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Defocusing of Radio Rays by the Troposphere

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When radio rays pass through the atmosphere, they are defocused due to its presence. This effect is measured by the divergence coefficient and general formulas are derived for D_1, the divergence coefficient of the direct ray, and D_2, the divergence coefficient of the reflected ray—assuming a smooth spherical earth. As examples, D_1 and D_2 are shown for some typical cases with an "exponential" atmosphere (troposphere).

1. Introduction

The field at a radio receiver within line-of-sight of the transmitter may be found by adding vectorially the field due to the direct ray and that due to the reflected ray. These fields depend directly upon the defocusing of the respective rays and this effect is measured by D_1, the divergence coefficient of the direct ray, and D_2, the divergence coefficient of the reflected ray.

The direct ray is the ray which goes directly from the transmitter to the receiver and it is defocused due to the change of refractive index, n, with height. This defocusing effect is given by:

$$D_1=\frac{E_1}{E_0} \tag{1.1}$$

where E_1 is the strength of the electric field due to the "direct" ray at the receiving point, P (in the atmosphere), and E_0 is the field strength which would be observed at the same distance if the system were located in a medium of constant refractivity.

The reflected ray strikes the earth (assumed spherical) and is reflected to the receiver. It receives some additional defocusing because of the earth's shape, and the total effect is given by:

$$D_2=\frac{E_2}{E} \tag{1.2}$$

where E_2 is the strength of the electric field due to this ray at the point P and E is the field strength that would be observed at that distance if the system were in a homogeneous medium and the earth were flat.

2. Divergence Coefficient of the Direct Ray

The divergence coefficient of the direct ray, D_1, may be derived by first considering a radio ray passing through the atmosphere unimpeded. The energy of this ray is

$$\text{Energy}=Cn_2E_1^2dq_1=Cn_1E_0^2dq_0 \tag{2.1}$$

where C is a constant of proportionality, E_1 is the field strength at the receiving point P—where the index of refraction is n_2—and dq_1 is the cross section of the ray at this point. E_0 and dq_0 are the corresponding quantities at the same distance if the system is in a medium whose index of refraction is n_1, that at the transmitter.

Then from (1.1)

$$D_1=\frac{E_1}{E_0}=\sqrt{\frac{n_1dq_0}{n_2dq_1}}. \tag{2.2}$$

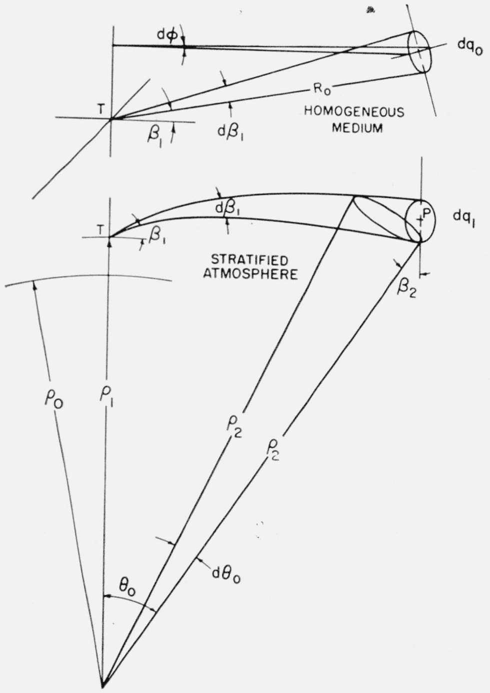

FIGURE 1. *Geometry of the direct ray.*

Referring to figure 1

$$dq_0 = \left|\frac{\pi R_0^2 (d\beta_1)^2}{4}\right| = \left|\frac{\pi R_0^2 \cos \beta_1 d\beta_1 d\phi}{4}\right| \tag{2.3}$$

and

$$dq_1 = \left|\frac{\pi \rho_2^2 \sin \beta_2 \sin \theta_0 d\theta_0 d\phi}{4}\right| \tag{2.4}$$

where ϕ is the angle the ray makes with the plane of the paper.

