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Statistical Distribution of the Amplitude and Phase of a Multiply Scattered Field

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The probability distribution of the amplitude and phase of the sum of a large number of random two-dimensional vectors is derived under the following general conditions: Both the amplitudes and the phases of the component vectors are random, the distributions being arbitrary within the validity of the Central Limit Theorem; in particular, the distributions of the individual vectors need not be identical, the amplitude and phase of each component vector need not be independent and the distributions need not be symmetrical. The distributions formerly derived by Rayleigh, Rice, Hoyt, and Beckmann are shown to be special cases of this distribution.

1. Introduction

The electromagnetic field scattered by randomly distributed scatterers necessarily always consists of the individually scattered waves which mutually interfere to form the resulting total field. We may thus write

$$E = re^{i\theta} = \sum_{j=1}^{n} A_j e^{i\phi_j} \tag{1}$$

where r is the amplitude and θ the phase of the resulting field and $A_j \exp(i\phi_j)$ are the elementary component waves. The amplitudes A_j and phases ϕ_j, and hence also r and θ, are random quantities.

The sum (1) may be represented in the complex plane as the sum of random vectors (fig. 1); the problem is also identical with the "random-walk problem" in mathematical statistics, for our task is to determine the probability distributions of r and θ if the distributions of the A_j and ϕ_j are known.

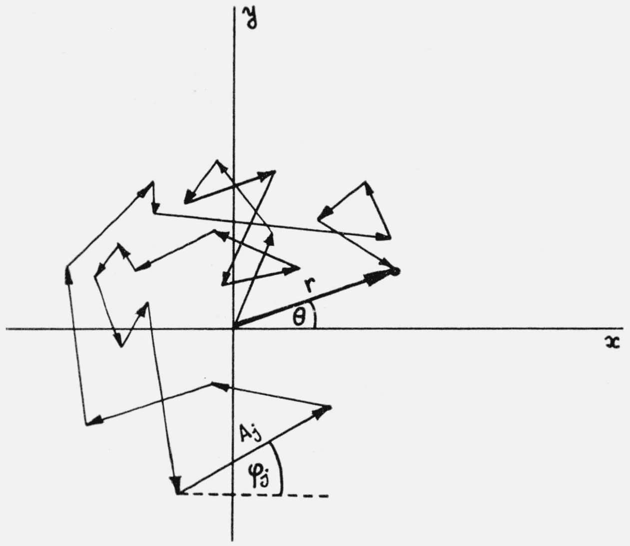

FIGURE 1. *Random vector sum in the complex plane.*

The problem in its most elementary form, when the A_j are constant and the ϕ_j are all distributed uniformly over an interval of length 2π, was solved by Rayleigh [1896] and leads to the well-known Rayleigh distribution

$$p(r) = \frac{2r}{n} e^{-r^2/n}. \tag{2}$$

Other special cases of the distribution of E were derived by Rice [1944, 1945] and by Hoyt [1947].

A more general solution, namely for all A_j constant and arbitrary symmetrical distributions of the phases was derived by Beckmann [1959]. The Rayleigh, Rice, and Hoyt distributions are included in this distribution as special cases.

The purpose of the present paper is to derive the distributions of r and θ in the general case when both the A_j and the ϕ_j are random and arbitrarily distributed. The distributions of the individual A_j or ϕ_j need not be identical, A_j and ϕ_j may be correlated and the distributions of the ϕ_j need not be symmetrical. The only restriction is that the distributions of the quantities $A_j \cos \phi_j$ and $A_j \sin \phi_j$ satisfy the conditions of the Central Limit Theorem [Gnedenko and Kolmogorov, 1949]. In physical practice these conditions are practically always satisfied provided that n, the number of interfering waves, is large and that these waves are statistically independent.[1]

Our problem, then, is to determine the probability distribution $p(r)$ and the (more rarely required) distribution $p(\theta)$ when the two-dimensional distributions $w_j(A,\phi)$ are given.

[1] Generalization for dependent variables is also possible [Bernstein, 1944].

