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Propagation of Plane Electromagnetic Waves Past a Shoreline [1]

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The problems of the diffraction of homogeneous plane waves and ground waves by a linear shoreline in a planar land-sea surface are discussed. The direction of propagation of these incident waves is assumed perpendicular, and that of their magnetic vectors parallel, to the shoreline. At the air-land interface, the customary impedance boundary condition is imposed while the sea is treated as a perfect conductor; atmospheric and ionospheric effects are ignored. Exact integral representations of the solutions are presented. In the case of homogeneous plane-wave excitation originating over the sea, the integral representations are employed to obtain expressions for the geometrical optics field and for the far-field form of the remaining scattered field, transition regions included. The possibility of coastal refraction is discussed.

1. Introduction

The first version of this paper appeared in July 1952 as a report [Bazer and Karp, 1952] which, however, has been out of print since early 1953. The present version has been solicited by the editor, who felt, in view of the continuing interest in the subject matter, that it would prove useful to make the essential contents of the report more generally available. At the suggestion of the past editor, the authors have extended their original historical survey to include mention of subsequent work by other authors on the same problem, and on some closely related problems. They have not, however, attempted to improve upon their own results; this paper is simply a survey of the results of their report.

The problems treated arise from the following model for propagation over a land-sea surface. The surface of the earth is taken to be the X–Z plane and the region $y>0$, the air (see fig. 1). The Z-axis (normal to the plane of the page) is taken to be the shoreline and

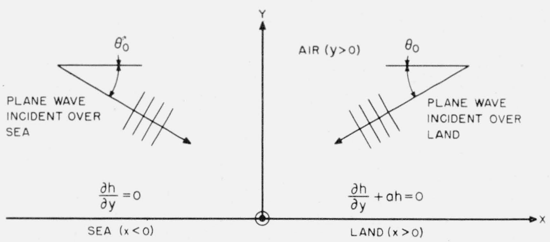

FIGURE 1. *Plane-wave excitation originating over the sea.*

the half-planes $x>0$, $y=0$, and $x<0$, $y=0$, to be the air-land and air-sea interfaces, respectively, of the earth-sea surface. It is assumed that this system is excited by plane waves—specifically by planar ground waves originating at $x=\infty$ or by homogeneous plane waves originating over the land or sea. It is further assumed that these plane waves are polarized with the magnetic vector parallel, and with the direction of propagation perpendicular to the shoreline so that the total field may be taken to be "two-dimensional." This field is specified by two vectors,

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an electric vector $\mathbf{E}^p=\mathbf{E}^p(x,y)$, parallel to the X–Y plane, and a magnetic vector $\mathbf{H}=h(x,y)\mathbf{z}_0$, $\mathbf{z}_0$, being the unit along the Z-axis. It follows from Maxwell's equations that [2]

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+k^2h=0, \qquad -\infty<x<+\infty, \qquad 0<y<\infty, \tag{1.1}$$

where

$$k=\frac{\omega}{c}[\mu\epsilon+(4\pi\mu\sigma i/\omega)]^{1/2}=|k|\exp(i\delta), \qquad 0<\delta<<1 \tag{1.2}$$

and that $\mathbf{E}^p$ is related to h by

$$\mathbf{E}^p=\frac{c}{i\omega[\epsilon+(4\pi\sigma i/\omega)]}\mathbf{z}_0\times\nabla_p h(x,y). \tag{1.3}$$

In these equations, ϵ and μ are the electric and magnetic inductive capacities; σ is the conductivity of air; ω is the angular frequency; c is the velocity of light; and ∇_p is the operator $\mathbf{x}_0\partial(\ \ )/\partial x+\mathbf{y}_0\partial(\ \ )/\partial y$, $\mathbf{x}_0$ and $\mathbf{y}_0$ being unit vectors along the X and Y axes. The total field is evidently determined, once $h(x,y)$ is known.

The sea is treated as a perfect conductor; this assumption leads to the boundary condition

$$\left.\frac{\partial h(x,y)}{\partial y}\right|_{y\downarrow 0}=0, \qquad x<0. \tag{1.4}$$

Further, if, as is here and hereafter assumed, the modulus of the ratio k/k_l of the propagation constant k of air to that of the land k_l is small and the conduction current is much larger than the displacement current in the earth, then the usual conditions on the transition of the tangential components of the electric and magnetic vectors through a surface of discontinuity lead to the following impedance boundary condition at the air-land interface:

$$\left.\frac{\partial h(x,y)}{\partial y}\right|_{y\downarrow 0}+a h(x,y)\bigg|_{y\downarrow 0}=0, \qquad x>0. \tag{1.5}$$

The quantity a is a complex constant whose argument is $\frac{\pi}{4}+\delta$, i.e.,

$$a=|a|\exp[i(\pi/4+\delta)]. \tag{1.6}$$

In terms of the electromagnetic constants ϵ, μ, σ and ϵ_l, μ_l, σ_l of the air and land respectively, a is given by

$$a=\pi d\mu^{-1}\mu_l\lambda^{-2}\exp[i(\pi/4+\delta)], \tag{1.7}$$

where

$$d=c(2\pi\omega\mu_l\sigma_l)^{-1/2}=\lambda(\nu/4\pi^2\mu_l\sigma_l)^{1/2}, \qquad \omega=2\pi\nu. \tag{1.7$'$}$$

Here, λ is the wavelength in air, d the "skin depth" of the land, and ν the frequency of the exciting field. If ν is less than or equal to 1,000 kc/s and σ_l is the conductivity of wet earth, one can easily verify (cf. Bazer and Karp [1952], Grünberg [1942, 1943]) that the approximations leading to the impedance boundary condition are valid. Furthermore, it can be shown that the ratio

$$|(a/k)|\leq 1/10, \tag{1.8}$$

and is $0(\nu^{\frac{1}{2}})$ for $\nu\leq$ 1,000 kc/s. It should be added that the boundary conditions of eqs (1.4) and (1.5) are not strictly valid in the neighborhood of the shoreline, so that one should not expect the fields derived from $h(x,y,)$ to describe the physical situation in this vicinity.

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[2] The time dependence $\exp(-i\omega t)$ and the Giorgi MKS system of units is employed throughout.

As indicated above, homogeneous plane and ground-wave excitation will be treated. For excitation of the homogeneous plane-wave type that originates over the sea, an incident magnetic field $h_0^s(x,y)$ of the form

$$h_0^s(x,y)=\exp\,[ik(x\cos\theta_0-y\sin\theta_0)],\qquad 0\leq\theta_0\leq\frac{\pi}{2}. \tag{1.9}$$

is assumed. For excitation of the same character originating over the land, an incident magnetic field of the form

$$h_0^l(x,y)=\exp\,[-ik(x\cos\theta_0+y\sin\theta_0)],\qquad 0<\theta_0\leq\frac{\pi}{2} \tag{1.10}$$

is assumed. For excitation of the ground wave type,[3] propagating in the negative x-direction from $x=+\infty$ to the shoreline, it is assumed that

$$h_0^g(x,y)=\exp\{-[ix(k^2+a^2)^{\frac{1}{2}}+ay]\},\qquad \mathrm{Im}\,(k^2+a^2)^{\frac{1}{2}}>0. \tag{1.11}$$

In all cases the corresponding electrical excitation is to be determined from eq (1.3).

To complete the formulation of the problem, it is necessary to specify appropriate conditions at the shoreline and at infinity. At the shoreline one assumes, whatever the excitation, that the total magnetic field $h(x,y)$ satisfies

$$\lim_{\rho\to 0}\int_0^{\pi} h^*(\rho\cos\theta,\ \rho\sin\theta)\,\frac{\partial h(\rho\cos\theta,\ \rho\sin\theta)}{\partial\rho}\,\rho d\theta=0. \tag{1.12}$$

Here, h^* denotes the complex conjugate of h, and $x=\rho\cos\theta$, $y=\rho\sin\theta$, $0\leq\theta\leq\pi$. This condition is equivalent to the statement that electromagnetic power is neither radiated nor absorbed at the shoreline.