Combining (2.2), (2.3), and (2.4):

$$D_1 = \frac{R_0}{\rho_2}\sqrt{\left|\frac{n_1 \cos \beta_1}{n_2 \sin \beta_2 \sin \theta_0 (d\theta_0/d\beta_1)}\right|}. \tag{2.5}$$

θ_0 is the central angle and, for a radio ray passing through a spherically stratified atmosphere, it is given by [Kerr, 1954; Counter, 1956]:

$$\theta_0 = \left|\int_{\rho_1}^{\rho_2} \frac{d\rho}{\rho[(n\rho/K)^2 - 1]^{1/2}}\right| \tag{2.6}$$

where the constant K is

$$K = n_i \rho_i \cos \beta_i, \tag{2.7}$$

which is Snell's Law. (See also appendix I.) In the above equations, ρ is the distance from the center of the earth to a point on the ray and n is the index of refraction at that point; ρ_i and n_i are the corresponding quantities at the end points and β_i are the angles $(-\pi/2 < \beta_i < \pi/2)$ between the ray at these points and the horizontal. In this paper, $i=1$ is associated with the transmitter and $i=2$ with the point P, although they may be interchanged in the final result, (2.5).

The derivative of (2.6) with respect to β_1 gives

$$|d\theta_0/d\beta_1| = \left|\frac{\tan \beta_1}{K^2}\int_{\rho_1}^{\rho_2} \frac{n^2 \rho d\rho}{[(n\rho/K)^2 - 1]^{3/2}}\right|. \tag{2.8}$$

(Again see appendix I.)

The distance R_0 is not completely defined; e.g., it may be the distance along the ray or the straight line distance to the point P—the only requirement being that

$$R_0=[\rho_1^2+\rho_2^2-2\rho_1\rho_2 \cos \theta_0]^{\frac{1}{2}} \tag{2.9}$$

when n is constant, since the ray becomes straight. In the calculations described below, R_0 is the straight line distance and therefore is defined by (2.9) for variable n also.

For constant n the integrals in (2.6) and (2.8) may be evaluated in closed form and, putting the results in (2.5), $D_1=1$, the value which is usually used.

In the troposphere n is of the form [Bean and Thayer, 1959]

$$n=1+(N_s\times10^{-6})\, e^{-ch} \tag{2.10}$$

where N_s is the refractivity on the earth's surface, c is the decay constant, and $h=\rho-\rho_0$ is the height above the earth; ρ_0 is the radial distance to the earth's surface and the following empirical relation reflects the average correspondence between N_s and ρ_0:

$$\rho_0 \text{ (km)}=6370+10 \ln [1+0.6 \exp (-3.35\times10^{-10}N_s^4)] \tag{2.11}$$

where 6,370 km is taken as the earth's radius. The refractivity decay constant c in (2.10) is given by [Bean and Thayer, 1959; Rice, Longley, and Norton, 1962]:

$$c \text{ (per km)}=\left\{\begin{matrix} \ln\left[\dfrac{N_s}{N_s-7.32 \exp (0.005577N_s)}\right]; N_s\geq 250 \\ N_s\times10^{-4}[7.939-0.01166N_s]; N_s<250 \end{matrix}\right\}. \tag{2.12}$$

Table 1 lists values of ρ_0 for N_s=200, 300, and 400. Using these values, the integrals in (2.6) and (2.8) were evaluated as shown in appendix II, R_0 was computed from (2.9), and D_1 was plotted logarithmically against the takeoff angle β_1 in figures 3 and 4. In figure 4, the cutoff is due to interference by the earth.

In these figures, h_1 is the height of the transmitter above the earth's surface (0 and 5 km) and h_2 is the height of the receiving point, P, above the earth's surface:

$$h_1=\rho_1-\rho_0; \tag{2.13a}$$

$$h_2=\rho_2-\rho_0. \tag{2.13b}$$

TABLE 1

N_s	ρ_0 (km)	c (per km)
200	6, 373. 008823	0. 11214
300	6, 370. 390117	. 1392842847
400	6, 370. 001131	. 1867197187

3. Divergence Coefficient of the Reflected Ray

The divergence coefficient of the reflected ray, D_2, may be derived by again considering a radio ray passing through the atmosphere but now reflecting from the earth before reaching the receiving point, P. Referring to figure 2, D_2 is developed in the same manner as D_1 and the result is

$$D_2=\frac{R}{\rho_2}\sqrt{\left|\frac{n_1 \cos \beta_1}{n_2 \sin \beta_2 \sin \theta(d\theta/d\beta_1)}\right|}. \tag{3.1}$$