2. General Solution

We introduce the quantities

$$x=r\cos\theta;\quad y=r\sin\theta \tag{3}$$

and denote the mean value by angular brackets $\langle\ \rangle$ and the variance by the symbol D. Then

$$\langle x\rangle=\alpha=\sum_{j=1}^{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_j(A,\phi)A\cos\phi\, dAd\phi, \tag{4}$$

$$\langle y\rangle=\beta=\sum_{j=1}^{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_j(A,\phi)A\sin\phi\, dAd\phi. \tag{5}$$

Denoting the individual terms of the sums (4) and (5) by [2]

$$\alpha_j=\iint w_j(A,\phi)A\cos\phi\, dAd\phi, \tag{6}$$

$$\beta_j=\iint w_j(A,\phi)A\sin\phi\, dAd\phi, \tag{7}$$

we have

$$D\{x\}=s_1=\sum_{j=1}^{n}\left[\iint w_j(A,\phi)A^2\cos^2\phi\, dAd\phi-\alpha_j^2\right], \tag{8}$$

$$D\{y\}=s_2=\sum_{j=1}^{n}\left[\iint w_j(A,\phi)A^2\sin^2\phi\, dAd\phi-\beta_j^2\right]. \tag{9}$$

The covariance of x and y is, after elementary manipulations,

$$\begin{aligned}\mathrm{cov}\,(x,y)=&\frac{1}{2}\sum_{j=1}^{n}\int w_j(A,\phi)A^2\sin 2\phi\, dAd\phi\\ &+\sum_{i=1}^{n}\sum_{j=1}^{n}{}'\iint w_i(A,\phi)A\cos\phi\, dAd\phi\\ &\iint w_j(A,\phi)A\sin\phi\, dAd\phi-\alpha\beta,\end{aligned} \tag{10}$$

where the apostrophe at the second sum indicates that the terms with $i=j$ are to be omitted.

We now temporarily assume that the distribution of ϕ is symmetrical about its mean value zero. Then the integrals in (5), (7), and (10) vanish owing to the factor $\sin\phi$ or $\sin 2\phi$ in the integrands, so that

$$\beta_j=\beta=\mathrm{cov}\,(x,y)=0. \tag{11}$$

Now according to the Central Limit Theorem the distribution of the sum of many independent random variables tends under certain conditions that in practical applications are almost invariably satisfied, to the normal (Gaussian) distribution.[3]

Since by (11) x and y are uncorrelated (and hence, as normal random variables, independent), the two-dimensional probability density of x and y is

$$W(x,y)=\frac{1}{2\pi\sqrt{s_1s_2}}\exp\left[-\frac{(x-\alpha)^2}{2s_1}-\frac{y^2}{2s_2}\right]. \tag{12}$$

Transforming to polar coordinates as in (3), we find the required distribution in the form

$$p(r)=\frac{r}{2\pi\sqrt{s_1s_2}}\int_0^{2\pi}\exp\left[-\frac{(r\cos-\alpha)^2}{2s_1}-\frac{r^2\sin^2\theta}{2s_2}\right]d\theta, \tag{13}$$

which after elementary manipulations may be written as

$$p(r)=Se^{-T}\int_0^{2\pi}e^{-P\cos^2\theta+Q\cos\theta}d\theta, \tag{14}$$

where S, T, P, and Q are functions of r, α, s_1, and s_2. The integral in (14) may be expressed as a series of modified Bessel functions of the first kind (I_m) as follows [Beckmann and Schmelovsky, 1958]:

$$\int_0^{2\pi}e^{-P\cos^2\theta+Q\cos\theta}d\theta=2\pi e^{-P/2}\sum_{m=0}^{\infty}(-1)^m\epsilon_m I_m\left(\frac{P}{2}\right)I_{2m}(Q) \tag{15}$$

where

$$\epsilon_m=\begin{cases}1 \text{ for } m=0,\\ 2 \text{ for } m\neq 0,\end{cases} \tag{16}$$

so that (13) may be expressed as follows:

$$p(r)=\frac{r}{\sqrt{s_1s_2}}e^{-\frac{\alpha^2}{2s_1}-\frac{s_1+s_2}{4s_1s_2}r^2}\times\sum_{m=0}^{\infty}(-1)^m\epsilon_m I_m\left(\frac{s_2-s_1}{4s_1s_2}r^2\right)I_{2m}\left(\frac{\alpha}{s_1}r\right). \tag{17}$$

Before normalizing the distribution (15), we calculate the mean square of r, which, apart from a constant factor ($1/120\pi$ in MKS units), equals the mean scattered power. From (3) and the formula for the variance of a random quantity

$$D\{r\}=\langle r^2\rangle-\langle r\rangle^2$$

and from the independence of x and y it follows that

$$\langle r^2\rangle=s_1+s_2+\alpha^2. \tag{18}$$

The RMS value of the scattered field equals the square root of this expression.

It is easily seen from (12) that the resulting vector $r\exp(i\theta)$ is the sum of a constant vector $\vec{\alpha}$ directed along the x-axis and a random vector $\mathbf{H}$ (fig. 2); the x and y components of this random (Hoyt) vector are normally distributed with mean values zero and unequal variances s_1 and s_2.

