hydrodynamic waves of the type described above are probably the main source of the short-period geomagnetic field variations. On the basis of Ginzburg's analysis one would expect the spectrum to peak in the region of 1 c/s or so. However, observations by Campbell [1959], and Berdichevsky and Brunelli [1959] on the surface of the ground indicate that the spectrum peaks around 0.03 c/s with a very rapid decrease for frequencies above about 0.2 c/s. The apparent discrepancy can be reconciled if the transmission characteristics of the ionosphere are considered [Akasofu, 1960].

For longitudinal propagation within the ionosphere the complex propagation constant Γ is given by Hines [1953] and Dungey [1955]

$$\Gamma^2\simeq-\frac{\omega^2}{(H^2\mu_0/\rho_p)}\frac{1-i\nu_p/\omega}{(1\mp i\nu_e/\Omega_e)(1\pm i\nu_p/\Omega_p)} \tag{8}$$

where ρ_p is the mass density of the ions, ν_e is the effective collision frequency from the electrons with the neutral particles, and ν_p is the effective collision frequency for the positive ions.

The transmission coefficient T for the ionosphere is then given approximately by

$$T\simeq-\int_{h_1}^{h_2}(\text{Im}\,\Gamma)dh \tag{9}$$

where h_1 and h_2 are the effective heights of the bottom and top of the ionosphere. This formula gives the ratio of the field below for a wave of unit amplitude incident from above. It is only valid when the medium is slowly varying. For frequencies 0.01, 0.1, and 1 c/s, the corresponding ratios are 0.32, 2.8×10^{-2}, and 8.2×10^{-6} if Dungey's model of the ionosphere is used. This indicates that the ionosphere is almost transparent for frequencies less than about 0.1 c/s. For the higher frequencies, the ionosphere becomes almost opaque. In fact, if the main source of micropulsations is above the F layer it means that the sources must be very intense.

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*The term gauss is actually a magnetic flux density, although it is frequently used in the literature for magnetic field. Note that for magnetic flux, one weber/m^2—10^4 gauss—10^9 gammas.

For these "higher" frequencies it is more probable that the sources are currents at E region heights in the ionosphere produced by some other mechanism [Baker and Martyn, 1953; Baker, 1953].[1] Also, at frequencies of the order of 1 c/s and higher, one may expect contributions from lightning discharges which are known as "sferics." Support for this view is obtained from the work of Holzer and Deal [1956], who find that there is a strong correlation between worldwide thunderstorm activity and the electromagnetic signals in audio and subaudio frequency range.

Because of the uncertainty and the conjectural theoretical nature of the source mechanisms, one could feel very insecure in attempting to interpret field measurements on the earth's surface in terms of crustal features. However, if the ratios of the mutually orthogonal components of the tangential electric and magnetic fields are considered, some very definite conclusions can be drawn about the crustal structure of the earth. Such studies are usually referred to as magneto-tellurics, since the analyses of geomagnetic and telluric (earth-current) variations are combined.

3. Review and Discussion of Previous Work on Magneto-Tellurics

Most investigations of magneto-telluric fields boil down to a study of the interrelation between the tangential components of the horizontal electric and magnetic fields at the surface of the earth. As far as this writer is able to ascertain the first definitive paper dealing with this subject appeared in 1950, and was authored by Tikhonov [1950]. He realized that rapid geomagnetic variations and earth currents, observed at the surface of the earth, must be connected by some definite relationship. He showed that, at low frequencies, the amplitude of the derivative of the component H_x of the magnetic variations is proportional to the (orthogonal) component of the electric field E_y.[2] This was in agreement with the experimentally established fact that there is a proportionality between these quantities. Tikhonov's model of the earth's crust is a planar layer $0 \leqq z \leqq l$ of finite conductivity σ lying upon an ideally conducting substrate. Implicitly in his analysis, it was assumed that horizontal gradients of the fields could be neglected. Thus, for a spectral component of frequency ω he found that, at $z=0$,

$$i\mu_0\omega H_x \cong E_y\gamma \coth(\gamma l) \tag{10}$$

where

$$\gamma = (i\sigma\mu_0\omega)^{\frac{1}{2}} \text{ and } \mu_0 \cong 4\times 10^{-7}.$$

Using previously published data on the observed diurnal variations at Tucson (Arizona) and Zui (USSR), Tikhonov computed the value of σ and l which best filled the first four harmonics. For Tucson, the values were about 4×10^{-3} mhos/m and 1,000 km, respectively. For Zui, the corresponding values were about 3×10^{-1} mhos/m and 100 km.

In a later paper [Tikhonov and Lipskaya, 1952], the horizontal gradient of the fields was allowed to be finite (although the earth's crust was still regarded as a horizontally stratified medium). The authors postulated that the field components of long period may be represented as a wave which is propagated from east to west with the velocity of the earth's rotation. Using the same data as mentioned above, they obtained revised estimates for the conductivity σ of the upper stratum of thickness l under the assumption of an ideally conducting substrate. For Tucson, the values were approximately 10^{-2} mhos/m and 1,100 km, where for Zui the corresponding values were about 7×10^{-1} mhos/m and 110 km. In this paper they also showed that the measured values of the *vertical* magnetic field variations were consistent with the model and the postulation of linear east-to-west motion of the fields.

In the third paper of this sequence [Lipskaya, 1953], the electromagnetic equations for the model described above were cast in a form to clearly demonstrate the various relationships between the field components.

[1] W.H.Campbell has recently proposed that small, short-period changes in the gross ionospheric pattern within the auroral electrojet are a principal cause of micropulsations (private communication, May 1962).

[2] The Cartesian coordinate system is chosen so that the z axis points downward and the earth's surface is $z=0$.

Subsequent to the Russian work mentioned above, Cagniard [1953] published a paper which has been extensively referenced since. His analysis which assumed plane wave incidence, developed formulas which related H_x and E_y on the surface of a stratified conducting medium. A discussion of the limitations in Cagniard's results appeared shortly thereafter [Wait, 1954]. The essential point made by this writer is that the proportionality between H_x and E_y is only valid if the fields themselves do not vary appreciably in a horizontal distance of the order of a "skin depth" in the ground. In the case of a homogeneous flat earth it was indicated that this distance was of the order of $|\gamma^{-1}|$. For example, at a frequency of 10^{-3} c/s, $\sigma \sim 10^{-3}$ mhos/m, $|\gamma^{-1}|$ is about 350 km. Consequently, the field should be uniform over a considerably broad area to permit the Cagniard interpretive procedure to be applied. This limitation is, of course, in addition to the requirement that the crustal layers themselves are uniform. Very recently Price [1962] has indicated that the limitation mentioned [Wait, 1954] becomes much more stringent when the magneto-telluric method is applied to a stratified earth in certain important instances.

In a later paper, Tikhonov and Shakhsuvarov [1956] discussed the methods for calculating the admittance H_x/E_y at the surface of a horizontally stratified earth of any number of layers. Again, horizontal field gradients were neglected. Actually, the formulas given were a special case of an earlier general analysis [Wait, 1953a and b] where the incidence was oblique. Tikhonov and Shakhsuvarov [1956] give a few curves of the ratio H_x/E_y for both two- and three-layer structures. Some asymptotic approximations of the impedance formula for parallel layers were discussed by Berdichevsky and Brunelli [1959]. The derivation of the basic formulas is very similar to that found in earlier papers [e.g., Wait, 1953a].

The problem has also been investigated by Scholte and Veldkamp [1955] in Holland. Their analysis of the variations of H_x and E_y is essentially the same as those of Tikhonov, Cagniard, and others. By a relatively simple method of data analysis they estimate the amplitude and phase of H_r/E_y without actually performing spectral analyses. They used the experimental data from the magnetic observatory at Witteveen for frequencies in the range from 10^{-4} to 10^{-1} sec. The resulting curves of amplitude and phase versus frequency fitted a two-layer earth model with the upper conductivity $\sigma_1=2$ mhos/m and the lower conductivity $\sigma_2=10^{-1}$ mhos/m. The thickness of the upper stratum was $d_1=600$ m. In the same paper Scholte and Veldkamp also discuss the influence of earth curvature on the ratio H_y/E_x. They conclude that the effect is very small. (A more general theory is given in the appendix.)

An analysis of the continuously varying conductivity profile was carried out by Bossy and De Vuyst [1959] in Belgium. They showed that for normal incidence the ratio H_x/E_y on the surface could be expressed in closed form when the conductivity $\sigma(z)$ varied with depth in the manner

$$\sigma(z)=\sigma_0\left(1+\frac{z}{a}\right)^{-\beta} \tag{11}$$

where σ_0 and a are constants. They used such a model to interpret experimental data of the phase of E_x/H_y for frequencies from about 3×10^{-3} to 1 c/s at Dourbes. However, they found it necessary to modify the model by having a (well conducting) homogeneous surface layer of thickness about 500 m overlying the inhomogeneous substrate of poor conductivity (with β about 9).

In all the work mentioned above, the natural magneto-telluric field has been studied only at frequencies less than about 1 c/s. Vladimirov [1960] has described an investigation of the use of higher frequencies in the range from 0.3 to 1,000 c/s. Field experiments were made in the Ryl'sk district of the Kursk region in the USSR. The area consists of sand and clay deposits underlain at a depth of approximately 500 m by a crystalline base of practically infinite resistivity. The measured values of the ratios $|E_y/H_x|^2$ and $|E_x/H_y|^2$ on the surface were consistent with an assumed two-layer structure where the upper conductivity $\sigma_1\cong5\times10^{-2}$ mhos/m and the lower conductivity $\sigma_2\cong0$. From the experimental curve, the thickness of the upper stratum was deduced to be 450 m. In a subsequent paper Vladimirov and Kolmakov [1960] discuss the resolving power of the magneto-telluric method. They confine their discussion to a three-layer model with an infinite resistive basement. The principle of equivalence stated by them is that for a given conductivity σ_1 and thickness h_1 of the uppermost layer, the theoretical three-layer curves of $|H_x/E_y|$ practically preserve their shape when the conductivity σ_2 and thickness h_2 of the intermediate layer vary over specified limits. (This question is discussed again in the present paper.) Vladimirov and Kolmakov then conclude that the same curve may characterize different geoelectric sections and can be interpreted only with known values of σ_2 and h_2. Some further comments on the same subject are given by Vladimirov and Nikiforova [1961] and Vladimirov and An [1961].

In an interesting paper, Chetaev [1960], also from the USSR, illustrates a procedure to obtain information about the anisotropy of a homogeneous stratum by measuring the surface impedance in orthogonal directions. The model he considers is a half-space with a medium whose longitudinal conductivity is σ_l and the transverse conductivity is σ_t. The angle of inclination of the anisotropy is α (i.e., $\alpha=0$ corresponds to a flat lying anisotropic medium with σ_t the horizontally directed conductivity). Chetaev shows that if the electric component E_y is transverse to the strike of the structure,

$$Z=E_y/H_x=-(i\mu\omega/\sigma_t)^{\frac{1}{2}}[1+[(\sigma_t/\sigma l)-1]\sin^2\alpha]^{\frac{1}{2}} \tag{12}$$

and if the electric component E_x is along the strike,

$$Z=E_x/H_y=(i\mu\omega/\sigma_l)^{\frac{1}{2}} \tag{13}$$

which is not affected by σ_t Since σ_t/σ_l is usually greater than one, the minimum value of the surface impedance $|Z|$, as a function of azimuth, is $(\mu\omega/\sigma_l)^{\frac{1}{2}}$ which occurs when the electric vector is along the strike.

It has been pointed out by Pokityanski [1961] that the interpretation of field results should take account of the anisotropy of the structure. In fact, in general, the tangential fields are related by

$$E_x=Z_{xx}H_y-Z_{xy}H_x \tag{14}$$

$$E_y=Z_{yx}H_y-Z_{yy}H_x \tag{15}$$

where the coefficients are the components of a surface impedance tensor.

In the case of parallel stratified or flat-lying media Z_{xy} and Z_{yx} vanish and $Z_{xx}=Z_{yy}$. Of course, the components of the impedance tensor depend on the choice of the (x,y) axes. For an anisotropic homogeneous half-space or for an inhomogeneous structure where the conductivity does not vary along one of the horizontal directions, it is possible to find special directions of x and y (denoted u and v, respectively), such that

$$E_u=Z_1H_v \tag{16}$$

and

$$E_v=-Z_2H_u \tag{17}$$

where Z_1 and Z_2 are the principal values of the tensor impedance. Consequently, if the u and v components of the tangential fields are measured, Z_1 and Z_2 may be simply calculated. Unfortunately, those principal directions are not always known beforehand. Pokityanski [1961] has described an ingenious graphical scheme to determine the u and v directions from measurements of the nonorthogonality of the tangential E and H vectors. Strictly speaking, the method is only valid for source fields which are linearly polarized. A more straightforward procedure would be to return to the surface impedance relations and write them in the form

$$Z_x=E_x/H_y=[Z_{xx}-Z_{xy}\alpha] \tag{18}$$

and

$$Z_y=E_y/H_x=-[Z_{yy}-Z_{yx}/\alpha] \tag{19}$$

where $\alpha=H_x/H_y$ and where Z_x and Z_y are the impedances as measured in the x and y directions, respectively.