Since it has been supposed that the air is slightly conducting (see eq (1.2)), it is to be expected, on physical grounds, that no scattered power will reach infinity. Writing

$$h(x,y)=h_0(x,y)+v(x,y), \tag{1.13}$$

where $h_0(x,y)$ is any one of the incident waves described above, one finds easily that no scattered power will reach infinity if the following requirement is met:

$$\lim_{\rho\to\infty}\int_0^{2\pi} v^*(\rho\cos\theta,\ \rho\sin\theta)\,\frac{\partial v(\rho\cos\theta,\ \rho\sin\theta)}{\partial\rho}\,\rho d\theta=0. \tag{1.14}$$

The problem is therefore to find solutions $h(x,y)$ of the form given in eq (1.13), which correspond to the various excitations introduced above and which satisfy: (1) the time-reduced wave equation (1.1), (2) the boundary conditions of eqs (1.4) and (1.5), and (3) the conditions at the shoreline (eq 1.12) and infinity (eq 1.14). It has been shown by Bazer and Karp [1952] that such solutions are uniquely determined. Here, it may be mentioned that neither existence of the solutions, nor questions of uniqueness depend critically upon the fact that arg $(a)=(\pi/4)+\delta$. For the unique existence of the solutions, it is sufficient to require merely that arg (a) be restricted to the range $\delta<\arg\,(a)<\pi+\delta$. Similarly, the small positive imaginary part of $|k|$, k, namely $|k|\sin\delta$ (cf. eq (1.2)), may be dispensed with and (1.14) replaced by the Sommerfeld condition. It is, however, retained here because its presence shortens the analysis.

For plane wave excitation originating over the sea, the formulation sketched above is a variant of that given by G. A. Grünberg [1942, 1943]. In these works he considers only the vertical component of the electric field, E_y, which from eq (1.3) is proportional to our $\frac{\partial h}{\partial x}$;

[3] The concern here is only with the mathematical possibility of such excitation, not with its physical realization.

E_y clearly satisfies the wave equation and the same boundary conditions as h. Grünberg reduces the problem of finding E_y to that of solving an integral equation of Wiener-Hopf type. While recognizing the possibility of exact solution by the Wiener-Hopf technique, he prefers an approximate method which leads directly to an expression for the far field of E_y on the land and to the conclusion that there is no refraction at large distances from the shoreline.

Problems closely related to those formulated above have been treated by several investigators, all employing the Wiener-Hopf technique. Among these are works by S. Edelberg [1952], T. B. A. Senior [1952], A. E. Heins and H. Feshbach [1954], and A. F. Kay [1957]. Edelberg's work relates in part to the diffraction of a normally-incident plane electromagnetic wave by an imperfectly conducting half-plane. Senior treats the same two-dimensional problem but allows arbitrary angles of incidence. Kay's investigation is a generalization of the present one in the sense that he employs the impedance boundary condition $\partial h/\partial y+a_0h=0$, $a_0\neq 0$ when $x<0$ and $\partial h/\partial y+a_1h=0$, $a_1\neq 0$, $a_1\neq a_0$ when $x>0$. Kay, however, assumes that a_0 and a_1 are real, whereas here the analog of a_1, namely a, is complex and a_0 vanishes. Heins and Feshback [1954] discuss the coupling of two half-planes. The common feature of all of these works is the "factorization" of a function $\sigma(\nu)$ the form $\sigma(\nu)=1+\kappa/\sqrt{k^2-\nu^2}$ (see section 2) where ν is a complex variable and κ is a suitable constant.

Concerning treatments of the land-sea problem not using the Wiener-Hopf technique, mention may be made of the works of P. C. Clemmow [1953], H. Bremmer [1954], K. Furutsu [1955], J. R. Wait [1956, 1957a, b, c, 1958], E. L. Feinberg [1959], and T. B. A. Senior [1956]. The reader is referred to these papers for additional bibliographical material and for the latest developments in the subject.

In the next section, solutions of the problems formulated above are given. These solutions, which are expressed in the form of integral representations, were originally derived by Bazer and Karp [1952] by a variant of the Wiener-Hopf procedure (see, in this connection, the work of S. N. Karp [1950a, b]). Here, for the sake of brevity, only the end results of the procedure, the integral representations, are presented. It is then verified by standard function-theoretic techniques that the integral representations actually do furnish solutions of the problems considered. An excellent survey of methods based on the Wiener-Hopf technique for the solution partial differential equations has been given by B. Noble [1958].

In section 3, the last section, formulas for the far field resulting from plane-wave excitation over the sea are summarized. In the special case of horizontal incidence, the far field on the air-land interface is shown to agree with that obtained by Grünberg in the works cited above. It is noted, in addition, that the behavior of far fields on the land for intermediate distances indicates the possibility of coastal refraction. All comments on this subject are, however, necessarily of a tentative nature, since, among other things, the vector character of the fields is not taken into account in the present two-dimensional treatment. For a thorough discussion of coastal refraction and related phenomena, based on methods which do not make explicit use of the Wiener-Hopf technique, the reader should consult the papers of P. C. Clemmow, E. L. Feinberg, and T. B. A. Senior, that were mentioned above.

2. Integral Representations of the Solutions—the Factorization

2.1. The Integral Representations

Hereafter, the symbols P_s and P_l will be used for abbreviated reference to the problems associated with homogeneous plane-wave excitation originating over the sea and land, respectively (cf. eqs (1.9) and (1.10)), and P_g will be similarly employed for the problem associated with ground-wave incidence. Let m, r, t, and b be defined by the equations

$$m=k\cos\theta_0,\qquad 0\leq\theta_0\leq\pi/2 \tag{2.1}$$

$$r=\frac{ik\sin\theta_0-a}{ik\sin\theta_0+a}, \tag{2.2}$$

$$t=2^{-1}i(1-r)(k^2-m^2)^{1/2}=\frac{ik\sin\theta_0 a}{ik\sin\theta_0+a}; \tag{2.3}$$

$$b=(k^2+a^2)^{1/2}, \qquad \mathrm{Im}\,(k^2+a^2)^{1/2}>0. \tag{2.4}$$

Let, in addition, $\sigma^+(\nu)$ and $\sigma^-(\nu)$ denote the "factors" of the function

$$\sigma(\nu)=1+[a/i(k^2-\nu^2)^{1/2}], \tag{2.5}$$

in the sense that

$$\frac{\sigma^+(\nu)}{\sigma^-(\nu)}=\sigma(\nu). \tag{2.6}$$

Explicit representations of these factors will be given later. For the following, it need only be required that $\sigma^+(\nu)$ and $\sigma^-(\nu)$ enjoy the following properties: (1) $\sigma^+(\nu)$ is regular, zeroless, and uniformly bounded in the (upper) half-plane defined by Im $(\nu)>-|k|\sin\delta$; (2) $\sigma^-(\nu)$ is regular, zeroless and uniformly bounded in the (lower) half-plane defined by Im $(\nu)<|k|\sin\delta$. Then, in terms of these quantities, the integral representations of the solutions may be expressed as follows:

$$P_s^-: h^s=2\exp(imx)\cos[y(k^2-m^2)^{\frac{1}{2}}]-\frac{a}{\pi i\sigma^+(m)}\int_{-\infty}^{\infty}\frac{\exp\{i[\nu x+y(k^2-\nu^2)^{\frac{1}{2}}]\}}{(\nu-m)[i(k^2-\nu^2)^{\frac{1}{2}}]}\sigma^-(\nu)d\nu; \tag{2.7}^-$$

$$P_s^+: h^s=2\exp(imx)\cos[y(k^2-m^2)^{\frac{1}{2}}]-\frac{a}{\pi i\sigma^+(m)}\int_{-\infty}^{\infty}\frac{\exp\{i[\nu x+y(k^2-\nu^2)^{\frac{1}{2}}]\}}{(\nu-m)[a+i(k^2-\nu^2)^{\frac{1}{2}}]}\sigma^+(\nu)d\nu; \tag{2.7}^+$$

$$\begin{aligned}P_l^-: h^l=&\exp(-imx)\{\exp[-iy(k^2-m^2)^{\frac{1}{2}}]+r\exp[iy(k^2-m^2)^{\frac{1}{2}}]\}\\&-\frac{t}{\pi i\sigma^-(-m)}\int_{-\infty}^{\infty}\frac{\exp\{i[\nu x+y(k^2-\nu^2)^{\frac{1}{2}}]\}}{(\nu+m)[i(k^2-\nu^2)^{\frac{1}{2}}]}\sigma^-(\nu)d\nu;\end{aligned} \tag{2.8}^-$$