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[2] To save space, we henceforward omit the integration limits $-\infty$, ∞; unless otherwise indicated, all integrals in this paper are therefore to be taken from $-\infty$ to ∞.

[3] For an exact statement of the Central Limit Theorem and the weakest (Lindeberg) conditions of its validity, see [Gnedenko and Kolmogorov, 1949, Ch. 5, Sec. 26], or [Richter, 1956, Ch. VII, Sec. 4]. An unnecessarily strong condition for the validity of the Central Limit Theorem is e.g. that the distributions of the terms be identical and their variance exist (Bernoulli condition).

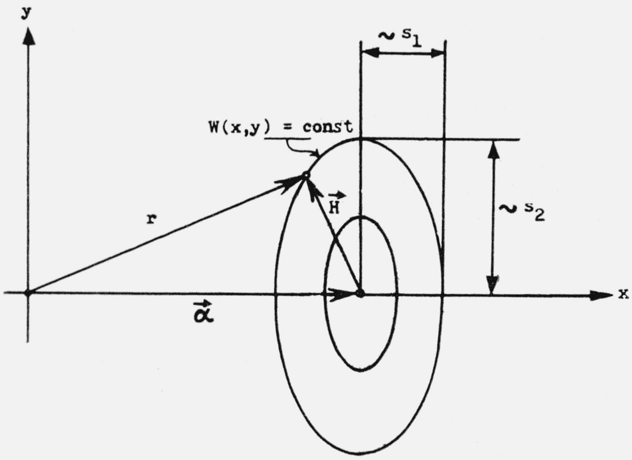

FIGURE 2. *Components of the resultant vector* r *and its equiprobability curves.*

We now normalize in accordance with (18) by using the ratio of the RMS values of the constant and random components

$$B=\frac{\alpha}{\sqrt{s_1+s_2}} \tag{19}$$

and introducing the "normalized amplitude"

$$\rho=\frac{r}{\sqrt{s_1+s_2}} \tag{20}$$

and the "asymmetry factor"

$$K=\sqrt{\frac{s_2}{s_1}}. \tag{21}$$

Using this normalization,[4] we obtain the required distribution by substituting (19) to (21) in (17) in the general form

$$p(\rho)=\frac{K^2+1}{K}\,\rho\exp\left[-\frac{1+K^2}{2}\left(B^2+\frac{1+K^2}{2K^2}\,\rho^2\right)\right]$$

$$\times\sum_{m=0}^{\infty}(-1)^m\epsilon_m I_m\left(\frac{K^4-1}{4K^2}\,\rho^2\right)I_{2m}\,[B(1+K^2)\rho]. \tag{22}$$

The distribution (22) is determined through the parameters B and K according to (19) to (21), which in turn are determined through α, s_1, and s_2 according to (4), (8), and (9).

The probability densities $p(\rho;B,K)$ according to (22) for various values of B and K are shown in figures 5 to 9.

The complement of the distribution function, i.e.,

$$P\{\rho>z;B,K\}=1-\int_0^z p(\rho;B,K)d\rho \tag{23}$$

[4] This normalization differs from that used by Beckmann, [1959 and 1960].