It is clear that Z_x and Z_y depend on α and, consequently, they will be a function of the source field. However, the elements of the surface impedance tensor can be calculated if at least two independent measurements are made. Thus, for a particular frequency, these will provide the following sets of values, $Z_{x,1}$, $Z_{y,1}$, α_1 and $Z_{x,2}$, $Z_{y,2}$, and α_2 where the subscript 1 or 2 is used to distinguish between the two measurements. Then, provided $\alpha_1 \neq \alpha_2$, it easily follows that

$$Z_{xy}=\frac{Z_{x,1}-Z_{x,2}}{\alpha_1-\alpha_2}, \tag{20}$$

$$Z_{yx}=\frac{Z_{y,1}-Z_{y,2}}{(1/\alpha_2)-(1/\alpha_1)}; \tag{21}$$

$$\begin{aligned} Z_{xx}&=Z_{x,1}+(Z_{xy}/\alpha_1)\\ &=Z_{x,2}+(Z_{xy}/\alpha_2); \end{aligned} \tag{22}$$

$$\begin{aligned} Z_{yy}&=-Z_{y,1}+(Z_{yx}/\alpha_1)\\ &=-Z_{y,2}+(Z_{yx}/\alpha_2). \end{aligned} \tag{23}$$

In principle these equations could be used to determine the elements of the surface impedance tensor for an elliptically polarized field. It is probable that the apparent scatter in the magneto-telluric data, as measured by Garland and Webster [1960] and Watt et al. [1962], is a consequence of the anisotropy or of tilting the structure.

In a recent paper, Kovtun [1961] has discussed, in a general way, the nature of the magneto-telluric fields for two-dimensional inhomogeneous structures. It is assumed that electrical properties vary only in the (x,z) plane; the surface of the earth is the (x,y) plane. It is stated incorrectly that the field can be decomposed into an *independent* set of TM (transverse magnetic) and TE (transverse electric) waves. While this may be true for the incident or primary field, it is overlooked that the boundary conditions couple these waves together [Wait, 1959] in the general case. Consequently, his subsequent discussion can only be considered approximate.

Recent activity in the United States on magneto-tellurics is localized mainly at the Massachusetts Institute of Technology and the University of Texas, and DECO Electronics, Inc., in Boulder, Colo. Unfortunately, little of the work has been published in the open literature.

In a short note, Cantwell and Madden [1960] at M.I.T. report some preliminary magneto-telluric measurements in Massachusetts covering the frequency range from 5×10^{-3} to 1 c/s. The analysis of the records made use of auto- and cross-correlation techniques so that estimates of the coherency between electric and magnetic signals were obtained. For the interpretation they reject records which have low coherency as they are claimed to result mainly from noise due to the instability of the coil mounts. According to Cantwell and Madden [1960],[3] the signals associated with high coherency between electric and magnetic signals yield consistent estimates of resistivity. They find that their data is compatible with a two-layer model with the upper, conductivity $\sigma_1 \simeq 1.2\times10^{-4}$ mhos/m and the lower conductivity $\sigma_2 < 1.2\times10^{-1}$ mhos/m. The thickness h_1 of the upper stratum is 70 km.

4. New Numerical Results for Horizontally Stratified Structure

Although this paper has been primarily a review of the theories and concepts in magneto-tellurics, it is considered to be worthwhile to include some specific curves which can be used to interpret future experimental data. The numerical results were actually obtained in connection with theoretical studies of radio wave propagation in the presence of stratified media. Here, for convenience, the results are given in dimensionless form to facilitate and broaden the application to field problems.

[3] Apparently the curves in figure 3 of Cantwell's and Madden's paper are mislabeled. The values indicated for ρ_2 should all be divided by 10.

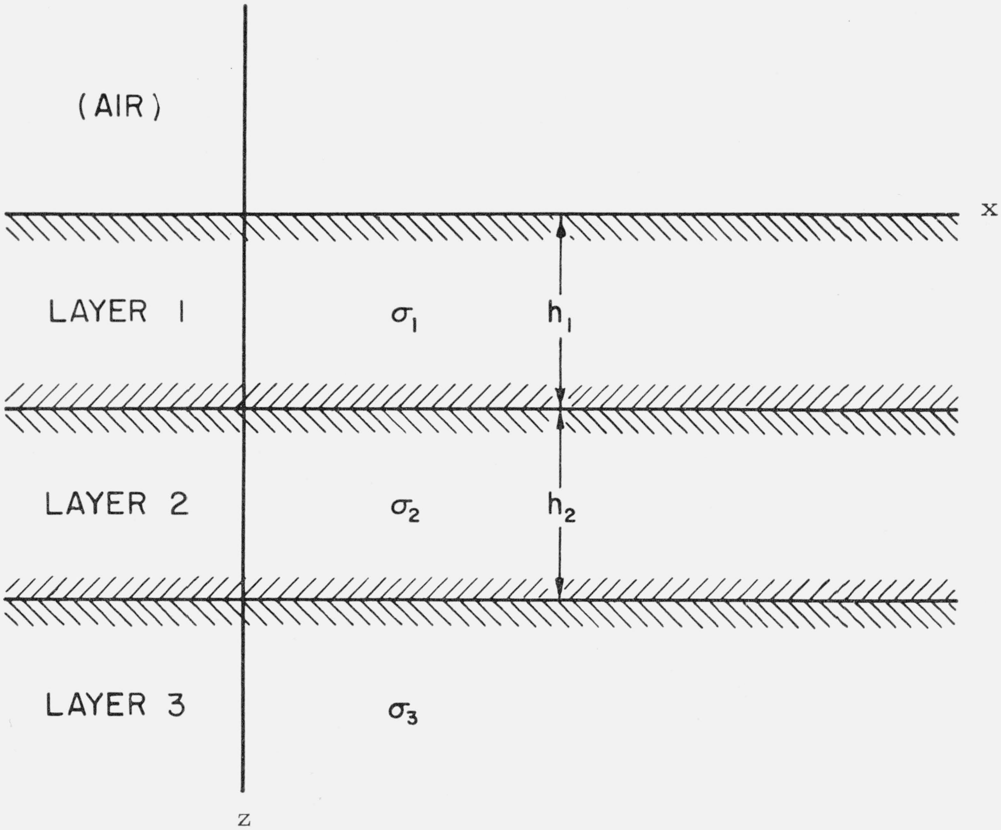


FIGURE 1. *Stratified model of the earth's crust.*

The model is quite simple, the earth is assumed to be horizontally stratified and consists of three homogeneous layers. The upper layer is of thickness h_1 with conductivity σ_1, the middle or intervening layer is of thickness h_2 with conductivity σ_2, and the bottom layer is of infinite thickness and has a conductivity σ_3. In terms of a Cartesian coordinate system (x,y,z) the earth's surface is $z=0$ and the interfaces between layers[4] are at $z=h_1$ and $z=h_1+h_2$. The magnetic permeability is assumed to be constant throughout and equal to μ_0. Furthermore, displacement currents are neglected in all three conducting layers. The problem is now to find an expression for the ratios of the tangential fields E and H at the boundary between free space and the earth. If the horizontal gradients of the fields are negligible the result can be written (for a time factor $e^{i\omega t}$)

$$Z=\frac{-E_y}{H_x}=\frac{E_x}{H_y}=\left(\frac{i\mu_0\omega}{\sigma_1}\right)^{\frac{1}{2}} Q\left(\sqrt{\sigma_1\mu_0\omega}h_1, \frac{\sigma_2}{\sigma_1}, \frac{\sigma_3}{\sigma_1}, \frac{h_1}{h_2}\right) \tag{24}$$

where Q is a function of the four variables indicated. Explicitly,

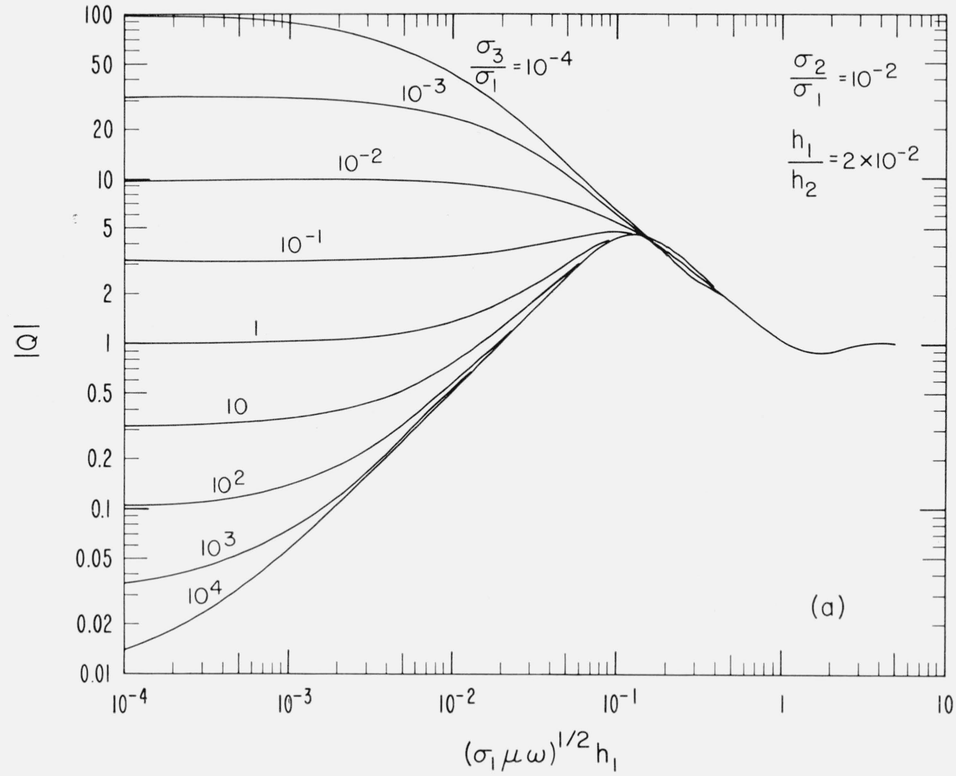
$$Q=\frac{\hat{Q}+(\sigma_2/\sigma_1)^{\frac{1}{2}}\tanh\left[(i\sigma_1\mu_0\omega)^{\frac{1}{2}}h_1\right]}{(\sigma_2/\sigma_1)^{\frac{1}{2}}+\hat{Q}\tanh\left[(i\sigma_1\mu_0\omega)^{\frac{1}{2}}h_2\right]} \tag{25}$$

where

$$\hat{Q}=\frac{1+(\sigma_3/\sigma_2)^{\frac{1}{2}}\tanh\left[(i\sigma_2\mu_0\omega)^{\frac{1}{2}}h_2\right]}{(\sigma_3/\sigma_2)^{\frac{1}{2}}+\tanh\left[(i\sigma_2\mu_0\omega)^{\frac{1}{2}}h_2\right]}. \tag{26}$$

This result follows directly from a previously derived general formula applicable to any number of layers [Wait, 1953a and b].

[4] See figure 1.



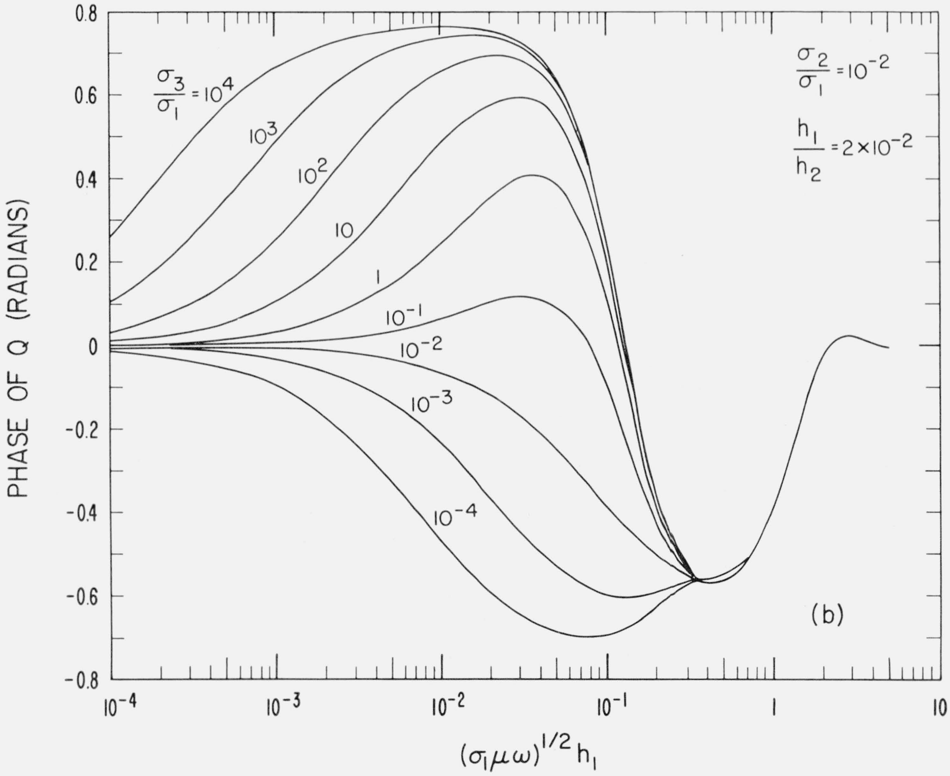


FIGURE 2. *Amplitude and phase of the complex quantity* Q *for a three-layer model plotted as a function of the dimensional frequency/depth factor* $(\sigma_1\mu\omega)^{\frac{1}{2}}h_1$.

[For this set of curves the exciting field is assumed to be uniform (see text).]