$$\begin{aligned}P_l^+: h^l=&\exp(-imx)\{\exp[-iy(k^2-m^2)^{\frac{1}{2}}]+r\exp[iy(k^2-m^2)^{\frac{1}{2}}]\}\\&-\frac{t}{\pi i\sigma^-(-m)}\int_{-\infty}^{\infty}\frac{\exp\{i[\nu x+y(k^2-\nu^2)^{\frac{1}{2}}]\}}{(\nu+m)[a+i(k^2-\nu^2)^{\frac{1}{2}}]}\sigma^+(\nu)d\nu;\end{aligned} \tag{2.8}^+$$

$$P_g^-: h^g=\exp[-(ibx+ay)]+\frac{a\sigma^+(b)}{2\pi i}\int_{-\infty}^{\infty}\frac{\exp\{i[\nu x+y(k^2-\nu^2)^{\frac{1}{2}}]\}}{(\nu+b)(k^2-\nu^2)^{\frac{1}{2}}}\sigma^-(\nu)d\nu; \tag{2.9}^-$$

$$P_g^+: h^g=\exp[-(ibx+ay)]-\frac{a\sigma^+(b)}{2\pi i}\int_{-\infty}^{\infty}\frac{\exp\{i[\nu x+y(k^2-\nu^2)^{\frac{1}{2}}]\}}{(\nu+b)[a+i(k^2-\nu^2)^{\frac{1}{2}}]}\sigma^+(\nu)d\nu. \tag{2.9}^+$$

In these equations, the P^+ representations are obtained from the corresponding P^- representations simply by replacing $\sigma^-(\nu)$ by the equivalent expression $\sigma^+(\nu)[1+a/i(k^2-\nu^2)^{\frac{1}{2}}]^{-1}$ (see eqs (2.5) and (2.6)). In each equation, the terms in the right member consist of an incident wave and when reflection is possible (as is the case in the P_s and P_l representations) a suitable reflected wave. Thus, in eq (2.7), the term $2\exp(imx)\cos[y(k^2-m^2)^{\frac{1}{2}}]$ is a superposition of the incident wave $\overset{s}{h_0}=\exp[ik(x\cos\theta_0-y\sin\theta_0)]$ (cf. eq (1.9)) and the wave $\exp[ik(x\cos\theta_0+y\sin\theta_0]$, reflected off the perfectly conducting sea. In eq (2.8) the first term is the incident wave originating over the land and the second term is the wave reflected off the imperfectly conducting air-land interface, the coefficient r being the reflection coefficient at this boundary. The exponential terms in the integrands of eqs (2.7)–(2.9) are factors of product solution, obtained from the wave equation by means of the standard separation-of-variables procedure. The remaining factors in the integrand are chosen so as to meet the boundary conditions at the air-sea and air-land interface (see eqs (1.4) and (1.5)).

To insure convergence of the integral representations and at the same time to satisfy the "damped radiation condition" at infinity (see eq (1.14)), it is necessary to fix the branch in such a manner that $\exp[iy(k^2-\nu^2)^{\frac{1}{2}}]$ is exponentially damped whenever ν is on the contour,

$-\infty < \mathrm{Re}\ \nu < \infty$, and $|\nu|$ exceeds $|k|$. For this purpose, one fixes the value of $(k^2-\nu^2)^{\frac{1}{2}}$ at $\nu=0$ by requiring that

$$(k^2-\nu^2)^{\frac{1}{2}}|_{\nu=0}=+k. \tag{2.10}$$

Choosing the branch cuts along the rays which go from $+k$ to $\infty \exp(i\delta)$ and $-k$ to $-\infty \exp(i\delta)$, one may then write

$$\begin{aligned} \nu-k&=r_1 \exp[i(\gamma_1+\delta)], \qquad 0<\gamma_1\leq 2\pi; \\ \nu+k&=r_2 \exp[i(\gamma_2+\delta)], \qquad -\pi<\gamma_2\leq \pi, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} r_1&=|\nu-k|, \qquad \gamma_1=\arg(\nu-k)-\arg\delta, \\ r_2&=|\nu+k|, \qquad \gamma_2=\arg(\nu+k)-\arg\delta. \end{aligned} \tag{2.12}$$

From eq (2.11), it then follows that

$$(k^2-\nu^2)^{\frac{1}{2}}=(r_1r_2)^{\frac{1}{2}} \exp[i(\gamma_1+\gamma_2+3\pi/2+\delta)], \quad \begin{array}{c} 0<\gamma_1\leq 2\pi, \\ -\pi<\gamma_2\leq\pi. \end{array} \tag{2.13}$$

It can now be verified that $\arg(k^2-\nu^2)^{\frac{1}{2}}\sim\pi/2$ for ν real and $|\nu|>k$. The behavior of $\arg(k^2-\nu^2)^{\frac{1}{2}}$ on the Re (ν)-axis and immediately above and below the cuts is depicted in figure 2.

It has thus far been assumed in the P_s and P_l representations that $\theta_0\neq\pi/2$. Physically, this angle $\theta_0=\pi/2$ corresponds to plane-wave excitation incident normally on the land-sea surface. This restriction may be removed as follows. Starting with the P_s^+-representation of eq $(2.7)^+$, one allows θ_0 to increase to $\pi/2$. The pole in the integrand at $\nu=m=k\cos\theta_0$ then approaches to contour, on the Re ν axis from above. In the limit as is well-known, half the residue of the pole at $\nu=\lim\limits_{\theta_0\to\pi/2} m=0$ is split off and the resulting integral is taken as a principal-value integral with respect to the pole at $\nu=0$. Letting $h^n(x,y)$ be the limiting solution, one obtains the following integral representation:

$$P_n^+: h^n=2\cos ky-\frac{a\exp(iky)}{(a+ik)}-\frac{a}{\pi i\sigma^+(0)}\int_{-\infty}^{\infty}\frac{\exp\{i[\nu x+y(k^2-\nu^2)^{\frac{1}{2}}]\}}{\nu[a+i(k^2-\nu^2)^{\frac{1}{2}}]}\sigma^+(\nu)\,d\nu, \tag{2.14$^+$}$$

the principal value being taken with respect to the pole at zero. P_n^+ refers to the problem associated with normal incidence and the function $\sigma^+(\nu)$. Writing $\sigma^+(\nu)$ in terms of $\sigma^-(\nu)$ by means of eqs (2.5) and (2.6), one then obtains the alternative P_n^--representation; namely,

$$P_n^-: h^n=2\cos ky-\frac{a}{a+ik}\exp(iky)-\frac{a}{\pi i\sigma^+(0)}\int_{-\infty}^{\infty}\frac{\exp i[\nu x+y(k^2-\nu^2)^{\frac{1}{2}}]}{\nu[i(k^2-\nu^2)^{\frac{1}{2}}]}\sigma^-(\nu)d\nu. \tag{2.14$^-$}$$

It can be shown, using a similar procedure, that the same results are obtained when the P_l^+-representation is taken as the point of departure.