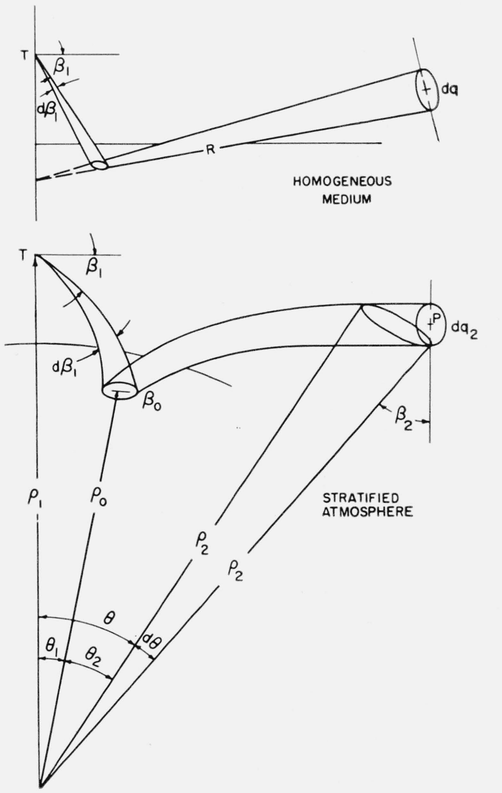

FIGURE 2. *Geometry of the reflected ray.*

The central angle θ is now the sum of the angles θ_1 and θ_2;

$$\theta = \theta_1 + \theta_2; \tag{3.2}$$

where θ_1 and θ_2 are [Kerr, 1954; Counter, 1956]:

$$\theta_1 = \left| \int_{\rho_0}^{\rho_1} \frac{d\rho}{\rho[(n\rho/K)^2 - 1]^{1/2}} \right| \tag{3.3a}$$

and

$$\theta_2 = \left| \int_{\rho_0}^{\rho_2} \frac{d\rho}{\rho[(n\rho/K)^2 - 1]^{1/2}} \right| \tag{3.3b}$$

if the atmosphere is spherically stratified. K is the same as before, i.e.,

$$K = n_i \rho_i \cos \beta_i \tag{3.4}$$

where ρ_i is again the distance from the center of the earth to a point on the ray, n_i is the index of refraction at that point, and β_i is the angle the ray makes with the horizontal at that point; $i=0$ is associated with the earth's surface, $i=1$ again with the transmitter, and $i=2$ with the point P, although these may be interchanged in (3.1).

The derivative of θ with respect to β_1 then gives

$$|d\theta/d\beta_1| = \left| \frac{\tan \beta_1}{K^2} \left\{ \int_{\rho_0}^{\rho_1} \frac{n^2 \rho d\rho}{[(n\rho/K^2) - 1]^{3/2}} + \int_{\rho_0}^{\rho_2} \frac{n^2 \rho d\rho}{[(n\rho/K)^2 - 1]^{3/2}} \right\} \right|. \tag{3.5}$$

R is defined only by the fact that it must be the slant range,

$$R = [\rho_0^2 + \rho_1^2 - 2\rho_0\rho_1 \cos \theta_1]^{1/2} + [\rho_0^2 + \rho_2^2 - 2\rho_0\rho_2 \cos \theta_2]^{1/2}, \tag{3.6}$$

when n is constant.

For the above case (n constant), the curvature of the earth is the only contributing factor in determining D_2, and

$$D_2 = \frac{R}{\rho_2}\sqrt{\frac{\sin\beta_0\cos^2\beta_1}{\sin\theta\,(\sin\beta_2\sin\theta_1+\sin\beta_1\sin\theta_2)}} \tag{3.7}$$

which is, as expected, the same as that derived by Van der Pol and Bremmer [1939] and Riblet and Barker [1948].

For small heights, ρ_1 and $\rho_2 \sim \rho_0$. (3.7) may be approximated by

$$D_2 \sim \left[1+\frac{2\theta_1\theta_2}{\theta\tan\beta_0}\right]^{-1/2}. \tag{3.8}$$

This is the formula which is usually used with the earth replaced by one with a radius $k(\rho_0)$ times the actual radius [Norton, 1941], where

$$k(\rho_0) \equiv \left.\frac{n}{n+\rho(dn/d\rho)}\right|_{\rho=\rho_0}. \tag{3.9}$$

It is evaluated for some typical cases with this assumption and the results are shown in figure 5, together with values of (3.7) without this assumption; in this figure, results are also shown for the same situations but assuming an exponential atmosphere (troposphere) with $N_s=200$, 300, or 400. (See (2.10) and table 1.) The transmitter height h_1 is 5 km.

It should also be noted that the curves in this figure are plotted against the reflection angle β_0 instead of the takeoff angle β_1. β_1 may be found, however, by using Snell's Law, (3.4), and β_0, for a particular path, may be found by applying an iterative method to (3.2).