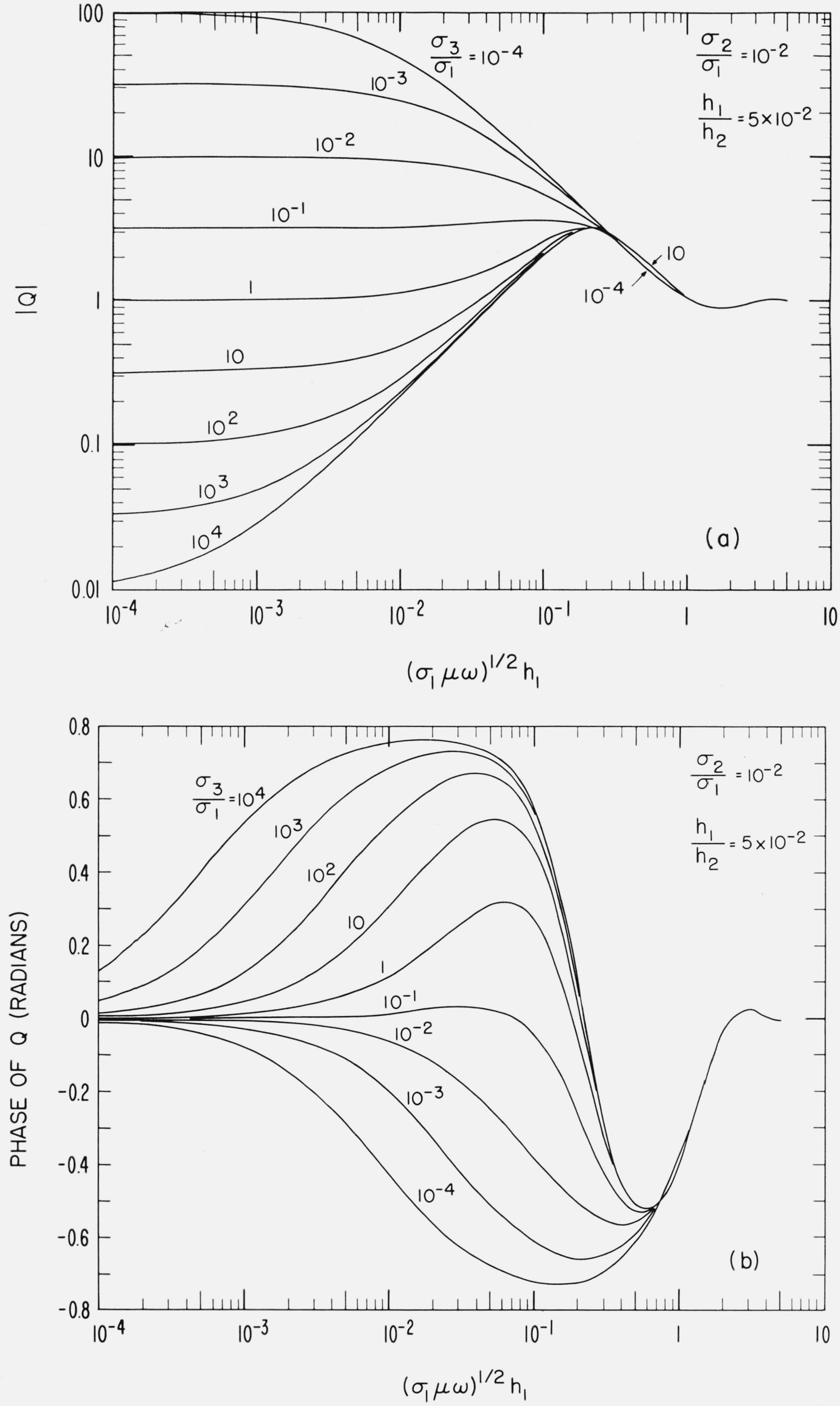


FIGURE 3. *Amplitude and phase of the complex quantity Q for a three-layer model plotted as a function of the dimensional frequency/depth factor* $(\sigma_1\mu\omega)^{\frac{1}{2}}h_1$.

[For this set of curves the exciting field is assumed to be uniform (see text).]

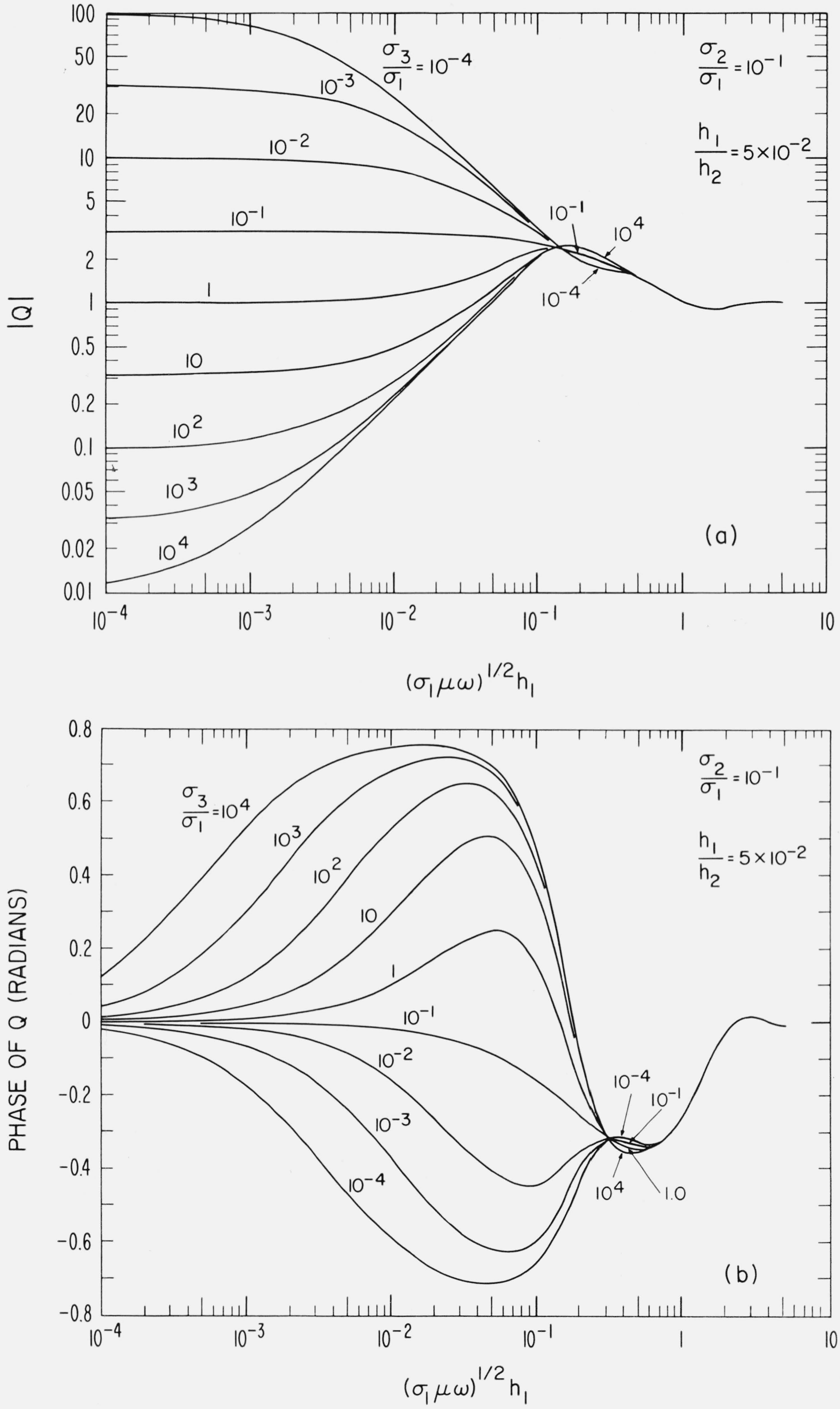


FIGURE 4. *Amplitude and phase of the complex quantity Q for a three-layer model plotted as a function of the dimensional frequency/depth factor* $(\sigma_1\mu\omega)^{\frac{1}{2}}h_1$.

[For this set of curves the exciting field is assumed to be uniform (see text).]

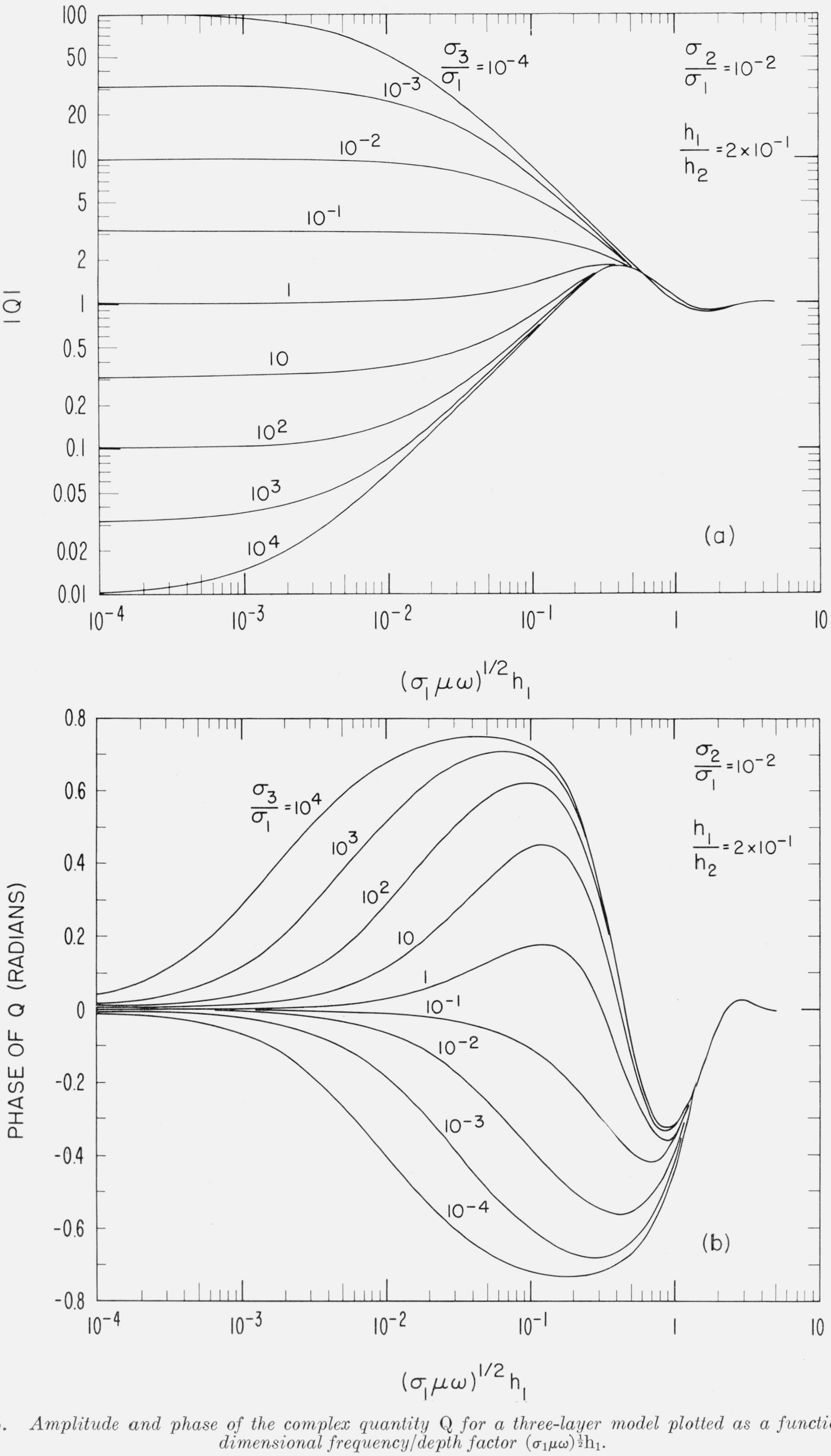


FIGURE 5. *Amplitude and phase of the complex quantity Q for a three-layer model plotted as a function of the dimensional frequency/depth factor* $(\sigma_1\mu\omega)^{\frac{1}{2}}h_1$.

[For this set of curves the exciting field is assumed to be uniform (see text).]