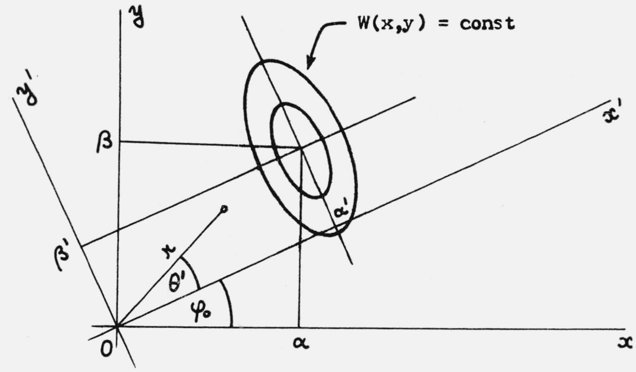

FIGURE 3. *Transformation of coordinates when* w(ϕ) *is not an even function.*

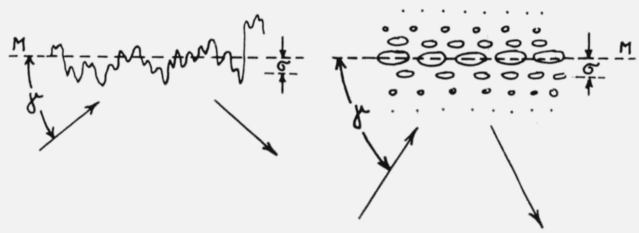

FIGURE 4. *Scattering by rough or turbulent layers in the troposphere.*

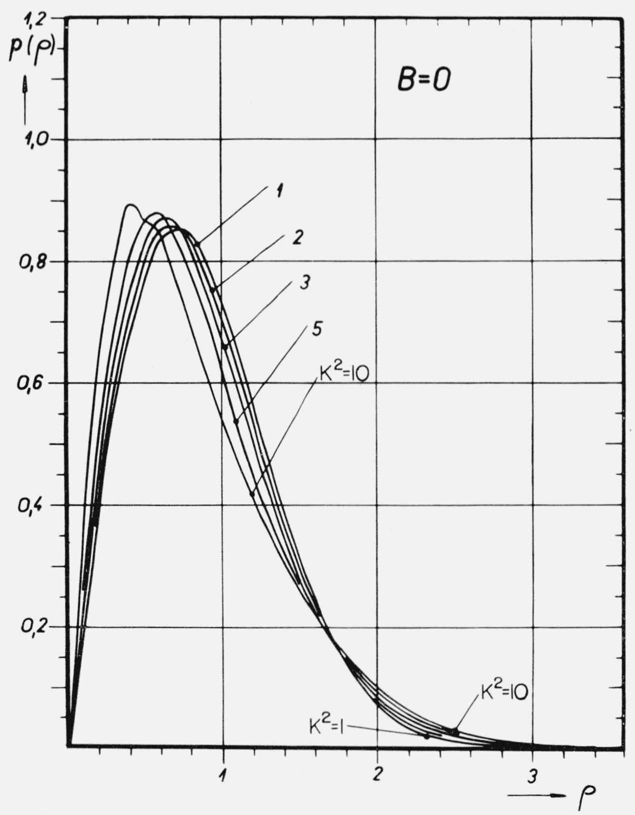

FIGURE 5. *Probability densities* p(ρ) *for* B=0.

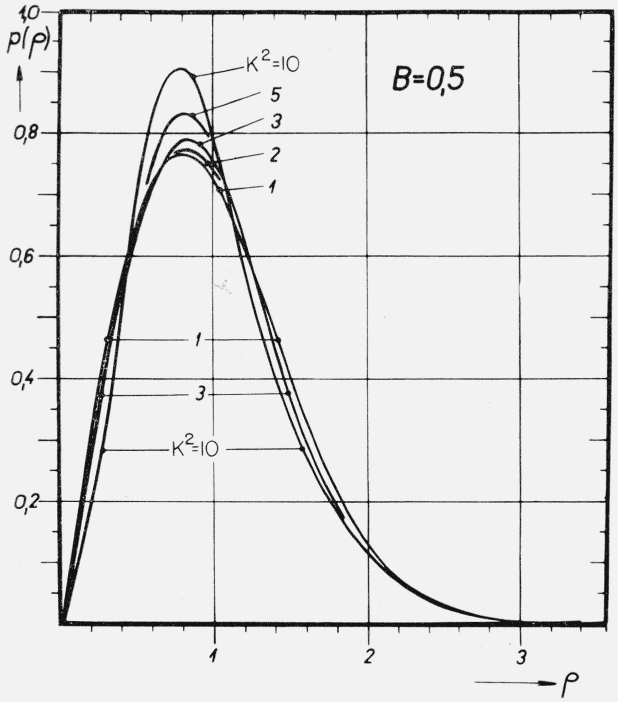

FIGURE 6. *Probability densities* p(ρ) *for* B=0.5.