It is now a simple matter to verify that the integrals appearing in eqs (2.7), (2.8), (2.9), and (2.14) are properly convergent and admit all operations under the integral signs, necessary for proving that the boundary conditions and the wave equation are satisfied. First, taking into account the boundedness of $|\sigma^+(\nu)|$ and $|\sigma^-(\nu)|$ on the contour (see properties (1) and (2) in the neighborhood of eq (2.6)), the behavior of $(k^2-\nu^2)^{\frac{1}{2}}$ on the contour, one sees that

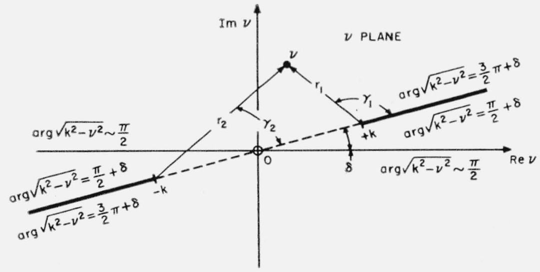

FIGURE 2. *The cut ν-plane.*

the integrands are at worst $0(\nu^{-2})$ as $|\nu|\rightarrow\infty$, independently of x and y. It follows that all the integrals converge uniformly for all x and y in the ranges $-\infty<x<\infty$, $0\leq y<\infty$ and that taking the limit $y\rightarrow 0$ under the integral signs is permissible operation. Next, when y is positive, the function $\exp\,[iy(k^2-\nu^2)^{\frac{1}{2}}]$ decays exponentially as $|\nu|\rightarrow\infty$ on the contour. This implies that one may differentiate any number of times under the integrals signs and still retain the absolute convergence of these integrals for all x and y, $-\infty<x<\infty$, $0<y<\infty$; the reduced-wave equation is therefore seen to be satisfied. Finally, if in $P^+(P^-)$ representations, all contours are bent upwards (downward) slightly so that, as $|\nu|\rightarrow\infty$ on the contour, the relation $0<\operatorname{Im}\nu<|k|\sin\delta(0>\operatorname{Im}\nu>|k|\sin\delta)$ is satisfied, then for $x>0(x<0)$ the function $\exp\,(i\nu x)$ is exponentially damped. It follows, in all cases, that differentiation under the integral signs is a permissible operation for all $y\geq 0$ provided $|x|>0$. Employing the P^+ representations when $x>0$ and the P^- representations when $x<0$, it is now a routine matter to verify the boundary conditions.

2.2. The Factorization

The crux of any boundary-value problem which is solved by the Wiener-Hopf technique is, as a rule, the factorization of a suitable function—$\sigma(\nu)$ in the present case. In fact, provided that the factorization is feasible, one can learn with a little practice how to write down integral representations of the solution (e.g., those of subsection 2.1) at once without resorting to the usual analytic procedures encountered in the early applications of the Wiener-Hopf method. Such an approach is exemplified in the papers by J. Bazer and S. N. Karp [1956] and S. N. Karp [1957]. Now as is well-known, formulas for the factors of a rather large class of functions have been given by N. Wiener and E. Hopf [1931]. The existence of factors with the desired properties is thus assured. However, these formulas express the factors in terms of rather complicated integrals involving contours of infinite extent and it is often found, for this reason, that the behavior of the factors in regions of interest is obscured. Thus, although possibility of factorization is in principle guaranteed, the problem of transforming the original Wiener-Hopf formulas into a more tractable form remains. This transformation was described in detail by Bazer and Karp [1952] for the function

$$\sigma(\nu)=1+[a/i(k^2-\nu^2)^{\frac{1}{2}}]. \tag{2.5$'$}$$

Here, only the end results of this transformation will be presented and some direct consequences of these results will be summarized.

First, it is to be noted that

$$\sigma^-(\nu)=\frac{1}{\sigma^+(-\nu)} \tag{2.15}$$

can be shown to hold for all ν so that it is enough to give the form of $\sigma^+(\nu)$. For this purpose, let the functions $f(\nu)$ and $L'(\nu)$ be defined by

$$f(\nu)=\frac{1}{(k^2-\nu^2)^{\frac{1}{2}}}\left[\frac{\pi}{2}-\operatorname{arc\,sin}\,(\nu/k)\right]=\frac{1}{(k^2-\nu^2)^{\frac{1}{2}}}\left\{\frac{\pi}{2}+i\log\left[\frac{i\nu+(k^2-\nu^2)^{\frac{1}{2}}}{k}\right]\right\}=\frac{\operatorname{arc\,cos}\,(\nu/k)}{(k^2-\nu^2)^{\frac{1}{2}}}, \tag{2.16}$$

and

$$L'(\nu)=\frac{-i\pi}{\nu+k}-a\left[\frac{f(\nu)-f(b)}{\nu-b}+\frac{f(\nu)-f(-b)}{\nu+b}\right], \tag{2.17}$$

where

$$b=(k^2+a^2)^{\frac{1}{2}},\qquad \operatorname{Im}b>0. \tag{2.18}$$

In eq (2.16), the principal values of the inverse trigonometric function and of the logarithm are intended and the branch of $(k^2-\nu^2)^{\frac{1}{2}}$ is specified as in eq (2.10). In terms of these functions, $\sigma^+(\nu)$ may be expressed as

$$\sigma^+(\nu)=\sigma^+(0)\,\exp\left\{\frac{1}{2\pi i}\int_0^\nu L'(\zeta)d\zeta\right\}, \tag{2.19}$$

where

$$\sigma^+(0)=[i+a/ik]^{\frac{1}{2}}. \tag{2.20}$$

Since $f(\nu)$ is regular at all points of the ν-plane except at the lower branch cut of $(k^2-\nu^2)^{\frac{1}{2}}$ [cf. figure 2], it follows, in virtue of eq (2.17) that $L'(\nu)$ is regular in this region. The same is therefore true of $\sigma^+(\nu)$ which may now be shown to have all the properties required of it in subsection 2.1.

The formulas for the far fields of problem P_s, P_l, and P_n involve the values of $\sigma^+(k\cos\theta)$ as θ varies in the interval $0\leq\theta\leq\pi$. It is therefore desirable to obtain a reliable approximation for σ^+ for θ in this range. That it is possible to do so is a consequence of the fact that $L'(\zeta)=L'[\zeta,(a/k)]$ (see eqs (2.16)–(2.18)) is a regular function of a/k when $-1<\zeta/k\leq 1$ and the fact that $|a/k|\leq 0.1$ (see eq (1.8)). Expanding the exponential term in eq (2.19) as a power series in a/k, one finds that

$$\sigma^+(k\cos\theta,a/k)=\sigma^+(0,a/k)\left\{1-\left[\frac{1}{\pi i}\left(\frac{\pi}{2}-\frac{\theta}{\sin\theta}\right)\right](a/k)\right.$$
$$\left.-\frac{1}{2}\left[\frac{1}{\pi^2}\left(\frac{\pi}{2}-\frac{\theta}{\sin\theta}\right)^2+\frac{1}{4}\frac{\cos\theta}{\cos^2(\theta/2)}\right](a/k)^2+R^+(\theta,a/k)\right\},\quad 0\leq\theta<\pi, \tag{2.21}$$

where $R^+(\theta,a/k)$ is the remainder of the series within the braces. In the same way, it is found that

$$\sigma^-(k\cos\theta,a/k)=\frac{1}{\sigma^+(0,a/k)}\left\{1+\left[\frac{1}{\pi i}\left(\frac{\pi}{2}-\frac{\chi}{\sin\chi}\right)\right](a/k)-\frac{1}{2}\left[\frac{1}{\pi^2}\left(\frac{\pi}{2}-\frac{\chi}{\sin\chi}\right)^2\right.\right.$$
$$\left.\left.-\frac{1}{4}\frac{\cos\chi}{\cos^2(\chi/2)}\right](a/k)^2+R^-(\chi,a/k)\right\},\qquad \chi=\pi-\theta,\ 0\leq\chi<\pi. \tag{2.22}$$

For θ and χ in the indicated range greater accuracy could be achieved by including more terms of the power series. However, in deriving the air-land boundary conditions relative errors in the fields of the order of magnitude of 0.01 are neglected. Greater accuracy would therefore be physically meaningless.

Bazer and Karp [1952] have shown that

$$\frac{|\sigma^+(0,a/k)|\,|R^+(\theta,a/k)|}{|\sigma^+(k\cos\theta,a/k)|}<0.01,\qquad 0\leq\theta\leq 140°, \tag{2.23}$$

$$\frac{|R^-(\chi,a/k)|}{|\sigma^+(0,a/k)|\,|\sigma^-(-k\cos\chi,a/k)|}<0.01,\qquad 0\leq\chi\leq 140°. \tag{2.24}$$

From the relation $\sigma^+(k\cos\theta)=(a+ik\sin\theta)\sigma^-(k\cos\theta)/i\sin\theta$ (cf. eq (2.5)), it follows easily that the relative errors in σ^+ and σ^- when calculated by means of eq (2.28) and (2.24) are less than 0.01 throughout the angular range $0<\theta<\pi$.