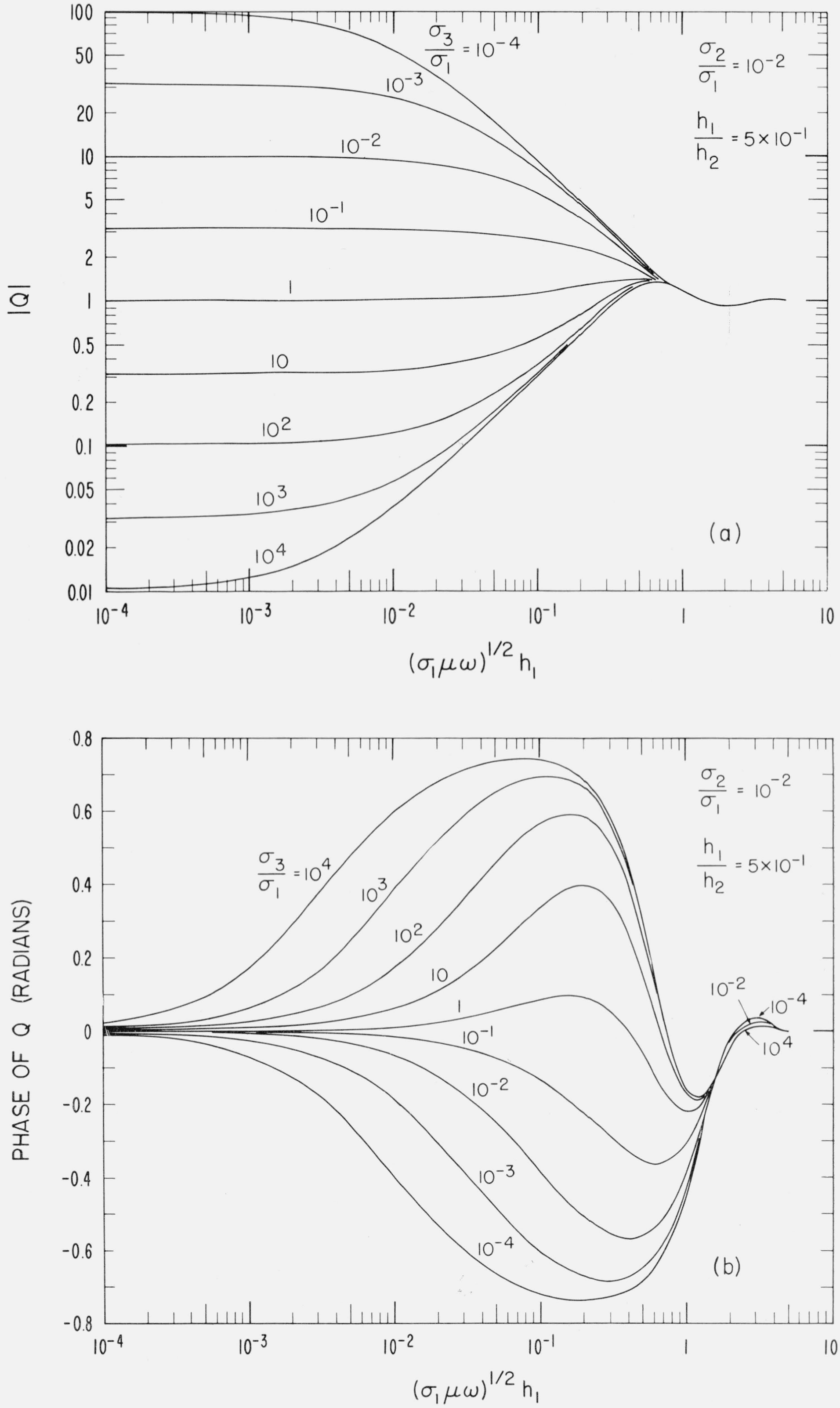


FIGURE 6. *Amplitude and phase of the complex quantity Q for a three-layer model plotted as a function of the dimensional frequency/depth factor* $(\sigma_1\mu\omega)^{\frac{1}{2}}h_1$.

[For this set of curves the exciting field is assumed to be uniform (see text).]

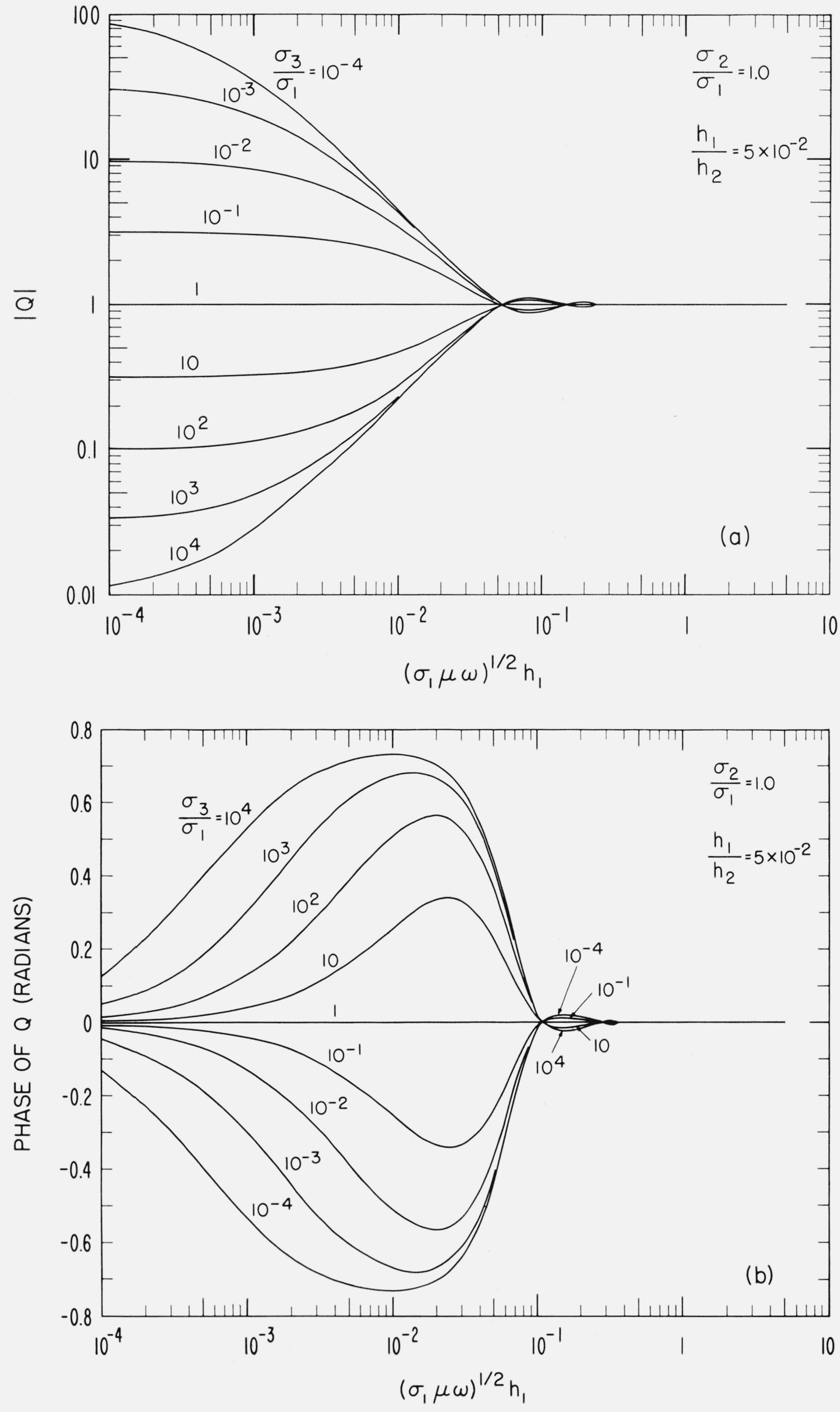


FIGURE 7. *Amplitude and phase of the complex quantity Q for a three-layer model plotted as a function of the dimensional frequency/depth factor* $(\mu_1\sigma\omega)^{\frac{1}{2}}h_1$.

[For this set of curves the exciting field is assumed to be uniform (see text).]

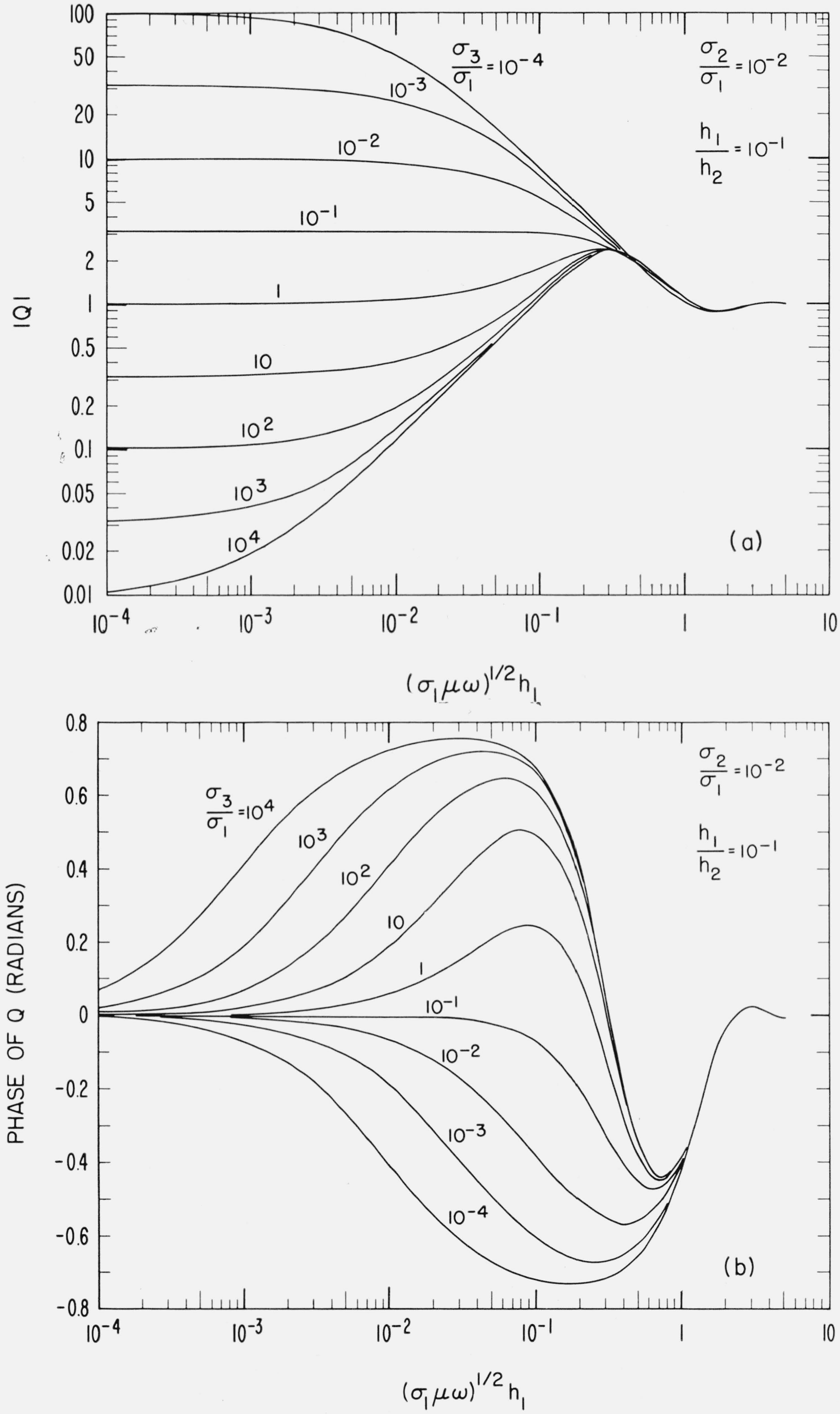


FIGURE 8. *Amplitude and phase of the complex quantity Q for a three-layer model plotted as a function of the dimensional frequency/depth factor* $(\sigma_1\mu\omega)^{\frac{1}{2}}h_1$.

[For this set of curves the exciting field is assumed to be uniform (see text).]

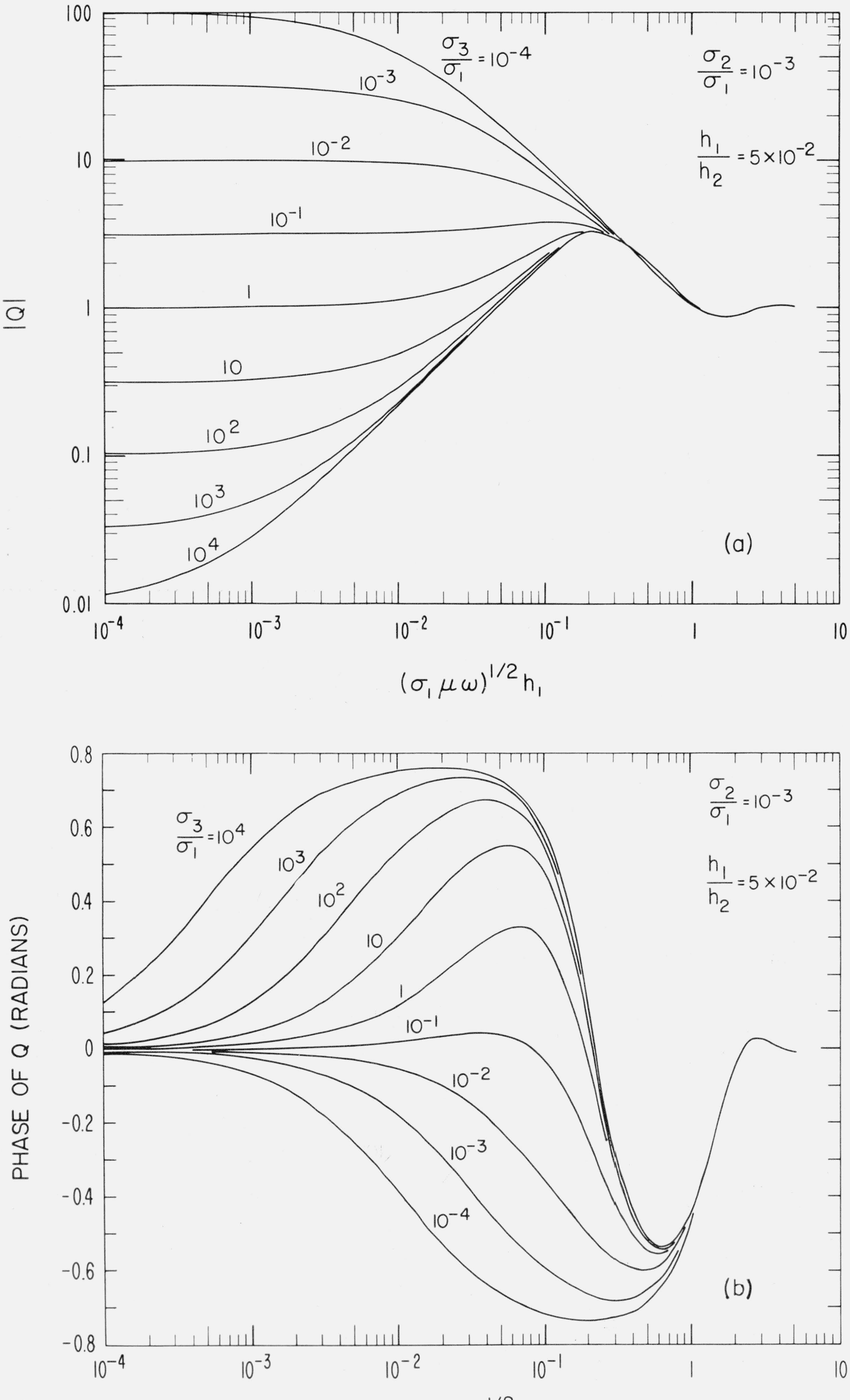


FIGURE 9. *Amplitude and phase of the complex quantity* Q *for a three-layer model plotted as a function of the dimensional frequency/depth factor* $(\sigma_1\mu\omega)^{\frac{1}{2}}h_1$.