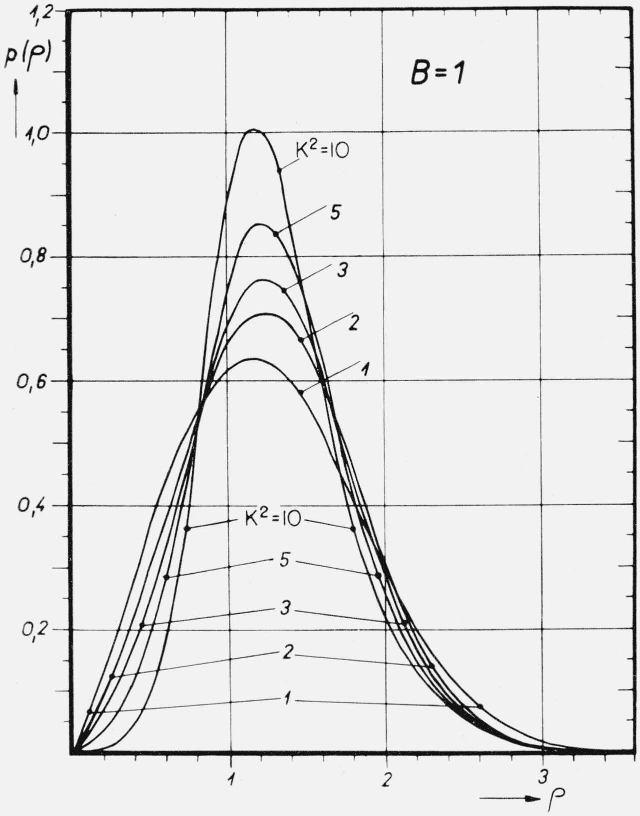

FIGURE 7. *Probability densities* p(ρ) *for* B=1.

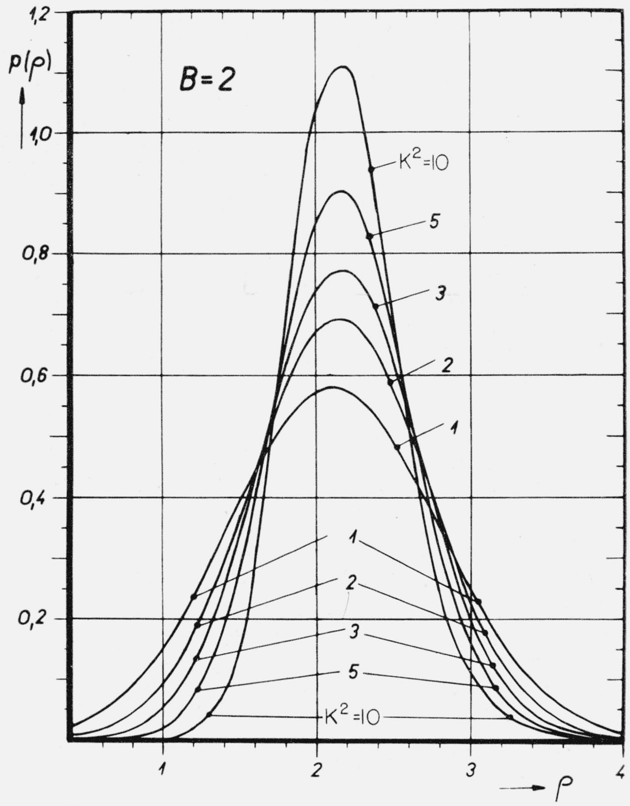

FIGURE 8. *Probability densities* p(ρ) *for* B=2.

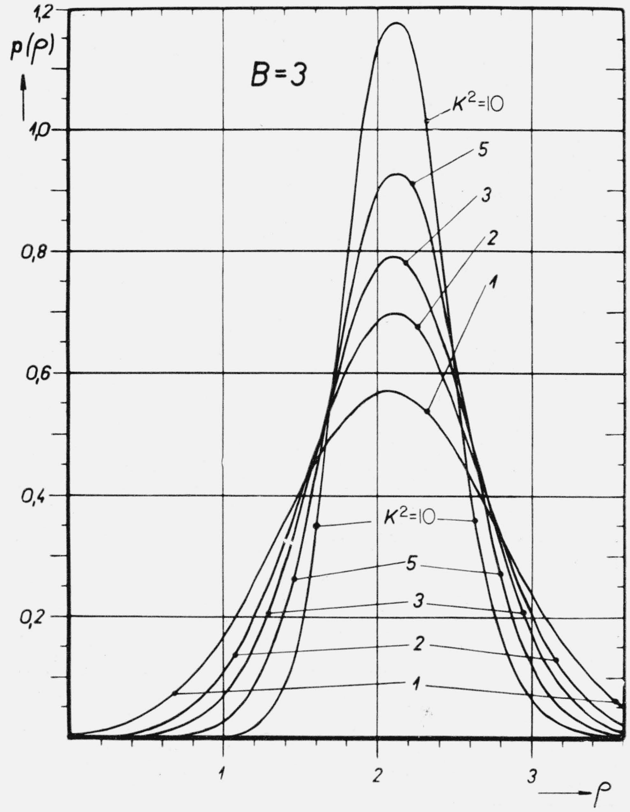

FIGURE 9. *Probability densities* p(ρ) *for* B=3.

for various values of B and K are plotted on probability paper in figures 10 to 14.[5]

[5] The curves in figs. 5 to 14 were calculated by direct numerical integration of (13) by punched card machine under the guidance of Mr. R. Vích.