Formulas for $\sigma^+(\nu)$ of the type given in (2.19) have also been derived by T. B. A. Senior [1952] and A. E. Heins and H. Feshbach [1954]. A derivation is also given in B. Noble's book [1958], pp. 91–92. It does not appear to be generally known, however, that V. Fock had already given the following compact formula for $\sigma^+(\nu)$ as early as 1944 [Fock, 1944, p. 45]:

$$\sigma^+(k\cos\theta)=\sqrt{\frac{\cos\theta+\cos\alpha}{1+\cos\theta}}\exp\left\{-\frac{1}{2\pi}\int_{\theta-\alpha}^{\theta+\alpha}\frac{u}{\sin u}\,du\right\}. \tag{2.25}$$

Here, the angles θ and α are defined by

$$\nu=k\cos\theta, \tag{2.26}$$

$$\sin\alpha=ia/k. \tag{2.27}$$

Fock merely states the result; but it is not difficult to show that this expression follows directly from eqs (2.19) and (2.17) on substituting $\cos\theta$ for ν/k and $\sin\alpha$ for ia/k (cf. eqs (2.26) and (2.27)).

It should be stressed finally that, in the problems under discussion, the constant a is assumed to have nonvanishing imaginary part (see eq (1.6)). If a is real then the absolute values of $\sigma^+(k \cos \theta)$ and $\sigma^-(k \cos \theta)$, $0 \leq \theta \leq \pi$, reduce to simple expressions. For, in this case, the angle α in eq (2.27) is purely imaginary and $\theta - \alpha$, the lower limit of integration in eq (2.25), is the complex conjugate of $\theta + \alpha$ in the upper. It follows easily from this fact and eqs (2.15), (2.25)–(2.27) that

$$|\sigma^-(k \cos \theta)| = \frac{1}{|\sigma^+(k \cos (\pi - \theta)|} = \left| \frac{\cos \theta - 1}{\cos \theta - \cos \alpha} \right| = \left| \frac{\nu - k}{\nu - \sqrt{k^2 + a^2}} \right|, \tag{2.28}$$

whenever ν/k is real and $|\nu/k| \leq 1$. These formulas were first derived by A. F. Kay [1957, eq 66]. Kay *incorrectly* implies that this simplification is valid for nonreal values of a, specifically, that eq (2.28) is also applicable to the problems discussed here.

3. The Far Fields-Coastal Refraction

3.1. The Far Fields

For the sake of brevity, only problem P_s will be discussed. The corresponding results for the remaining problems are given by Bazer and Karp in their report [1952].

Write

$$x = \rho \cos \theta, \qquad y = \rho \sin \theta, \qquad 0 \leq \theta \leq \pi, \tag{3.1}$$

and introduce the new variable of integration

$$\beta = \text{arc} \cos (\nu/k), \qquad 0 \leq \beta \leq \pi \tag{3.2}$$

into eq $(2.7)^+$. The result is

$$P_s^+ : h^s = \exp [ik\rho \cos (\theta - \theta_0)] + \exp [ik\rho \cos (\theta + \theta_0)]$$

$$+ \frac{a}{\pi i \sigma^+(m)} \int_C \frac{\exp [ik\rho \cos (\beta - \theta)] \sigma^+(k \cos \beta) \sin \beta d\beta}{(\cos \beta - \cos \theta_0)(a + ik \sin \beta)}, \tag{3.3}$$

where the contour C is the image in the β-plane of the contour Im $\nu = 0$ in the ν-plane. It is easy to verify that C is essentially the contour shown in figure 3. C is symmetric with respect to the point $\beta = \pi/2$, and as Im $\beta \to i\infty$ on C, Re $\beta \to \delta = \arg k$. The lower portion of the upper cut in the ν-plane goes over into the upper half of the line Re $\beta = 0$ (darkened in the figure), whereas the upper portion of the upper cut goes into the lower half of the line Re $\beta = 0$. The segment traced from 0 to $+k$ in the ν-plane goes over into the arc traced from $\pi/2$ to 0 in the

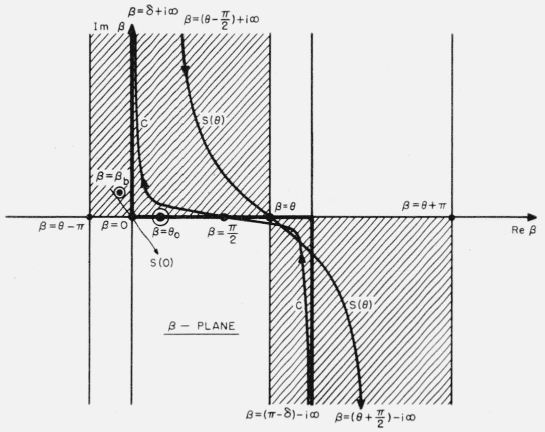

FIGURE 3. *The complex β-plane.*

β-plane. Symmetry with respect to $\beta=\pi/2$ determines the rest of the figure. The deviation of C from the darkened rectilinear path, which is due to the presence of a small imaginary part in k, has been greatly exaggerated for the sake of clarity. The poles in the integrand of the P_s^+ representation are at $\beta=\theta_0\leq\pi/2$ and at $\beta_b=\text{arc sin }(ia/k)\sim(ia/k)=|a/k|\exp(i3\pi/4)$. These poles are shown enclosed in full circles. One might expect a pole at $\beta\sim\pi-(ia/k)$. However, on substituting $\nu=k\cos\beta$ into eqs (2.5) and (2.6) it is found that $i\sin\beta\,\sigma^+(k\cos\beta)/(a+ik\sin\beta)=\sigma^-(k\cos\beta)$. Thus the integrand in the P_s^+ representation is regular near $\beta=\pi$ (since $\sigma^-(k\cos\beta)$ has this property). Note, since $\arg(ia/k)\sim3\pi/4$, β_b is outside the strip $0\leq\text{Re}\,\beta\leq\pi$. Nevertheless, this pole makes its presence felt owing to its proximity to the saddle point at $\theta=0$.[4]

Now that the positions relative to C of the poles at $\beta=\theta_0$ and $\beta=\text{arc sin }(ia/k)$ have been established, it will be supposed, for the sake of simplicity, that $\delta=\arg k$ is equal to zero. The path C then becomes the heavy rectilinear contour of figure 3 except at the pole $\beta=\theta_0$ which it avoids by means, say, of a small semicircular arc. The shaded portions of the figure are those regions where, for fixed θ, $\text{Im}\,[i\cos(\beta-\theta)]<0$; one may therefore deform the contour freely in this region.

The steepest descent contour $S(\theta)$ (see fig. 3) is defined by the equation

$$S(\theta):\cos(\beta-\theta)=1+is^2, \tag{3.4}$$

where s is real. Solving for s in terms of β one finds on selecting the proper branch that

$$s=+2^{\frac{1}{2}}\exp(i\pi/4)\sin[(\beta-\theta)/2]. \tag{3.5}$$

By deforming C into $S(\theta)$ one obtains from eq (3.3), in addition to the appropriate combination of incident and reflected waves, an integral term of the form

$$J(\theta)=\int_{S(\theta)}\exp[ik\rho\cos(\beta-\theta)]A(\beta)d\beta \tag{3.6}$$

which, in virtue of eq (3.4), becomes

$$J(\theta)=2^{\frac{1}{2}}\exp[i(k\rho-\pi/4)]\int_{-\infty}^{\infty}\exp(-k\rho s^2)B(s)ds \tag{3.7}$$

where

$$B(s)=\frac{A[\beta(s)]}{\cos[(\beta(s)-\theta)/2]}. \tag{3.8}$$

Let k be assumed fixed and let $k\rho>>1$ when $\rho\geq\rho_0$; here ρ_0 is a sufficiently large positive number. Then, in the evaluation of the integral term of eq (3.7) only the values of $B(s)$ in the neighborhood of $s=0$ are important owing to the presence of the exponential factor $\exp(-k\rho s^2)$. To obtain a reliable and tractable approximate expression for $J(\theta)$, one therefore is led to replace $B(s)$ by an approximation chosen so as (1) to represent $B(s)$ well near the origin and (2) to make the resulting termwise integration simply evaluable in terms of known functions. Suppose first that $B(s)$ is regular in a sufficiently large neighborhood of the origin. In this case a power series expansion for $B(s)$ meets the above requirements, since it leads to integrals of the form

$$\int_{-\infty}^{\infty}\exp(-k\rho s^2)s^{2n}ds$$

which can be evaluated in terms of inverse powers of $k\rho$ with gamma-function coefficients. The leading term of this expansion gives the same result as the usual saddle point approximation [$s=0$ corresponds to the saddle point at $\beta=\theta$, since $s=(1+i)\sin\frac{1}{2}(\beta-\theta)$]. On the other hand, suppose $B(s)$ possesses a pole at $s=s_0$, where $|s_0|$ may be made arbitrarily close (or even