[For this set of curves the exciting field is assumed to be uniform (see text).]

It can be seen from eq (25) that if $h_1 \to \infty$, Q tends to unity. This suggests that, for the general case, one defines an apparent conductivity σ_a such that

$$Z = \left(\frac{i\mu_0\omega}{\sigma_a}\right)^{\frac{1}{2}} = \left(\frac{i\mu_0\omega}{\sigma_1}\right)^{\frac{1}{2}} Q. \tag{27}$$

Thus

$$Q = (\sigma_1/\sigma_a)^{\frac{1}{2}} \text{ or } \sigma_a/\sigma_1 = Q^{-2}.$$

Clearly, σ_a must be complex in order to admit such an equivalence with a homogeneous half-space. The concept of an equivalent or complex conductivity has been discussed previously in connection with radio propagation over stratified media [Wait, 1953b, 1958].

Extensive numerical results for Q over a wide range of the four parameters are available [Jackson et al., 1962]. These greatly extend the set of curves of $|\sigma_a/\sigma_1|$, for $\sigma_3=0$ only, published by Yungul [1961]. A sample of some of these numerical data is given in this paper in graphical form. In the first set shown in figures 2a to 6b, the ratio σ_2/σ_1 is fixed at 1/100. This corresponds to an intervening layer of very poor conductivity. In each case, the abscissa is the dimensionless ratio $(\sigma_1\mu_0\omega)^{\frac{1}{2}}h_1$. For these five sets of curves, the values of h_1/h_2 are 1/50, 1/20, 1/10, 1/5, and 1/2, respectively. Thus, in all cases, the upper layer is thin compared with the middle or intervening layer.

It is interesting to note that for the low frequencies [corresponding to small values of $(\sigma_1\mu_0\omega)^{\frac{1}{2}}h_1$], σ_a always approaches σ_3 the conductivity of the bottom layer. At the high frequencies, σ_a approaches σ_1 as noted above. In the intermediate region an interesting transition takes place. The right portion of the curves is roughly characteristic of a two-layer structure and the shape is determined mainly by the characteristics of the upper layer.

In the second set of curves shown in figures 7a to 9b, the ratio h_1/h_2 is fixed at 1/20 and the conductivity ratio σ_2/σ_1 takes the values 1, 0.1, and 10^{-3}. In the first set shown in figures 7a and 7b, the upper two layers combined in a single layer of thickness (h_1+h_2). For this case, the results are identical to the standard two-layer curves [Cagniard, 1953; Wait, 1953a]. The interesting thing about the three sets of curves, in figures 8a to 9b, is their similarity. Here it appears that the conductivity of the intervening layer σ_2 plays a very small role. Actually, this is a general characteristic of all three-layer magneto-telluric curves provided σ_2 is somewhat less than σ_1.

The relative insensitivity of the factor Q to the conductivity and thickness of a poorly conducting intervening layer can be demonstrated directly from the basic equations given above. For example, under the conditions that

$$(\sigma_2\mu_0\omega)^{\frac{1}{2}}h_2 << 1$$

$$\hat{Q} \simeq (\sigma_2/\sigma_3)^{\frac{1}{2}}$$

and thus eq (25) becomes

$$Q \simeq \frac{1+(\sigma_3/\sigma_1)^{\frac{1}{2}} \tanh\,[(i\sigma_1\mu\omega)^{\frac{1}{2}}h_1]}{(\sigma_3/\sigma_1)^{\frac{1}{2}} + \tanh\,[(i\sigma_1\mu\omega)^{\frac{1}{2}}h_1]}. \tag{28}$$

Therefore, under the restriction stated above, the three-layer structure is equivalent to a two-layer model whose parameters are σ_1, σ_3, and h_1. The result is essentially independent of σ_2 and h_2.

It is worth pointing out that a dual three-layer model exists which broadens the application of these numerical results. The parameters of the dual model are indicated by primed symbols and they are related to the original problem as follows:

$$\sigma_2'/\sigma_1' = \sigma_1/\sigma_2$$

$$\sigma_3'/\sigma_1' = \sigma_1/\sigma_3$$

$$(\sigma_1'\mu_0\omega)^{\frac{1}{2}}h_1' = (\sigma_1\mu_0\omega)^{\frac{1}{2}}h_1$$

$$(\sigma_2'\mu_0\omega)^{\frac{1}{2}}h_2' = (\sigma_2\mu_0\omega)^{\frac{1}{2}}h_2.$$

It then immediately follows from eqs (25) and (26) that

$$Q'\left(\sqrt{\sigma_1\mu_0\omega}h_1', \frac{\sigma_2'}{\sigma_1'}, \frac{\sigma_3'}{\sigma_1'}, \frac{h_1'}{h_2'}\right)=\frac{1}{Q\left(\sqrt{\sigma_1'\mu_0\omega}h_1, \frac{\sigma_2}{\sigma_1}, \frac{\sigma_3}{\sigma_1}, \frac{h_1}{h_2}\right)}. \tag{29}$$

The apparent conductivity σ_a' in the dual or transformed problem is given by

$$\sigma_a'/\sigma_1'=(Q')^{-2}=Q^2.$$

Consequently, the ordinate in figures 2a, 3a, to 9a can be regarded as the quantity $|(\sigma_a'/\sigma_1')^{\frac{1}{2}}|$ in the transformed problem. The conductivity ratios *are all inverted* and

$$\frac{h_1'}{h_2'}=\frac{\sigma_1}{\sigma_2}\frac{h_1}{h_2}.$$

As is obvious, the ordinate in the phase curves in figures 2b, 3b, to 9b is the *negative* of the phase of Q'.

In the five sets of curves from figures 2a to 6b, the values of h_1'/h_2' are thus 2, 5, 10, 20, and 50, respectively, and the conductivity ratio σ_2'/σ_1' is fixed at 100.

The dual property of magneto-telluric curves for horizontally stratified structures has been discussed recently by Kolmakov [1961]. His analysis seems to be unnecessarily involved. Actually, using our correction factors (i.e., amplitude and phase of Q), the duality is almost self-evident.

The curves shown in the preceding figures are based on the important and often overlooked assumption that the exciting field is effectively uniform. Stated in another way, the fields should not vary appreciably in the horizontal direction. To indicate the significance of this assumption for a horizontally stratified medium, it is desirable to consider a two-dimensional spatially periodic field. The coordinate system may then be chosen so that $\partial/\partial y=0$ and $\partial/\partial x=-ikS$ where $k=2\pi$/free-space wavelength$=\omega/c$ and S is a dimensionless quantity which may be complex. Therefore, the fields are of the form.

$$u(x,z)\sim f(z)\exp(-ikSx) \tag{30}$$

where $f(z)$ is some function of z. The usual assumption in magneto-telluric studies is to set $S=0$ insofar as the fields in the earth are concerned. Cagniard [1953, 1954] justifies the assumption by saying that kx is always small compared with unity; consequently, $\exp(-ikSx)$ may be replaced by unity. The fallacy in his argument is that S may be very large when the source of the field is in the near zone. Cagniard does not recognize this important fact and, rather surprisingly, he imagines the source field to be a uniform plane wave at a real angle of incidence. In this trivial case S is then the sine of the (real) angle of incidence and, of course, it could never exceed unity.

To demonstrate that S may be greater than one, we need only choose a very simple model for the source. It is a uniform line current I at some height H above the surface of the earth. For simplicity, it will be located at $z=-H$ and runs parallel to the y axis of the coordinate system (x,y,z). The primary field (neglecting the influence of the earth) of this line current has only y component of the electric field. It is well known and is given, in terms of a modified Bessel function, by [Wait, 1959]

$$E_y=\frac{i\mu_0\omega I}{2\pi}K_0[ikR] \tag{31}$$

where $R=[(z+H)^2+x^2]^{\frac{1}{2}}$ is the distance to the current line from the observer at (x,z). If, rather hypothetically, $kR>>1$, the Bessel function may be approximated by the first term of its asymptotic expansion. Then

$$E_y=-\frac{i\mu_0\omega I}{2\pi}\left(\frac{\pi}{2ikR}\right)^{\frac{1}{2}}e^{-ikR}. \tag{32}$$

Thus the magnitude of the field is proportional to $(kR)^{-\frac{1}{2}}$ and the wave front is nearly plane. However, unfortunate as it may be, this limiting form is never achieved in the micropulsation region. For example, at $\omega/2\pi=10^{-2}$ the wavelength $2\pi/k=3\times10^7$ km, which is rather large. In nearly all cases of practical interest the observer is in the near zone where $kR<<1$. Thus

$$E_y\simeq-\frac{i\mu_0\omega I}{2\pi}\,[\log\,(kR/2)+0.5772\ldots], \tag{33}$$

which bears little similarity to a plane wave.

To demonstrate the existence of S values greater than unity, it is desirable to write the equation in the form of an integral [Wait, 1953b], thus

$$E_y=\frac{i\mu_0\omega I}{2\pi}\int_0^\infty\frac{\exp\,[-ikC|z+H|]}{C}\,e^{-ikSx}dS \tag{34}$$

where $C=(1-S^2)^{\frac{1}{2}}$. In the far field, where $(kR)>>1$, the important values of S are in the range between 0 and 1 and the integral can be evaluated by the method of stationary phase to yield eq (32). However, if kR is small the important contributions in the integral are for large values of S.

To focus attention on the physical aspects of the problem, a single harmonic component of the spectrum is considered. Also, initially the earth is assumed to be a homogeneous half-space of conductivity σ_1 and permeability μ_0. The earth curvature is neglected and the assumption is justified in the appendix. Thus, for a source field which varies as exp $(-ikSx)$ and when the electric field has only x component, the surface impedance is easily found to be [Wait, 1953b, 1958]

$$Z_1=-E_y/H_x]_{z=0}=\left(\frac{i\mu_0\omega}{\sigma_1}\right)^{\frac{1}{2}}(1-i\beta)^{-\frac{1}{2}} \tag{35a}$$

where

$$\beta=k^2S^2/\sigma_1\mu_0\omega.$$

If $\beta<<1$

$$Z_1\simeq\left(\frac{i\mu_0\omega}{\sigma_1}\right)^{\frac{1}{2}} \tag{35b}$$

which is the value appropriate for a homogeneous half-space under the assumption of negligible horizontal gradients. Since kS can be identified physically with $\partial/\partial x$ it follows that the condition $\beta<<1$ is equivalent to saying that the surface fields vary in the x direction in a distance small compared with δ where

$$\delta=[2/(\sigma_1\mu\omega)]^{\frac{1}{2}}$$

is the skin depth of the conducting medium. At a frequency of 0.1 c/s and for a conductivity $\sigma_1\simeq10^{-3}$ mhos/m, δ is approximately 50 km. Consequently, if the fields vary appreciably in a distance of the order of 50 km over the surface of a homogeneous earth, some departure from eq (35b) is to be expected at frequencies less than 0.1 c/s. This limitation was pointed out previously [Wait, 1954] using a somewhat different argument.

When the earth becomes horizontally stratified, the limitation that β is small remains. However, the stringency of this condition varies in a manner depending on the stratification as pointed out by Price [1962]. To demonstrate this interesting phenomenon, the surface impedance for a two-layer earth is considered. The model consists of a homogeneous surface layer of conductivity σ_1 and thickness h_1 and semi-infinite lower layer of conductivity σ_2. Then on the assumption that $\frac{\partial}{\partial x}=-ikS$ and $\frac{\partial}{\partial y}=0$, it follows from previous work [Wait, 1953b] that

$$Z_1=\left(\frac{i\mu\omega}{\sigma_1}\right)^{\frac{1}{2}}(1-i\beta)^{-\frac{1}{2}}Q \tag{36}$$

where

$$Q = \frac{G + \tanh \chi}{1 + G \tanh \chi} \tag{37}$$

where

$$G = \left[\frac{1 - i\beta}{(\sigma_2/\sigma_1) - i\beta}\right]^{\frac{1}{2}} \tag{38}$$

and

$$\chi = (i\sigma_1\mu\omega)^{\frac{1}{2}} h_1 (1 - i\beta)^{\frac{1}{2}}. \tag{39}$$

If h_1 tends to infinity, Q approaches unity and eq (36) reduces to (35a). Thus Q can be regarded again as a correction factor which accounts for the stratification in the earth's crust. If β is replaced by zero, the correction factor becomes

$$Q = \frac{(\sigma_1/\sigma_2)^{\frac{1}{2}} + \tanh\left[(i\sigma_1\mu\omega)^{\frac{1}{2}} h_1\right]}{1 + (\sigma_1/\sigma_2)^{\frac{1}{2}} \tanh\left[(i\sigma_1\mu\omega)^{\frac{1}{2}} h_1\right]}, \tag{40}$$

which is a special case of eq (25) when $h_3 = \infty$.