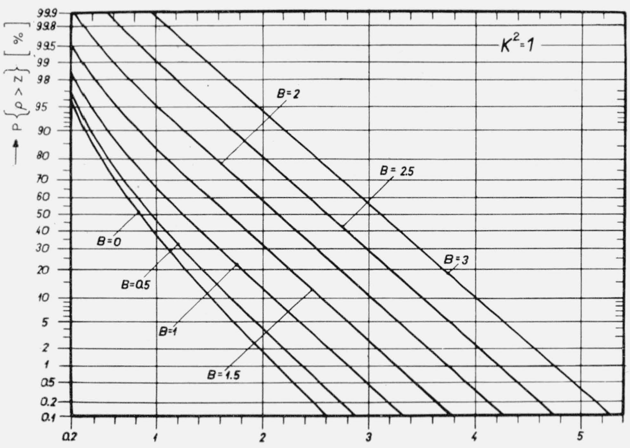

FIGURE 10. *Distribution curves of* $\mathrm{P}\{\rho>\mathrm{z}\}$ *for* $\mathrm{K}^2=1$.

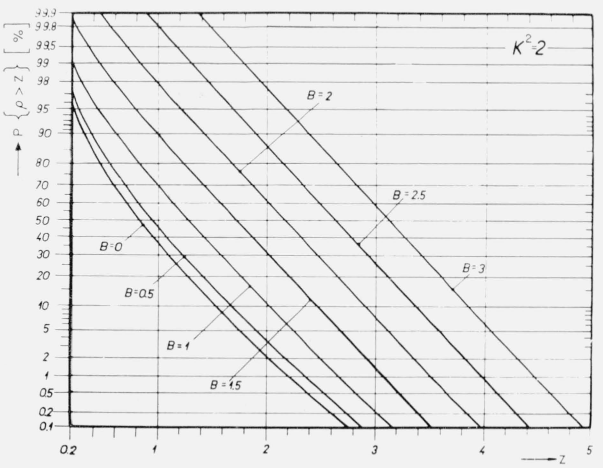

FIGURE 11. *Distribution curves of* $\mathrm{P}\{\rho>\mathrm{z}\}$ *for* $\mathrm{K}^2=2$.

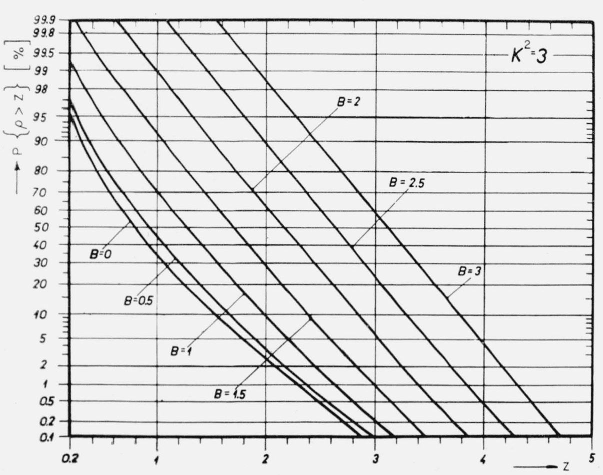

FIGURE 12. *Distribution curves of* $\mathrm{P}\{\rho>\mathrm{z}\}$ *for* $\mathrm{K}^2=3$.

As may be seen from figures 10 to 14, the distribution becomes practically normal for $B\geq 3$, $K\leq 3$, as may also be shown theoretically (cf. appendix). The mean value and variance in this case is given by

$$\langle\rho\rangle\approx B,\quad D\{\rho\}\approx\frac{1}{1+K^2}. \tag{24}$$

From (18), (19), and (20) we have

$$\langle\rho^2\rangle=1+B^2;\quad \rho_{\mathrm{RMS}}=\sqrt{1+B^2}. \tag{25}$$

Hence

$$P\left\{\frac{r}{r_{\mathrm{RMS}}}>z;B,K\right\}=P\left\{\frac{\rho}{\rho_{\mathrm{RMS}}}>z;B,K\right\}$$
$$=1-\int_0^{z\sqrt{1+B^2}}p(\rho;B,K)d\rho. \tag{26}$$

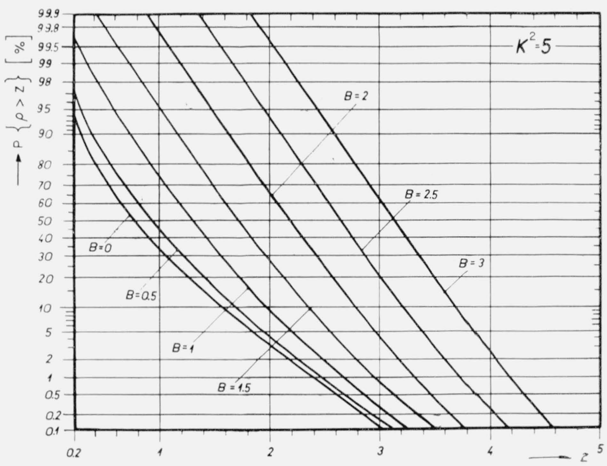

FIGURE 13. *Distribution curves of* $\mathrm{P}\{\rho>\mathrm{z}\}$ *for* $\mathrm{K}^2=5$.