4. Conclusion

The defocusing effect of the atmosphere may be of importance for small takeoff angles, β_1. (See figs. 3 and 4.) This defocusing effect, however, is usually small compared to that arising from the earth's curvature. This is indicated by comparing the values of D_2 (atmosphere plus the earth's curvature, fig. 5) with the corresponding values of D_1 (atmosphere only, fig. 3).

Especially for large antenna heights, a good approximation of D_2, the divergence coefficient of the reflected ray, is then given by (3.7)—no atmospheric effect—with β_0 being the reflection angle of the *true* ray. This is also shown in figure 5.

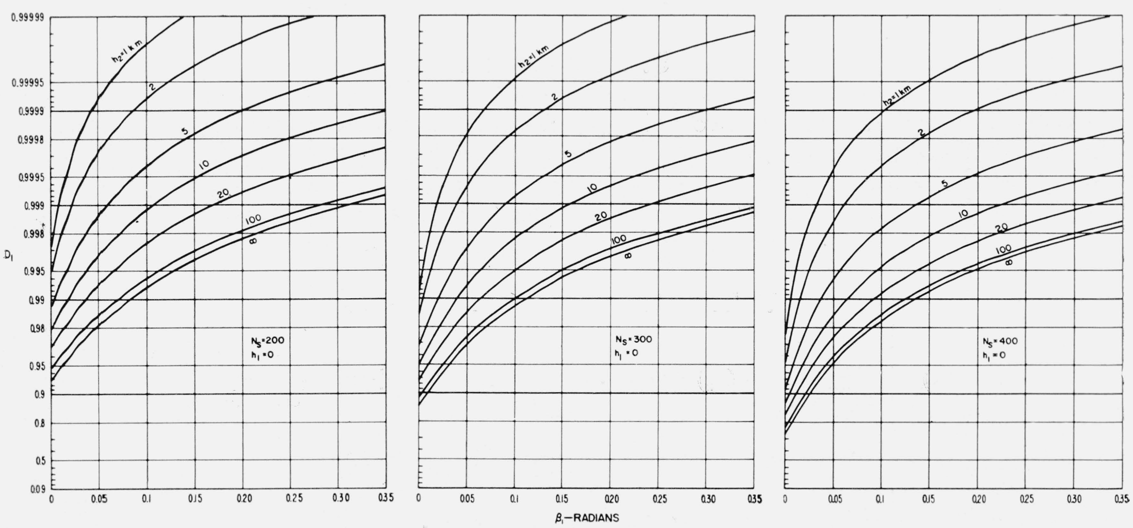

FIGURE 3. *The divergence coefficient of the direct ray with one terminal on the ground.*

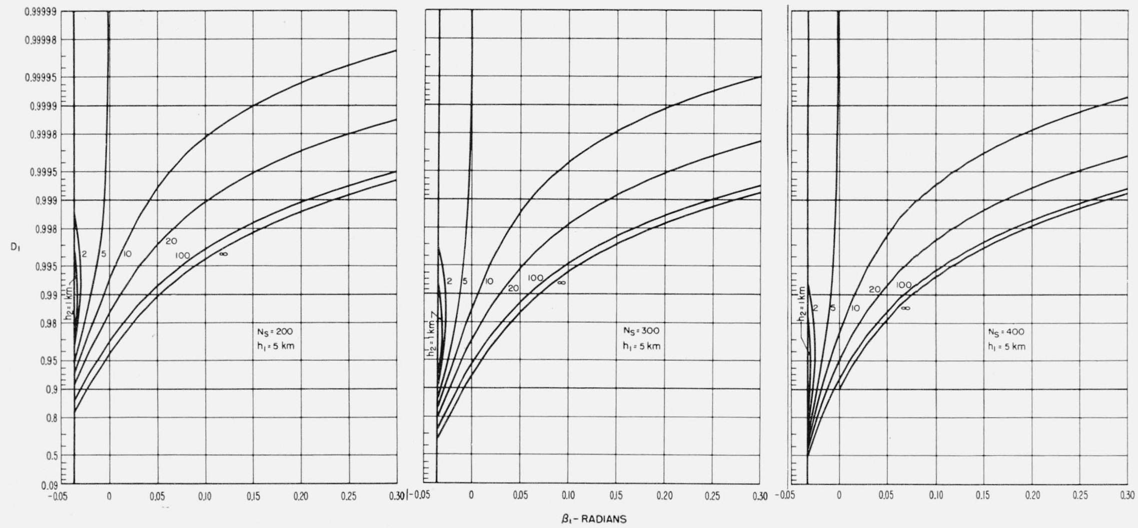

FIGURE 4. *The divergence coefficient of the direct ray with one terminal elevated 5 km.*