To illustrate the influence of finite β on the behavior of Q, a number of calculations were carried out on a digital computer. Extensive tabulations of this quantity for a range of values of the parameters are available [Jackson et al., 1962]. Some of the results are shown graphically in figures 10a to 12b. In the first pair, σ_2 is effectively zero corresponding to a conducting layer (of thickness h) lying on an insulating substratum. It is apparent from these curves that a finite value of β leads to a major change in the shape of the curves. The same behavior is evident when the lower layer is finitely conducting as can be evidenced in figures 11a and 11b where $\sigma_1/\sigma_2 = 25$. However, in this case, the effect is not so pronounced. In fact, when the lower layer is relatively highly conducting, the influence of finite β is relatively small as can be seen from figures 12a and 12b.

To discuss the significance of the results shown in figures 10a to 12b, it is desirable to introduce a scale distance L defined by

$$\left|\frac{\partial}{\partial x}\right| = kS = \frac{1}{L}.$$

It is a measure of the horizontal distance in which the field changes by an appreciable amount. For a periodic disturbance, L is the (spatial) wavelength. Thus

$$\beta = \delta^2/(2L^2)$$

where δ is the skin depth in the upper layer. For a highly insulating substratum, it appears that β must be very small or that δ must be much less than L if the standard magneto-telluric interpretation is to be applied. For example, if

$$h_1 = 1.4 \text{ km}, \ \sigma_1 = 10^{-3} \text{ mhos/m}, \ f = 0.1 \text{ c/s and } \sigma_2 << \sigma_1,$$

it follows that

$$\delta = 50 \text{ km and } (\sigma_1\mu\omega)^{\frac{1}{2}} h_1 = \sqrt{2} h_1/\delta \cong 0.02.$$

Then from figure 10a it can be seen that if $\beta < 10^{-3}$, $|Q|$ is within 10 percent of its value for $\beta = 0$. This condition is equivalent to

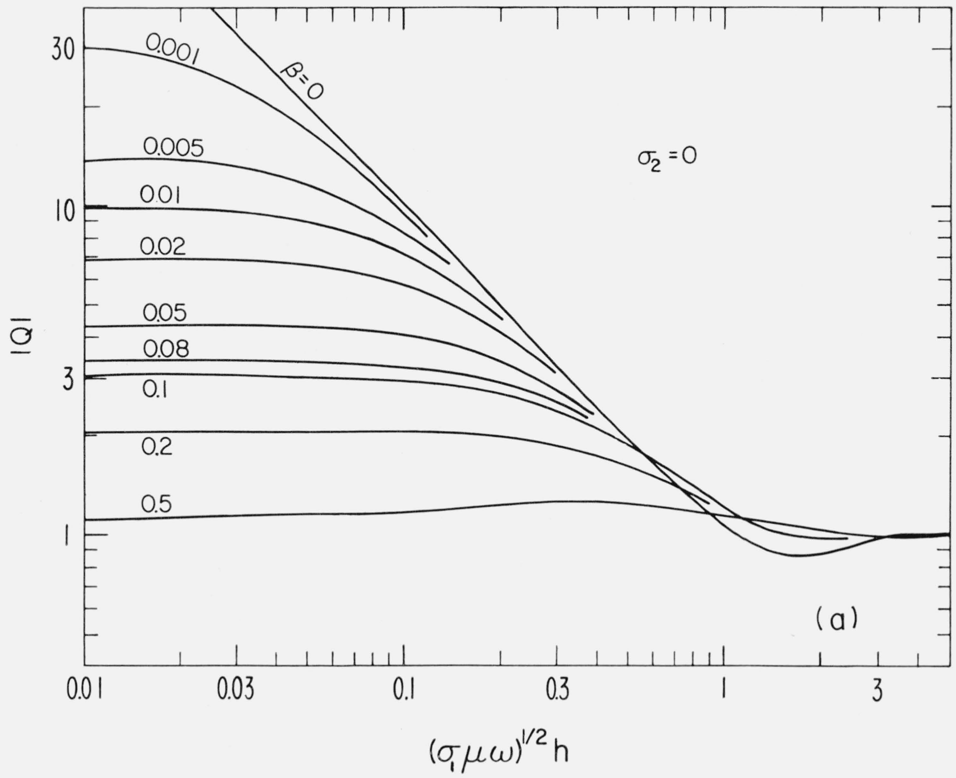
$$L > \delta/(\beta\sqrt{2})$$

or

$$L > 1{,}100 \text{ km}.$$

The condition is even more stringent for the phase.

Uniformity of the exciting fields over distances of the order of 1,000 km would not be very common, particularly in higher latitudes where the currents could be quite localized. Also, at low latitudes the presence of equatorial electrojet [Vestine, 1960] at heights of the order of 100



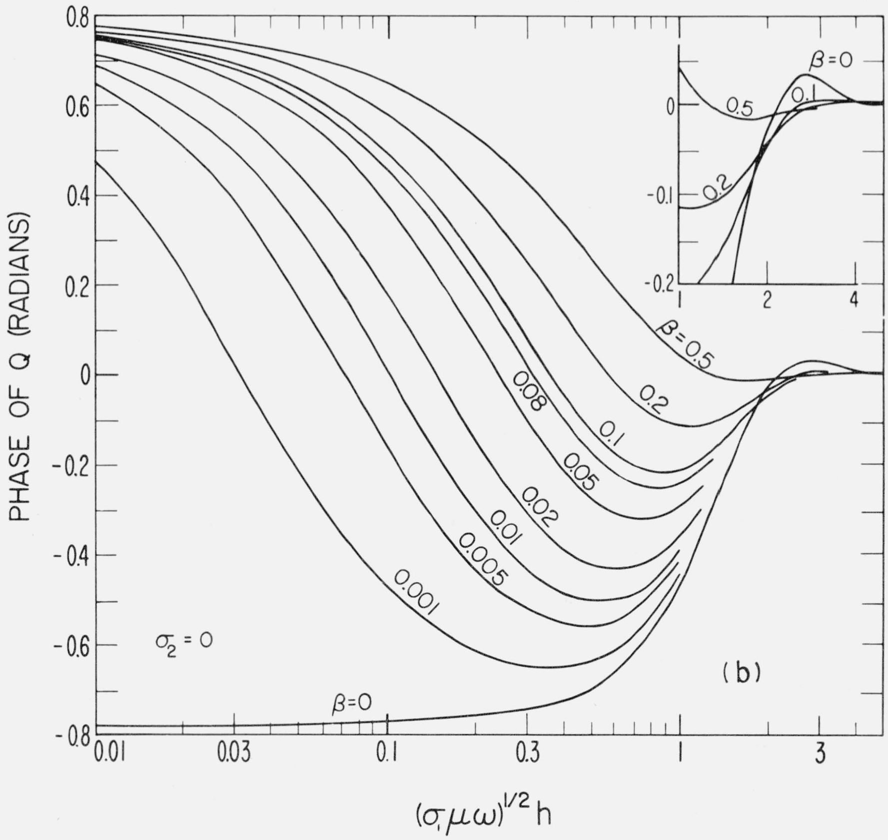


FIGURE 10. *Amplitude and phase of Q for a two-layer model plotted as a function of* $(\sigma_1\mu\omega)^{\frac{1}{2}}h_1$.

[For this set of curves the exciting field is non-uniform (i.e., $\beta>0$, see text).]

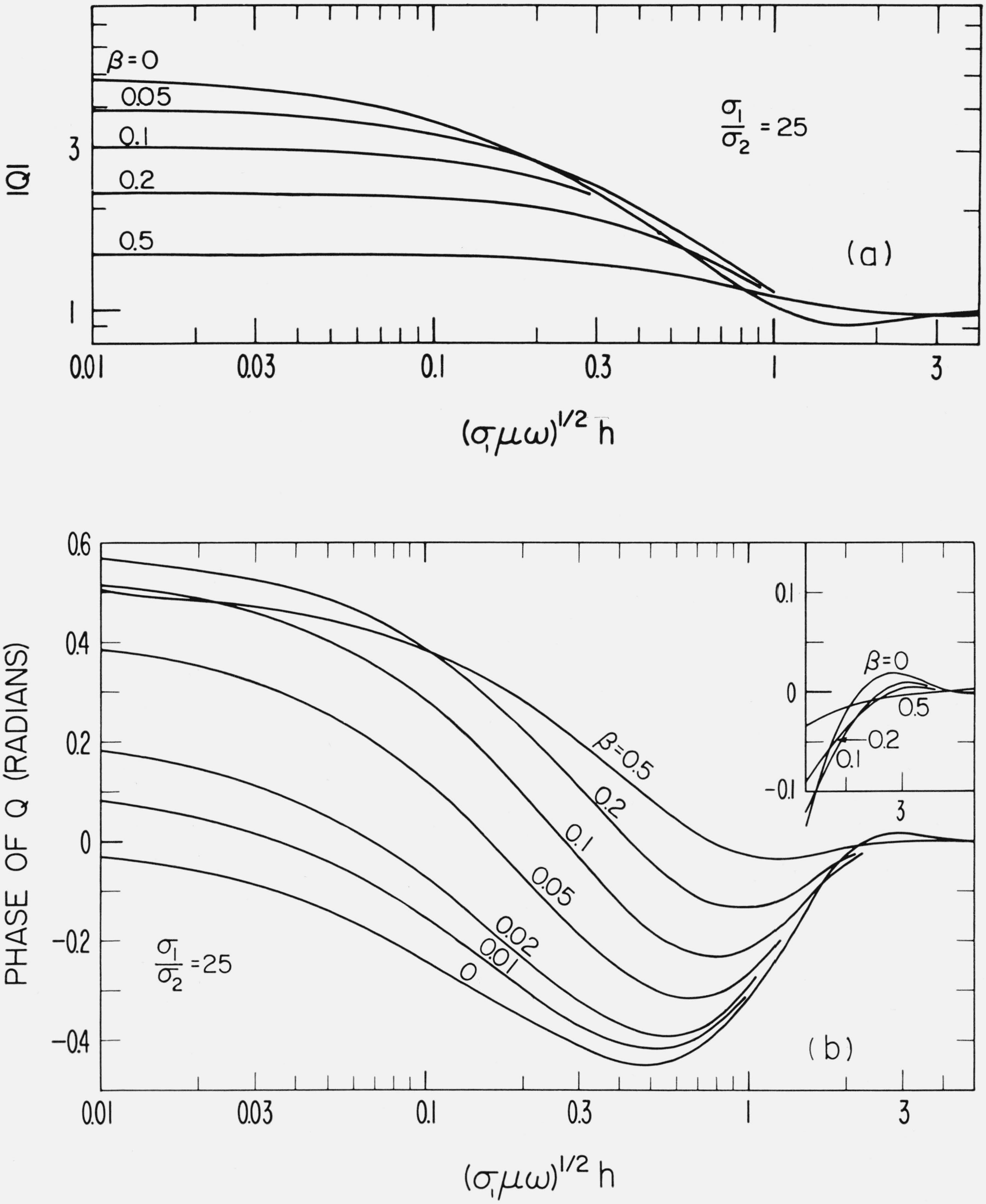


FIGURE 11. *Amplitude and phase of* Q *for a two-layer model plotted as a function of* $(\sigma_1\mu\omega)^{\frac{1}{2}}h_1$.
[For this set of curves the exciting field is non-uniform (i.e., $\beta>0$, see text).]

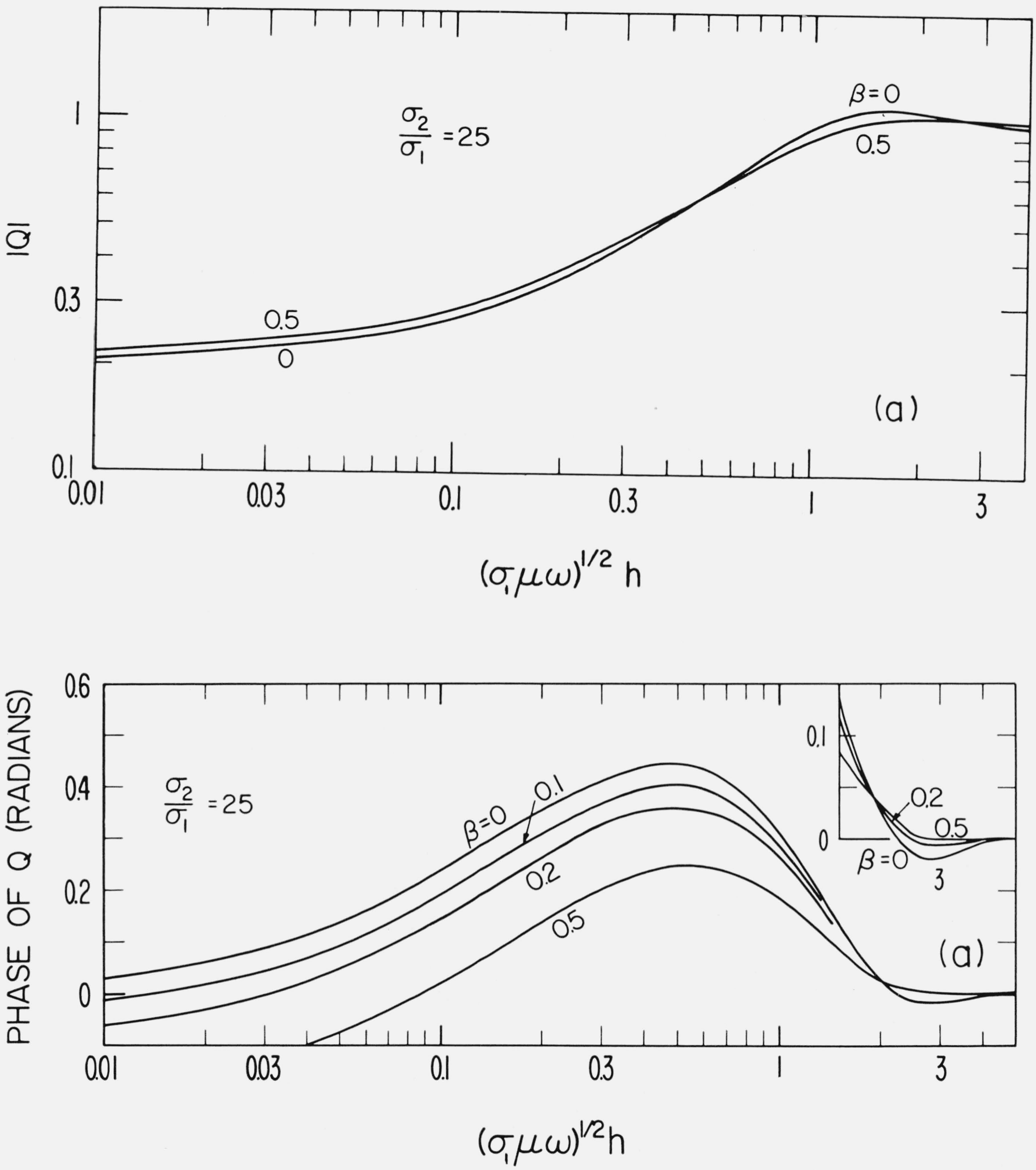


FIGURE 12. *Amplitude and phase of* Q *for a two-layer model plotted as a function of* $(\sigma_1 \mu \omega)^{\frac{1}{2}} h_1$.