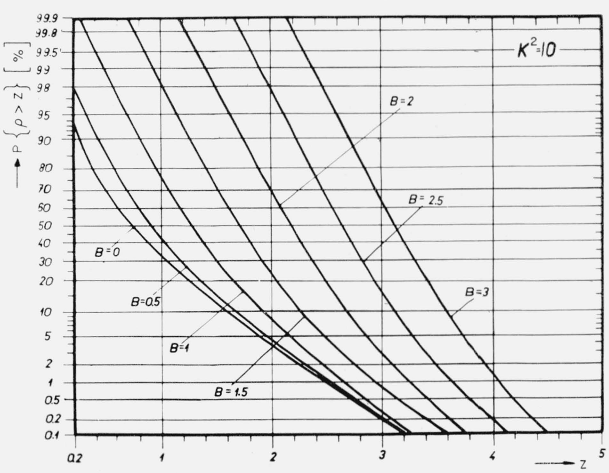

FIGURE 14. *Distribution curves of* $\mathrm{P}\{\rho>\mathrm{z}\}$ *for* $\mathrm{K}^2=10$.

Figures 15 to 19 show this relation plotted on Rayleigh paper (on which the Rayleigh distribution appears as a straight line with 45° slope).

The less often required statistical distribution of the phase is found from (13) by integrating over r from 0 to ∞ instead of over θ from 0 to 2π. After a somewhat tedious calculation one obtains

$$p(\theta)=\frac{Ke^{-\frac{1}{2}B^2(1+K^2)}}{2\pi(K^2\cos^2\theta+\sin^2\theta)}\,[1+G\sqrt{\pi}e^{G^2}(1+\text{erf}\,G)], \tag{27}$$

where

$$G=BK\sqrt{\frac{1+K^2}{2(K^2\cos^2\theta+\sin^2\theta)}}$$

and

$$\text{erf}\,G=\frac{2}{\sqrt{\pi}}\int_0^G e^{-t^2}dt.$$

We now return to the case when the distributions of the phases ϕ_j are not symmetrical about zero. The general formulas (4) to (9) are still valid, but neither β in (5) nor the covariance cov (x,y) in (10) will vanish, so that x and y are no longer independent and our derivation breaks down from (11) onward. In this case we proceed as follows.

We calculate the covariance cov (x,y) according to (10) and determine the correlation coefficient

$$C=\frac{\text{cov}\,(x,y)}{\sqrt{s_1s_2}}. \tag{28}$$

We now introduce new coordinate axes x' and y' which are turned through an angle ϕ_0 with respect to the original axes x,y (fig. 3). The angle ϕ_0 is so chosen that the quantities $x'=r\cos\theta'$ and $y'=r\sin\theta'$ are uncorrelated, where

$$\theta'=\theta-\phi_0. \tag{29}$$

For x' and y' to be uncorrelated (and hence, as normal random variables, independent), it is sufficient that the two-dimensional distribution $W(x',y')$ have an axis of symmetry parallel to one of the coordinate axes. Since the curves $W(x,y)=$const are concentric ellipses with center $x=\alpha$, $y=\beta$, it is therefore sufficient to choose the x' and y' axes parallel to the axes of the ellipses. The required angle ϕ_0 then follows from [Hristow 1961, p. 125]

$$\tan 2\phi_0=\frac{2C\sqrt{s_1s_2}}{s_1-s_2}. \tag{30}$$

In this new coordinate system we then proceed as before; the only difference is that β' does not vanish now. Instead of (13) we therefore have

$$p(r)=\frac{r}{2\pi\sqrt{s_1s_2}}\int_0^{2\pi}\exp\left[-\frac{(r\cos\theta-\alpha)^2}{2s_1}-\frac{(r\sin\theta-\beta)^2}{2s_2}\right]d\theta, \tag{31}$$