[4] It should be recalled here that $|a/k|<1/10$ in the range of frequencies studied (see eq (1.8)).

equal) to zero. Here, for fixed $k\rho \geq k\rho_0 >> 1$ a series expansion may possess too small a radius of convergence to extend readily into the range where the influence of the exponential decay of $\exp(-k\rho s^2)$ is felt. In this case, following the methods of H. Ott [1943] and B. L. Van der Waerden [1951], one isolates the pole in a suitable fashion. The method of Ott differs from that of Van der Waerden in the manner of isolating the pole. In their report [1952], the authors employ the procedure of Van der Waerden. The following is a summary of the end results of this procedure for problem P_s.

3.2. Summary of Results

a. Characterization of the Regions of Figures 4(a) and 4(b)

The poles at $\beta=\theta_0$ and $\beta=\beta_b$ are the poles which play a role in the far field formulas. A careful examination of the relative positions of $S(\theta)$ and β_h reveals that $S(\theta)$ sweeps over β_h as θ approaches zero.[5] The value of θ for which $S(\theta)$ actually passes through β_h is called θ_h. Expressed in terms of the variable s, the poles at $\beta=\theta[$ and $\beta=\beta_h$ are located respectively at (see eq (3.5)).

$$s_0=2^{\frac{1}{2}} \exp(i\pi/4) \sin[(\theta_0-\theta)/2] \tag{3.9}$$

and

$$s_0=2^{\frac{1}{2}} \exp(\pi/4) \sin[(\beta_b-\theta)/2]. \tag{3.10}$$

Let $k\rho \geq k\rho_0 >> 1$. Then the value exponential term is appreciable for those values of θ such that $k\rho s_0^2 << 1$ and the Van der Waerden procedure applies. Let it be assumed for the present that $\theta_0 \neq 0$. The region included within the parabola (see figure 4(a))

$$B: \qquad |(2k\rho)^{\frac{1}{2}} \sin[(\theta_0-\theta)/2]|=K_1<<1 \tag{3.11}$$

is called region B. Similarly, the equation

$$D: \qquad |(2k\rho)^{\frac{1}{2}} \sin[(\beta_b-\theta)/2]|=K_2<<1 \tag{3.12}$$

defines a parabola-like region D in the vicinity of $\theta=\theta_b$. In figure (4a) the region D is indicated roughly by the shading above the positive x-axis. Since β_b has a small imaginary part, this region, unlike region B, cannot extend to infinity. The regions A and C of figure (4a) are defined by the relations

$$A: \qquad \pi \geq \theta \geq \theta_0+\Delta\theta_0 \tag{3.13}$$

$$C: \qquad \theta_0-\Delta\theta_0 \geq \theta \geq \theta_b+\Delta\theta_b \tag{3.14}$$

where $\Delta\theta_0$ and $\Delta\theta_b$ are small positive angles. Since β is bounded away from the poles when θ lies in A or C it is possible to choose ρ_0 so large that $|(2k\rho) \sin[(\theta-\theta_0)/2]| >> 1$ and $|(2k\rho)^{\frac{1}{2}} \sin[(\beta_b-\theta)/2]| >> 1$ whenever $\rho \geq \rho_0$. In these regions the usual saddle point approximation applies.

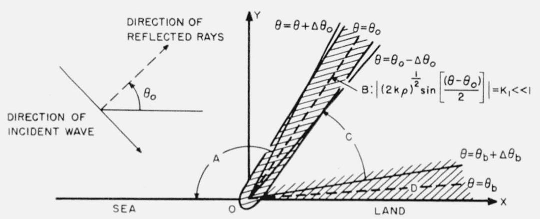

FIGURE 4(a) *Regions* A, B, C, D—*characterized by differing asymptotic field behaviors.*

The excitation is assumed to originate over the sea and to make an angle of θ_0, $0<\theta_0<\pi/2$ with the positive x-axis.

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[5] This fact is proved as follows. Let $\beta_b=x_b+iy_b$, x_b, y_b real and write $\beta=x_0+iy_0$ for that point on $S(0)$ which has the same ordinate as β_b, so that $y_0=y_b$. It is then easy to show that $x_0<x_b$.

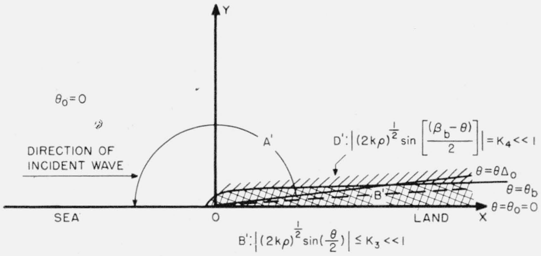

FIGURE 4(b) *Regions* A′ *and* B′—*characterized by different asymptotic field behaviors.*

The excitation progresses parallel to the sea towards the shoreline.

Figure (4b) is obtained from figure (4a) by setting $\theta_0=0$. Regions A' and B' are defined by eqs (3.13) and (3.14) with $\theta_0=0$.

b. Notation

The following notation will be employed in the next two sections.

$$g_1(\beta,\theta,\theta_0) = -\frac{2^{\frac{1}{2}} \exp(i\pi/4)(a-ik\sin\beta)\sigma^+(k\cos\beta)\sin\beta\cos[(\beta+\theta_0-2\theta)/4]}{2\cos[(\beta-\theta_0)/4]\sin[(\beta+\theta_0)/2]\cos[(\beta-\theta_0)/2]} \tag{3.15}$$

$$b_1 = b_1(\theta,\theta_0) = 2^{\frac{1}{2}} \exp(i\pi/4)\sin[(\theta_0-\theta)/2], \tag{3.16}$$

$$l_1^{\frac{1}{2}} = l_1^{\frac{1}{2}}(\theta,\theta_0) = 2^{\frac{1}{2}} \exp(i\pi/4)\,|\sin[(\theta_0-\theta)/2]|; \tag{3.17}$$

$$g_2(\beta,\theta,\theta_0) = \frac{2^{\frac{1}{2}} \exp(i\pi/4)\cos^2(\beta/2)\sigma^+(k\cos\beta)\sin\beta\cos[(\beta-2\theta+\beta_b)/4]}{ik\cos[(\beta-\beta_b)/4)]\cos[(\beta+\beta_b)/2]\cos[(\beta-\theta)/2]}, \tag{3.18}$$

$$b_2 = b_2(\theta,\beta_b) = 2^{\frac{1}{2}} \exp(i\pi/4)\sin[(\beta_b-\theta)/2], \tag{3.19}$$

$$\sin\beta_b = ia/k, \qquad \cos\beta_b = \frac{(k^2+a^2)^{\frac{1}{2}}}{k}, \qquad \mathrm{Im}\cos\beta_b > 0, \tag{3.20}$$

$$l_2^{\frac{1}{2}} = l_2^{\frac{1}{2}}(\theta,\beta_b) = \mp 2^{\frac{1}{2}} \exp(i\pi/4)\sin[(\beta_b-\theta)/2], \begin{Bmatrix} -\text{ when } \theta>\theta_b \\ +\text{ when } \theta<\theta_b \end{Bmatrix}, \tag{3.21}$$

$$1(x) = \begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases} \tag{3.22}$$

The constants c and r are defined by the equations

$$c = a/(a^2+k^2\sin\theta) \tag{3.23}$$

$$r = (ik\sin\theta_0 - a)/(ik\sin\theta_0 + a). \tag{3.24}$$

The function erfc $(z^{\frac{1}{2}})$, with z complex, is defined by

$$\mathrm{erfc}\,(z^{\frac{1}{2}}) = 1 - \mathrm{erf}\,(z^{\frac{1}{2}}) = \frac{1}{2} - \pi^{-\frac{1}{2}} \int_0^{z^{\frac{1}{2}}} \exp(-\zeta^2)\,d\zeta, \tag{3.25}$$

where $z^{\frac{1}{2}}$ is defined to be positive on the positive part of the Re (z) axis.