[For this set of curves the exciting field is non-uniform (i.e., $\beta > 0$, see text).]

km would be expected to produce considerable nonuniformity. Actually, the two-layer model with a highly insulating substratum is a most unfavorable circumstance. Furthermore, it is somewhat hypothetical since the poorly conducting crystalline rock will also be of finite depth. Consequently, a three-layer model should again be adopted for a more realistic appraisal. In this case, eq (36) may be generalized [Wait, 1953b] to

$$Q=\frac{G\hat{Q}+\tanh \chi}{1+G\hat{Q}\tanh \chi}$$

where

$$\hat{Q}=\frac{\hat{G}+\tanh \hat{\chi}}{1+\hat{G}\tanh \hat{\chi}}$$

$$\hat{G}=\left[\frac{\sigma_2/\sigma_1-i\beta}{\sigma_3/\sigma_1-i\beta}\right]^{\frac{1}{2}} \text{ and } \hat{\chi}=(i\sigma_2\mu_0\omega)^{\frac{1}{2}}h_2(1-i\beta\sigma_1/\sigma_2)^{\frac{1}{2}}.$$

Now, if the intervening stratum is poorly conducting such that

$$(\sigma_2\mu_0\omega)^{\frac{1}{2}}h_2<<1 \text{ and } \beta\sqrt{\sigma_1\mu_0\omega}h_2<<1$$

it readily follows that

$$Q\simeq\frac{\hat{G}G+\tanh \chi}{1+\hat{G}G\tanh \chi}$$

where

$$\hat{G}G=\left[\frac{1-i\beta}{\sigma_3/\sigma_1-i\beta}\right]^{\frac{1}{2}}.$$

The latter equation for Q does not depend on σ_2 or h_2 and thus the three-layer structure is equivalent to a two-layer model whose constants are σ_1, σ_3, and h_1. If, *in addition*, $\beta<<1$ this formula for Q reduces to eq (29).

In summary, the restrictions on the use of the curves of Q given in figures 2a to 9b can be applied to a three-layer structure provided the single condition

$$|\beta Q^2|<<1$$

is met. Thus, for highly insulating substrata, the condition can become quite stringent since $|Q^2|$ is then large compared with unity. On the other hand, for highly conducting substrata, $|Q^2|$ may be small and the condition is not at all stringent.

5. Appendix. Surface Impedance of a Spherically Stratified Conductor

In many boundary-value problems involving waves in stratified media the solutions may be quickly obtained if the analogies with transmission line theory are exploited. In general, the analogous transmission line is nonuniform in the sense that the characteristic impedances may be different in the two directions on the line. Although Schelkunoff [1943] has discussed nonuniform transmission line theory, it seems worth while to include a short exposition of its essential features.

The starting point is the equations which connect the transverse voltage V between two parallel wires and the longitudinal current I in the lower wire. In terms of distance x down the line, these are

$$\frac{dV}{dx}=-ZI \text{ (A1) and } \frac{dI}{dx}=-YI \text{ (A2)}$$

where Z is the distributed series impedance per unit length and Y is the distributed series admittance per unit length. Allowing Z and Y to be both functions of x one readily finds that

$$\frac{d^2V}{dx^2}-\frac{Z'}{Z}\frac{dV}{dx}-YZV=0. \tag{A3}$$

$$\frac{d^2I}{dx^2}-\frac{Y'}{Y}\frac{dI}{dx}-YZI=0. \tag{A4}$$

Second-order differential equations of this type possess two linearly independent solutions. The general solution is a linear combination of these solutions. For example

$$V(x)=AV^+(x)+BV^-(x) \tag{A5}$$

where A and B are independent of x and V^+ and V^- are the fundamental wave functions. Similarly,

$$I(x)=AI^+(x)+BI^-(x) \tag{A6}$$

where I^+ and I^- are the corresponding fundamental current wave functions. It is evident that the wave functions individually satisfy eqs (A1) and (A2).

A characteristic wave impedance may now be associated with each pair of wave functions; thus

$$K^+(x)=\frac{V^+(x)}{I^+(x)}=-\frac{1}{YI^+}\frac{dI^+}{dx}=-\frac{ZV^+}{dV^+/dx} \tag{A7}$$

and

$$K^-(x)=-\frac{V^-(x)}{I^-(x)}=\frac{1}{YI^-}\frac{dI^-}{dx}=\frac{ZV^-}{dV^-/dx}. \tag{A8}$$

It is a convenient physical artifice to consider the waves associated with K^+ as propagating in the positive x direction and those associated with K^- as waves propagating in the negative x direction. In the case of a uniform line where Z and Y are constant, the meaning of these wave functions is quite clear. In the presence of any local nonuniformity, a reflected wave would be generated and the individual wave functions no longer are purely propagating. Nevertheless, in many cases, the elementary wave functions bear considerable resemblance to "traveling" waves. For example, this happens when Z and Y are slowly varying functions of x.

For present purposes the label progressive is used to describe a fundamental wave on a nonuniform line. In the limiting case of a uniform line the progressive waves become traveling waves.

It is important to note that there is some arbitrariness in the selection of the wave functions on nonuniform lines. The choice is usually made on the basis of convenience.

A useful quantity is the ratio of the wave functions at two points x_1 and x_2 on the line. Thus, by definition,

$$\chi_V^+(x_1,x_2)=\frac{V^+(x_2)}{V^+(x_1)}, \qquad \chi_V^-(x_1,x_2)=\frac{V^-(x_2)}{V^-(x_1)} \tag{A9}$$

$$\chi_I^+(x_1,x_2)=\frac{I^+(x_2)}{I^+(x_1)}, \qquad \chi_I^-(x_1,x_2)=\frac{I^-(x_2)}{I^-(x_1)}. \tag{A10}$$

We are now in the position to study the reflection in nonuniform lines. For example, a semi-infinite nonuniform line is terminated in an impedance Z. If $Z=K^+$, the voltage associated with this incident wave is exactly equal to the voltage across the terminal impedance and thus the entire incident current wave flows through this impedance. Hence, no reflection occurs. However, if $Z\neq K^+$, the incident current wave is not completely absorbed and reflection occurs. If the incident "progressive" waves are characterized by the wave functions $V^+(x)$ and $I^+(x)$, the problem is to calculate the reflection coefficient at the terminal impedance.

These are defined by

$$q_I = \frac{I^-}{I^+} \text{ and } q_V = \frac{V^-}{V^+} \tag{A11}$$

for current and voltage, respectively. Now,

$$V_t = V^+ + V^- \text{ and } I_t = I^+ + I^- \tag{A12}$$

where V_t and I_t are the voltage current at the terminals of Z_t. Furthermore,

$$V^+ = K^+ I^+,\ V^- = K^- I^-,\ V_t = Z_t I_t. \tag{A13}$$

The latter two sets of equations are readily solved to give

$$q_V = \frac{M^+ - Y_t}{M^- + Y_t} \text{ where } M^{\pm} = \frac{1}{K^{\pm}} \text{ and } Y_t = \frac{1}{Z_t} \tag{A14}$$

and

$$q_I = \frac{K^+ - Z_t}{K^- + Z_t}. \tag{A15}$$

A basic problem in nonuniform lines is to form wave functions $V(x)$ and $I(x)$ at $x = x_1$ in terms of the specified impedance at $x = x_2$. For convenience, $x_2 > x$. Using the preceding conventions one may easily write

$$V(x) = V^+(x) + V^+(x_2) q_V(x_2) \frac{V^-(x)}{V^-(x_2)} \tag{A16}$$

and

$$I(x) = I^+(x) + I^+(x_2) q_I(x_2) \frac{I^-(x)}{I^-(x_2)}. \tag{A17}$$

Here V^+ and I^+ are regarded as the "incident" progressive wave and V^- and I^- are the "reflected" progressive wave.

The impedance at the point $x = x_1$ is then given by

$$Z(x_1) = \frac{V(x_1)}{I(x_1)} = K^+(x_1) \frac{1 + q_V(x_2) \chi_V^+(x_1, x_2) \chi_V^-(x_2, x_1)}{1 + q_i(x_2) \chi_I^+(x_1, x_2) \chi_I^-(x_2, x_1)}. \tag{A18}$$

This result is most useful in reflection-type problems.

To complete this very brief survey of nonuniform transmission line theory, the corresponding formulas for transmission coefficients are also given.

The transmission coefficients at impedance discontinuity are defined by

$$p_V = \frac{V_t}{V^+} \text{ and } p_I = \frac{I_t}{I^+} \tag{A19}$$

being analogous to the previous definitions of q_V and q_I. Then from eqs (A12) and (A13),

$$p_V = \frac{M^- + M^+}{M^- + Y_t} \tag{A20}$$

and

$$p_I = \frac{K^- + K^+}{K^- + Z_t}. \tag{A21}$$

As expected, these coefficients become unity when $Z_t = K^+$ (or $Y_t = M^+$).

The transmission coefficient across a section (x_1, x_2) of *another* line inserted between the original line and impedance Z_t is readily obtained. The result, given by Schelkunoff [1943], is

$$T = p(1 + q\chi + q^2\chi^2 + \ldots)\chi^+(x_1,x_2)$$

$$= \frac{p}{1-q\chi}\chi^+(x_1,x_2) \qquad \text{(A22)}$$

where

$$p = p^+(x_1)p^+(x_2),\; q = q^-(x_1)q^+(x_2) \qquad \text{(A23)}$$

and $\chi = \chi^+(x_1,x_2)\chi^-(x_2,x_1)$. The result holds for both current and voltage waves which explains the absence of subscripts on I and V. The physical meaning of the results is very clear: p is the product of the two transmission coefficients at the discontinuities x_1 and x_2, q is the product of the reflection coefficient for a progressive wave incident from the right on the junction x_1, and the reflection coefficient for a progressive wave incident from the left. (Here x increases towards the right.) In the absence of multiple reflections T would be equated to $p\chi^+(x_1,x_2)$. Thus, the factor $(1+q\chi+q^2\chi^2+\ldots)$ represents the influence of multiple reflections.

We now proceed to apply nonuniform transmission line theory to a spherically stratified medium. In particular, let us consider the earth (of outer radius a_1) as consisting of a homogeneous core of radius a_2 of electrical constants σ_1, ϵ_1, and μ_1 surrounded by a homogeneous mantle of electrical constants σ_2, ϵ_2, and μ_2. It is assumed that the sources of the field are completely exterior to the earth. Thus, within the concentric homogeneous regions the fields may be derived from two scalar potential functions U and V. The first set which are TM (transverse magnetic) are derivable from U and the second set are TE (transverse electric), derivable from V.

For TM waves

$$E_r = \frac{1}{\sigma + i\epsilon\omega}\left(\frac{\partial^2 U}{\partial r^2} - \gamma^2 U\right), \qquad H_r = 0 \qquad \text{(A24)}$$

while for the TE waves

$$H_r = \frac{1}{i\mu\omega}\left(\frac{\partial^2 V}{\partial r^2} - \gamma^2 V\right), \qquad E_r = 0. \qquad \text{(A25)}$$

Both U and V satisfy

$$r^2\frac{\partial^2 U}{\partial r^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2} = \gamma^2 r^2 U \qquad \text{(A26)}$$

as can be readily ascertained from Maxwell's equations.

Since any field may be expressed as a superposition of these two sets, it is sufficient to discuss them separately.

Using the standard separation-of-variables technique, it is assumed that

$$U = u(\theta,\phi)\hat{u}(r). \qquad \text{(A27)}$$

On substituting this into eq (A26) one finds that u and $\hat{u}$ must satisfy

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{\partial^2 u}{\partial\phi^2} + \nu(\nu+1)\sin^2\theta u = 0 \qquad \text{(A28)}$$

and

$$\frac{d^2\hat{u}}{dr^2} - \left[\gamma^2 + \frac{\nu(\nu+1)}{r^2}\right]\hat{u} = 0 \qquad \text{(A29)}$$

where ν is a constant.