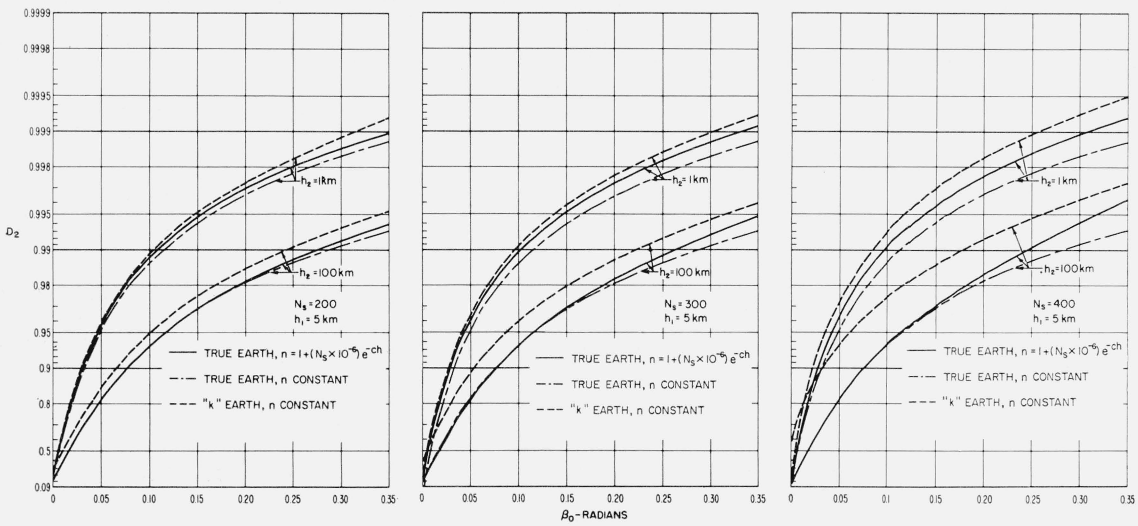

FIGURE 5. *The divergence coefficient of the reflected ray with one terminal elevated 5 km.*

5. Appendix I

When $\beta_1 < 0$, there may be two points on a ray at the same height and, for $\beta_2 > 0$, (2.6) should be replaced by the sum

$$\theta_0 = \int_r^{\rho_1} \frac{d\rho}{\rho[(n\rho/K)^2-1]^{1/2}} + \int_r^{\rho_2} \frac{d\rho}{\rho[(n\rho/K)^2-1]^{1/2}} \tag{I.1}$$

where r is such that $rn(r) = K$. Then (2.8) should be replaced by

$$|d\theta_0/d\beta_1| = \left| k(\rho_1) - \frac{\cos\beta_2 \sin\beta_1}{\cos\beta_1 \sin\beta_2} k(\rho_2) + \tan\beta_1 \int_r^{\rho_1} \frac{(dk/d\rho)d\rho}{[(n\rho/K)^2-1]^{1/2}} + \tan\beta_1 \int_r^{\rho_2} \frac{(dk/d\rho)d\rho}{[(n\rho/K)^2-1]^{1/2}} \right|, \tag{I.2}$$

where $k(\rho_i)$ is again defined as [CCIR Study Group IV–C, 1959]:

$$k(\rho_i) \equiv \left. \frac{n}{n + \rho(dn/d\rho)} \right|_{\rho = \rho_i}. \tag{I.3}$$

6. Appendix II

In finding θ_0 and $|d\theta_0/d\beta_1|$ for an exponential atmosphere, the integrals were evaluated by first letting $r \cos z = \rho_1 \cos \beta_1$ and then using Gaussian quadrature.

The only times when this method could not be used were in finding $|d\theta_0/d\beta_1|$ when $\beta_1=0$ or $\beta_2=0$. For these cases integration by parts was employed, with $n\rho$ being the variable, and it was found that

$$|d\theta_0/d\beta_1| = k(\rho_1) \tag{II.1}$$

when $\beta_1=0$, and that

$$|\sin \beta_2(d\theta_0/d\beta_1)| = |k(\rho_2) \tan \beta_1| \tag{II.2}$$

when $\beta_2=0$. For the definition of $k(\rho_i)$ see appendix I.

7. References

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