The far field expressions given below will be expressed in terms of $\sigma^+(k\cos\theta)$. When $\theta\to\pi$, $\sigma^+(k\cos\theta)$ is singular. It is often useful, in this case, to make the substitution (see eqs (2.5) and (2.6)).

$$\sigma^+(k\cos\theta) = [1 + (a/ik\sin\theta)]\sigma^-(k\cos\theta). \tag{3.26}$$

Power series expansions of $\sigma^+(k\cos\theta) = \sigma^+[k\cos\theta, (a/k)]$ and $\sigma^-(k\cos\theta) = \sigma^-[k\cos\theta, (a/k)]$ in terms of (a/k), $|(a/k)| < 0.1$, are given in eqs (2.21) and (2.22).

c. Far Fields—Problem $P_s(0<\theta\leq\pi)$

In this case, the incident wave originates over the sea. The angle θ_0 between the direction of incidence and the positive x-axis (see figure 4a) is assumed not to vanish; horizontal incidence is therefore excluded.

$$\text{Region A: } |(2k\rho)^{\frac{1}{2}}\sin[(\theta-\theta_0)/2]|>>1; \qquad \pi\geq\theta\geq\theta_0+\Delta\theta_0$$

$$h^s\sim\{\exp[ik\rho\cos(\theta-\theta_0)]+\exp[ik\rho\cos(\theta+\theta_0)]\}$$
$$-\frac{2a\sigma^+(k\cos\theta)\sin\theta}{i\sigma^+(k\cos\theta_0)(\cos\theta-\cos\theta_0)(a+ik\sin\theta_0)}\,\frac{\exp[i(k\rho-\pi/4)]}{(2\pi k\rho)^{\frac{1}{2}}}. \tag{3.27}$$

The term in braces is the sum of the incident plane wave and the plane wave which is reflected from the sea. The last term is obtained by a standard saddle point evaluation.

$$\text{Region B: } |(2k\rho)^{\frac{1}{2}}\sin[(\theta-\theta_0)/2]|\leq K_1<<1$$

$$h^s\sim\{\exp[ik\rho\cos(\theta-\theta_0)]+\exp[ik\rho\cos(\theta+\theta_0)]\}-\frac{(2a)1(\theta_0-\theta)}{a+ik\sin\theta_0}\exp[ik\rho\cos(\theta-\theta_0)]$$
$$+ck^2\frac{2\sigma^+(k\cos\theta)\sin\theta\,(\cos\theta+\cos\theta_0)}{\sigma^+(k\cos\theta_0)(a+ik\sin\theta)}\,\frac{\exp[i(k\rho-\pi/4)]}{(2\pi k\rho)^{\frac{1}{2}}}+\frac{2c[g_1(\theta,\theta,\theta_0)-g_1(\theta_0,\theta,\theta)]}{i\sigma^+(k\cos\theta_0)b_1(\theta,\theta_0)}\,\frac{\exp[i(k\rho-\pi/4)]}{(2\pi k\rho)^{\frac{1}{2}}}$$
$$+\frac{2^{\frac{3}{2}}cg_1(\theta_0,\theta,\theta_0)}{i\sigma^+(k\cos\theta_0)b_1(\theta,\theta_0)}\times\exp\left[i\left(k\rho-\frac{\pi}{4}\right)\right]\exp(l_1k\rho)\,l_1^{\frac{1}{2}}(\theta,\theta_0)\operatorname{erfc}[(l_1(\theta,\theta_0)k\rho)^{\frac{1}{2}}]. \tag{3.28}$$

The first, third, and fourth terms in this expression are continuous across $\theta=\theta_0$. On the other hand, the second and fifth are discontinuous across $\theta=\theta_0$ because of the presence factors $1(\theta-\theta_0)$ and $[l_1^{\frac{1}{2}}(\theta,\theta_0)/b_1(\theta,\theta_0)]$ respectively (see eqs (3.16), (3.17)). If we write

$$\operatorname{erfc}[(l_1(\theta,\theta_0)k\rho)^{\frac{1}{2}}]=\frac{1}{2}-\pi^{\frac{1}{2}}\int_0^{(l_1k\rho)^{\frac{1}{2}}}\exp(-\zeta^2)d\zeta,$$

see eq (3.25)), then it is easy to show that the discontinuity in the second term is just compensated by that in the fifth term. It should also be mentioned that at $\theta=\theta_0$ the expression for h^s is independent of $\sigma^+(k\cos\theta_0)$ or $\sigma^-(k\cos\theta_0)$.

Region C: $|(2k\rho)^{\frac{1}{2}}\sin[(\theta-\theta_0)/2]|>>1,\ \theta_0+\Delta\theta_0\geq\theta\geq\theta_b+\Delta\theta_b$

$$h^s\sim\exp[ik\rho\cos(\theta-\theta_0)]+r\exp[ik\rho\cos(\theta+\theta_0)]$$
$$+\frac{2a\sigma^+(k\cos\theta)\sin\theta}{i\sigma^+(k\cos\theta_0)(\cos\theta-\cos\theta_0)(a+ik\sin\theta)}\,\frac{\exp[i(k\rho-\pi/4)]}{(2\pi k\rho)^{\frac{1}{2}}}. \tag{3.29}$$

The expression within the braces is the sum of the incident plane wave $\exp[ik\rho\cos(\theta-\theta_0)]$ and the plane wave $\exp[ik\rho\cos(\theta+\theta_0)]$ which is reflected off the land; r is the reflection coefficient appropriate to the land (cf. eqs (3.24) and (2.2)).

Region D: $|(2k\rho)^{\frac{1}{2}}\sin[(\beta_b-\theta)/2]|<<1,\ \theta\sim\theta_b\sim0$

$$h^s=\{\exp[ik\rho\cos(\theta-\theta_0)]+r\exp[ik\rho\cos(\theta+\theta_0)]\}$$
$$+\frac{1(\theta-\theta_b)2a\sin\beta_b\sigma^+(k\cos\beta_b)\exp[ik\rho\cos(\beta_b-\theta)]}{ik\sigma^+(k\cos\theta_0)(\cos\beta_b-\cos\theta_b)\cos\beta_b}$$
$$-\frac{2c\sigma^+(k\cos\theta)\sin\theta(a-ik\sin\theta)}{i\sigma^+(k\cos\theta_0)(\cos\theta-\cos\theta_0)}\,\frac{\exp[i(k\rho-\pi/4)]}{(2\pi k\rho)^{\frac{1}{2}}}$$
$$+\frac{2ck^2[g_2(\theta,\theta,\theta_0)-g_2(\beta_b,\theta,\theta_0)]}{i\sigma^+(k\cos\theta_0)b_2(\theta,\beta_b)}\,\frac{\exp[i(k\rho-\pi/4)]}{(2\pi k\rho)^{\frac{1}{2}}}$$
$$+\frac{2^{3/2}ck^2g_2(\beta_b,\theta,\theta_0)}{i\sigma^+(k\cos\theta_0)b_2(\theta,\beta_b)}\exp[i(k\rho-\pi/4)]\exp(l_2k\rho)\,l_2^{\frac{1}{2}}(\theta,\beta_b)\operatorname{erfc}[(l_2(\theta,\beta_b)k\rho)^{\frac{1}{2}}]. \tag{3.30}$$

This expression remains valid in region C. Note that the last term contains the factors $\exp(ik\rho)\exp(l_2k\rho)=\exp\{ik\rho[1-2\sin^2[(\beta_b-\theta)/2]]\}=\exp[ik\rho\cos(\beta_b-\theta)]$. As θ passes through θ_b, $l_2^{\frac{1}{2}}(\theta,\beta_b)$ changes sign (see eq (3.21)). From this fact, and eq (3.25), one finds that the jump in the third term of eq (3.30) is just compensated by the jump in the last term, the remaining terms being continuous across $\theta=\theta_b$.

d. Far Fields—Problem P_s—($\theta_0=0$)

Again the incident wave originates over the sea, but in the present case the direction of incidence is horizontal.