The u's are Legendre functions and the general form may be expressed as

$$u(\theta,\phi) = \sum_m \int [F_m(\nu)P_\nu^m(\cos\theta) + G_m(\nu)P_\nu^m(-\cos\theta)]\,d\nu \times [F_m \cos m\phi + g_m \sin m\phi] \qquad \text{(A30)}$$

where the integration contour is suitably chosen in the complex ν plane and m is an integer. The constants $F_m(\nu)$, $G_m(\nu)$, f_m, and g_m depend on the nature of the source field. For example, if the source is a vertical electric dipole at $\theta=0$, $F_m(\nu)=0$, and $g_m=0$. Also, for $m\neq 0, f_m=0$. Thus

$$u=\int G_0(\nu)P_\nu(-\cos\theta)d\nu. \tag{A31}$$

It is known that integrals of this type can be deformed so as to enclose the poles of $G_0(\nu)$ and thus u can be represented as a series of residues. Therefore,

$$u\cong 2\pi i\sum_{\nu_s}[\text{Residues of } G_0(\nu_s)]\,P_{\nu_s}(-\cos\theta) \tag{A32}$$

where ν_s are the solutions of $[G_0(\nu)]^{-1}=0$. Provided $|\nu_s|>>1$ and θ is not near 0 or π

$$P_{\nu_s}(-\cos\theta)\cong\frac{\text{const}}{(\sin\theta)^{\frac{1}{2}}}\left[e^{-ika_1\theta S_s}+e^{i\pi/2}e^{-ika_1(2\pi-\theta)\overline{S_s}}\right] \tag{A33}$$

where $\nu_s(\nu_s+1)\cong\nu_s^2\cong ka_1S_s$. Here the angular function u has the physical character of two waves traveling in opposite directions around the cylinder. In analogy to waves on a flat surface, S_s can be interpreted as the sine of a complex angle of incidence where k is the wave number in free space.

It is clearly apparent from this simple example that ν is related to the azimuthal variation of the source field. In the general case, ν depends both on the longitudinal and latitudinal variations.

Solutions of the radial equation are conveniently expressed in the form

$$\hat{u}=A\hat{I}_\nu(\gamma r)+B\hat{K}_\nu(\gamma r) \tag{A34}$$

where $\hat{I}_\nu$ and $\hat{K}_\nu$ could be any two independent solutions of eq (A29). Here $\gamma=[i\mu\omega(\sigma+i\epsilon\omega)]^{\frac{1}{2}}$ where $\text{Re}\,\gamma>0$. For convenience $\hat{I}_\nu(\gamma r)$ is chosen to be a solution which is finite or zero at $r=0$ and $\hat{K}_\nu(\gamma r)$ is chosen to vanish as r tends to infinity. In terms of modified (cylindrical) Bessel functions

$$\hat{I}_\nu(z)=\left(\frac{\pi z}{2}\right)^{\frac{1}{2}}I_{\nu+1/2}(z) \tag{A35}$$

and

$$\hat{K}_\nu(z)=\left(\frac{2z}{\pi}\right)^{\frac{1}{2}}K_{\nu+1/2}(z) \tag{A36}$$

where I and K have their conventional meaning.

In the present problem, the waves associated with $\hat{I}_\nu(\gamma r)$ are regarded as being "incident," whereas the waves associated with $\hat{K}_\nu(\gamma r)$ are regarded as "reflected."

The surface impedance at $r=a_1$ is defined by

$$Z_\nu=\left[-\frac{E_\theta^\nu}{H_\phi^\nu}\right]_{r=a_1} \tag{A37}$$

which, in general, depends on ν. The medium between the limits $r=a_1$ and a_2 is now regarded as a nonuniform transmission line of length $l=a_1-a_2$. The transverse voltage V_T and the current I on the line are then analogous to the electric field $-E_\theta^\nu$ and the magnetic field H_ϕ^ν, respectively, where the superscript ν is to indicate the possible dependence on ν.

The characteristic impedance of the line looking inward is then

$$K^+(\gamma_1 r)=\eta_1\frac{\hat{I}'_\nu(\gamma_1 r)}{\hat{I}_\nu(\gamma_1 r)} \tag{A38}$$

and the impedance looking outward is

$$K^-(\gamma_1 r) = -\eta_1 \frac{\hat{K}'_\nu(\gamma_1 r)}{\hat{K}_\nu(\gamma_1 r)}. \tag{A39}$$

In the above

$$\gamma_1 = [i\mu_1\omega(\sigma_1 + i\epsilon_1\omega)]^{\frac{1}{2}}$$

and

$$\eta_1 = [i\mu_1\omega/(\sigma_1 + i\epsilon_1\omega)]^{\frac{1}{2}}.$$

The line is now considered to be terminated by an impedance Z_t where

$$Z_t = \left[\eta_2 \frac{\hat{I}'_\nu(\gamma_1 r)}{\hat{I}_\nu(\gamma_1 r)}\right]_{r=a_1}. \tag{A40}$$

From the analogy with transmission line theory, one readily finds that

$$Z_\nu = K^+(\gamma_1 a_1)\left[\frac{1+q_e\chi_e(a_2,a_1)\chi_e(a_1,a_2)}{1+q_h\chi_h(a_2,a_1)\chi_h(a_1,a_2)}\right] \tag{A41}$$

where

$$q_e = \frac{(1/Z_t) - 1/K^+(\gamma_1 a_1)}{(1/Z_t) + 1/K^-(\gamma_1 a_1)}, \tag{A42}$$

$$q_h = \frac{Z_t - K^+(\gamma_1 a_1)}{Z_t + K^-(\gamma_1 a_1)}, \tag{A43}$$

$$\chi_e(a_2, a_1) = \frac{a_1 \hat{I}'_\nu(\gamma_1 a_2)}{a_2 \hat{I}'_\nu(\gamma_1 a_1)}, \tag{A44}$$

$$\chi_e(a_1, a_2) = \frac{a_2 \hat{K}'_\nu(\gamma_1 a_1)}{a_1 \hat{K}'_\nu(\gamma_1 a_2)}, \tag{A45}$$

$$\chi_h(a_2, a_1) = \frac{a_1 \hat{I}_\nu(\gamma_1 a_2)}{a_2 \hat{I}_\nu(\gamma_1 a_1)}, \tag{A46}$$

$$\chi_h(a_1, a_2) = \frac{a_2 \hat{K}_\nu(\gamma_1 a_1)}{a_1 \hat{K}_\nu(\gamma_1 a_2)}. \tag{A47}$$

The preceding results can be greatly simplified under the assumption that $\gamma_1 a_1$ and $\gamma_2 a_2$ are large compared with unity. For example, noting that $\hat{I}_\nu(z)$ satisfies the equation

$$\frac{d^2\hat{I}_\nu(z)}{dz^2} = \left[1 + \frac{\nu(\nu+1)}{z^2}\right]\hat{I}_\nu(z), \tag{A48}$$

it readily follows that $K^+(\gamma r)$ satisfies

$$K^2 + \eta_1 \frac{dK}{dz} = \left[1 + \frac{\nu(\nu+1)}{z^2}\right]\eta_1^2, \tag{A49}$$

where $K = K^+(\gamma_1 r)$ and $z = \gamma_1 r$. For a first approximation, the derivative term may be neglected. Thus

$$K^{+}(\gamma_1 r) \cong \eta_1 \left[1+\frac{\nu(\nu+1)}{(\gamma_1 r)^2}\right]^{\frac{1}{2}}. \tag{A50}$$

This approximate result, obtained in a very simple fashion, corresponds to the use of the Debye or the second order result. The theory of Bessel functions indicates that its validity, in this form, depends on the conditions Re $\gamma_1 r >> 1$ and $|\gamma_1 r|^2$ somewhat greater than $\nu(\nu+1)$ [Sommerfeld, 1949]. Similarly,

$$K^{-}(\gamma_1 r) \cong \eta_1 \left[1+\frac{\nu(\nu+1)}{(\gamma_1 r)^2}\right]^{\frac{1}{2}} \tag{A51}$$

and

$$K^{+}(\gamma_2 r) \cong \eta_2 \left[1+\frac{\nu(\nu+1)}{(\gamma_2 r)^2}\right]^{\frac{1}{2}}. \tag{A52}$$

To within the same approximation, the reflection coefficients at $r=a_1$ may then be written

$$q_e = \frac{(1/\eta_2)\left[1-\left(\frac{\gamma_0 a_1}{\gamma_2 a_2}S\right)^2\right]^{-\frac{1}{2}} - (1/\eta_1)\left[1-\left(\frac{\gamma_0 a_1}{\gamma_1 a_2}S\right)^2\right]^{-\frac{1}{2}}}{(1/\eta_2)\left[1-\left(\frac{\gamma_0 a_1}{\gamma_2 a_2}S\right)^2\right]^{-\frac{1}{2}} + (1/\eta_1)\left[1-\left(\frac{\gamma_0 a_1}{\gamma_1 a_2}S\right)^2\right]^{-\frac{1}{2}}} \tag{A53}$$

and

$$q_h = \frac{\eta_2\left[1-\left(\frac{\gamma_0 a_1}{\gamma_2 a_2}S\right)^2\right]^{\frac{1}{2}} - \eta_1\left[1-\left(\frac{\gamma_0 a_1}{\gamma_1 a_2}S\right)^2\right]^{\frac{1}{2}}}{\eta_2\left[1-\left(\frac{\gamma_0 a_1}{\gamma_2 a_2}S\right)^2\right]^{\frac{1}{2}} + \eta_1\left[1-\left(\frac{\gamma_0 a_1}{\gamma_1 a_2}S\right)^2\right]^{\frac{1}{2}}} \tag{A54}$$

where $\nu(\nu+1) \cong \nu^2 \cong -\gamma_0^2 S^2 = k^2 S^2$. The corresponding (approximate) form for the product of the transmission coefficients is easily shown to be

$$\begin{aligned} \chi_e(a_2,a_1)\chi_e(a_1,a_2) &\cong \chi_h(a_2,a_1)\chi_h(a_1,a_2) \\ &\cong \exp\left[-2\gamma_1 \int_0^l \left[1-\frac{\gamma_0^2 a_1^2 S^2}{\gamma_1^2 r^2}\right]^{\frac{1}{2}} dr\right] \end{aligned} \tag{A55}$$

where $l = a_1 - a_2$.

Using eqs (A50), (A53), (A54), and (A55), eq (A41) for Z_ν can be expressed in a fairly convenient form despite its complicated appearance. A great simplification can be made when the thickness l of the shell is small compared with a_1. Thus, the ratios a_1/a_2 and a_1/r in the preceding expressions may be replaced by unity. The surface impedance may then be written

$$Z_\nu = \eta_1 \left[1-\frac{\gamma_0^2}{\gamma_1^2}S^2\right]^{\frac{1}{2}} \frac{\frac{\gamma_1^2}{\gamma_2^2}\left[\frac{\gamma_2^2-\gamma_0^2 S^2}{\gamma_1^2-\gamma_0^2 S^2}\right]^{\frac{1}{2}} + \tanh\left[(\gamma_1^2-\gamma_0^2 S^2)^{\frac{1}{2}} l\right]}{1+\frac{\gamma_1^2}{\gamma_2^2}\left[\frac{\gamma_2^2-\gamma_0^2 S^2}{\gamma_1^2-\gamma_0^2 S^2}\right]^{\frac{1}{2}} \tanh\left[(\gamma_1^2-\gamma_0^2 S^2)^{\frac{1}{2}} l\right]} \tag{A56}$$

where it has been assumed that $\mu_1 = \mu_2 = \mu_0$. This result is *identical* to what one would obtain on a planar two-layer earth for a plane wave incident at an angle of incidence arc sin S [Wait, 1958]. Consequently, in this limiting case the influence of earth curvature vanishes.

The specific results developed above for the surface impedance are restricted to TM waves. The corresponding results for TE waves are obtained by exactly the same method. In fact the results are completely analogous if admittances are used in place of impedances. The surface admittance defined by

$$Y_\nu = \left.\frac{H_\theta}{E_\phi}\right]_{r=a_1}$$

has precisely the same form as eq (A41) for Z_v if $K^+(\gamma_1 a)$ is now replaced by the admittance $M^+(\gamma_1 a)$ and q_e and q_h have the same form as eqs (A53) and (A54) except that η_1 and η_2 are replaced by their reciprocals everywhere. The bracketed terms are unchanged.

Under the assumptions that Re $\gamma_1 a_1$ and Re $\gamma_1 a_2 >> 1$, $\mu_1 = \mu_2 = \mu_0$, and that $a_1 - a_2 << a_1$, one easily finds that

$$Y_v = \frac{1}{\eta_1}\left[1 - \frac{\gamma_0^2 S^2}{\gamma_1^2}\right]^{\frac{1}{2}} \frac{\left[\frac{\gamma_2^2 - \gamma_0^2 S^2}{\gamma_1^2 - \gamma_0^2 S^2}\right]^{\frac{1}{2}} + \tanh\left[(\gamma_1^2 - \gamma_0^2 S^2)^{\frac{1}{2}} l\right]}{1 + \left[\frac{\gamma_2^2 - \gamma_0^2 S^2}{\gamma_1^2 - \gamma_0^2 S^2}\right]^{\frac{1}{2}} \tanh\left[(\gamma_1^2 - \gamma_0^2 S^2)^{\frac{1}{2}} l\right]}. \tag{A57}$$

This is equivalent to the result given by eq (36) in the main body of the text.

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