Region A′: $|(2k\rho)^{\frac{1}{2}}\sin(\theta/2)|>>1$, $\pi\geq\theta\geq\theta_b+\Delta\theta_b$

$$\frac{1}{2}h^2\sim\exp(ik\rho\cos\theta)+\frac{a\sigma^+(k\cos\theta)}{i\sigma^+(k)\tan(\theta/2)(a+ik\sin\theta)}\frac{\exp[i(k\rho+\pi/4)]}{(2\pi k\rho)^{\frac{1}{2}}}. \tag{3.31}$$

This expression follows from a straightforward application of the saddle-point method.

Region B′: $|(2k\rho)^{\frac{1}{2}}\sin(\theta/2)|\leq K_3<<1$ and $|(2k\rho)^{\frac{1}{2}}\sin[(\beta_b-\theta)/2]|\leq K_4<<1$

$$\begin{aligned}\frac{1}{2}h^s\sim&\exp(ik\rho\cos\theta)+\frac{[g_1(\theta,\theta,0)-g(0,\theta,0)]}{ai\sigma^+(k)b_1(\theta,0)}\frac{\exp[i(k\rho-\pi/4)]}{(2\pi k\rho)^{\frac{1}{2}}}\\&+\frac{2^{\frac{1}{2}}g_1(0,\theta,0)}{ai\sigma^+(k)b_1(\theta,0)}\exp[i(k\rho-\pi/4)]\exp(l_1(\theta,0)k\rho)l_1^{\frac{1}{2}}(\theta,0)\times\operatorname{erfc}[(l_1(\theta,0)k\rho)^{\frac{1}{2}}]\\&+1(\theta-\theta_b)\frac{2^{\frac{1}{2}}k^2g_2(\beta_b,\theta,0)}{a\sigma^+(k)}\exp(-i\pi/4)\exp[ik\rho\cos(\beta_b-\theta)]\\&+\frac{k^2[g_2(\theta,\theta,0)-g_2(\beta_b,\theta,0)]}{ai\sigma^+(k)b_2(\theta,\beta_b)}\frac{\exp[i(k\rho-\pi/4)]}{(2\pi k\rho)^{\frac{1}{2}}}\\&+\frac{2^{\frac{1}{2}}k^2g_2(\beta_b,\theta,0)}{ia\sigma^+(k)b_2(\theta,\beta_b)}\exp[i(k\rho-\pi/4)]\exp(l_2k\rho)l_2^{\frac{1}{2}}(\theta,\beta_b)\times\operatorname{erfc}[(l_2^{\frac{1}{2}}(\theta,\beta_b)k\rho)^{\frac{1}{2}}].\end{aligned} \tag{3.32}$$

This expression gives the far field for h^s in region A' also and reduces to the expression given in eq (3.31) when $|(2k\rho)^{\frac{1}{2}}\sin(\theta/2)|>>1$.

Important simplifications occur when $\theta=0$—i.e., on the land surface. If the influence of the complex pole at $\beta=\beta_b$ is neglected or equivalently if it is assumed that $k\rho_0$ is so large that $|(2k\rho_0)^{\frac{1}{2}}\sin(\beta_b/2)|>>1$ then the last three terms are $0[(k\rho_0)^{-3/2}]$ whenever $\rho>\rho_0$. Furthermore, the third term just cancels the first term. As a result, it is found that

$$\frac{1}{2}h^s(\rho,0)\sim-\frac{2k}{a}\frac{\exp[i(k\rho-\pi/4)]}{(2\pi k\rho)^{\frac{1}{2}}}=\frac{i\lambda^{3/2}}{2\pi^2d_2}\frac{\exp(ik\rho)}{\rho^{\frac{1}{2}}}. \tag{3.33}$$

Here, λ is the wavelength in air and d_2 is defined in eq (1.7). This formula was first obtained by Grünberg [1942, 1943]. Note that this result is independent of the function σ^+.

On the other hand, when $\theta=0$ and $|(2k\rho)^{\frac{1}{2}}\sin(\beta_b/2)|$, with $\rho>\rho_0$, is small compared to unity, all the terms of eq (3.32) must be taken into account. The result is

$$\begin{aligned}\frac{1}{2}h^s(\rho,0)\sim&\frac{2k}{a}\left[1-\frac{\sigma^+(k\cos\beta_b)\cos^2(\beta_b/2)}{\sigma^+(k)\cos\beta_b}\right]\frac{\exp[i(k\rho-\pi/4)]}{(2\pi k\rho)^{\frac{1}{2}}}+\frac{\sigma^+(k\cos\beta_b)\cos^2(\beta_b/2)}{\sigma^+(k)\cos\beta_b}\\&\times\exp(ik\rho\cos\beta_b)\times\left[1+\frac{2}{(\pi)^{\frac{1}{2}}}\int_0^{(k\rho)^{\frac{1}{2}}(1+i)\sin(\beta_b/2)}\exp(-\zeta^2)d\zeta\right]+0[(k\rho)^{-3/2}].\end{aligned} \tag{3.34}$$

(The terms four and six in eq (3.32) lead, with the help of eq (3.25) to the second term above and the remaining terms of eq (3.32) yield the first term above.) Since $\cos\beta_b=[1+(a/k)^2]^{\frac{1}{2}}$

and $|(a/k)|\leq 0.1$, we have the following estimate for $h^s(\rho,0)$ on the land

$$\frac{1}{2}h^s(\rho,0)\sim\exp\left[i\rho(k^2+a^2)^{\frac{1}{2}}\right]\left[1+\frac{2}{(\pi)^{\frac{1}{2}}}\int_0^{(k\rho)^{\frac{1}{2}}(1+i)\sin(\beta_b/2)}\exp(-\zeta^2)d\zeta\right]+0(a/k)+0[(k\rho)^{-3/2}]. \qquad (3.35)$$

Thus for sufficiently small $|(2k\rho)^{\frac{1}{2}}\sin(\beta_b/2)|$, $h^s(\rho,0)$ is a ground wave (cf., eq (1.11)).

3.3. Some Comments Regarding Coastal Refraction

As was pointed out by Bazer and Karp [1952], had problem P_s been formulated in terms of E_y, the vertical component of the electric field, instead of h (see eqs (1.1)–(1.3)), it would have been possible to treat the case of an exciting plane wave incident parallel to the sea but making an angle, $\frac{1}{2}\pi-\phi$ say $(0\leq\phi<\pi/2)$ with the shoreline. (In fact, it was in terms of the angle ϕ that G. A. Grünberg [1942, 1943] formulated his discussion of the problem of coastal refraction (see section 1).) It would then have been found, employing the method of the paper, that E_y varies as $(\cos\phi)^{\frac{1}{2}}h^s(\rho,0)$ for $|[(k\cos\phi)\rho]^{\frac{1}{2}}\sin(\beta_b/2)|>>1$ on the land, [6] a distance ρ from the shoreline. Here h^s is the expression in eq (3.33). This result is in complete agreement with Grünberg's and shows, at least within the present theoretical context, that there is no refraction at large distances from the shoreline. The possibility of coastal refraction still exists, however, when $|(k\rho\cos\phi)^{\frac{1}{2}}\sin(\beta_b/2)|<<1$, since, in this case, the presence of a ground wave of the form $\exp\left[i\rho\sqrt{k^2\cos^2\phi+a^2}+k\cos\phi\, z\right]$ would be felt (see eq (3.35)). The attention of the authors was drawn to this range by H. G. Booker,[7] to whom they are further indebted for bringing to their notice an early work by Eckersley [1920] which bears directly on this problem. To sum up, the appearance of a ground wave for "moderate" values of $(k\cos\phi)\rho$ leads us to believe that coastal refraction may be explained by an extension of the methods of our paper. What is required is a vector treatment of the problem for the case of oblique incidence and a careful investigation of the error terms in the oblique-incidence analogs of the results presented in the above summary.

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[6] In the present discussion $\beta_b=\text{arc}\cos\left[\frac{\sqrt{k^2\cos^2\phi-a^2}}{k\cos\phi}\right]$, cf. eq (3.20).

[7] In a conversation with S. N. Karp.

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