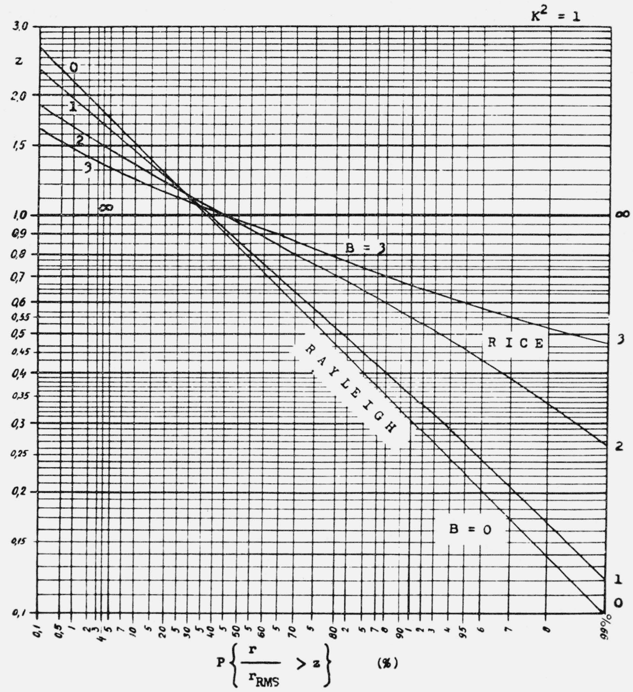

FIGURE 15. *Distribution curves of* $P\{r/r_{RMS}>z\}$ *for* $K^2=1$ *plotted on Rayleigh paper.*

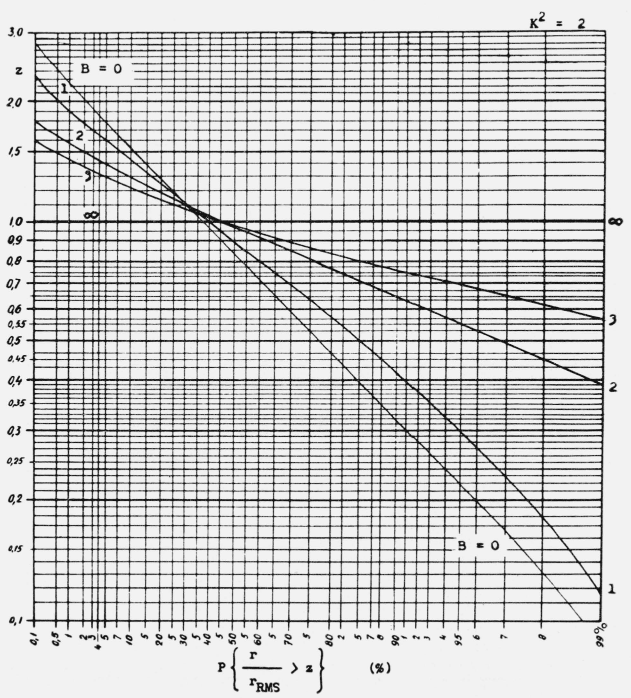

FIGURE 16. *Distribution curves of* $P\{r/r_{RMS}>z\}$ *for* $K^2=2$ *plotted on Rayleigh paper.*

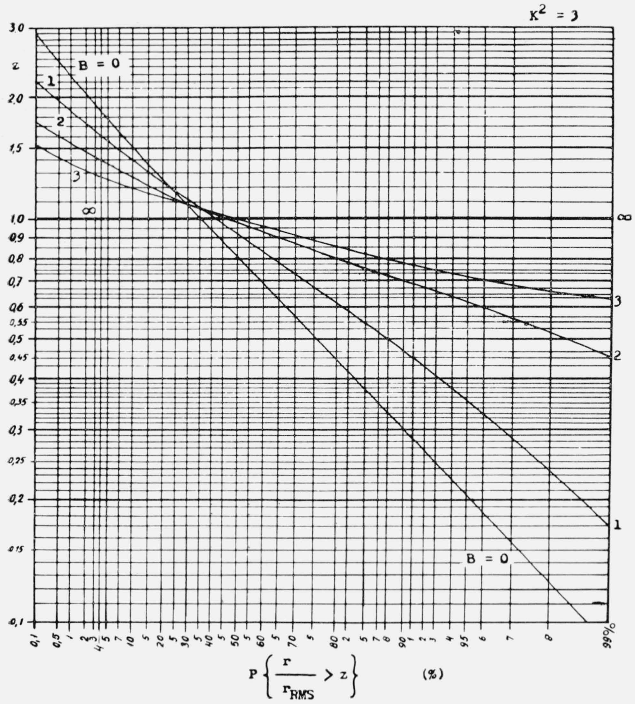

FIGURE 17. *Distribution curves of* $P\{r/r_{RMS}>z\}$ *for* $K^2=3$ *plotted on Rayleigh paper.*

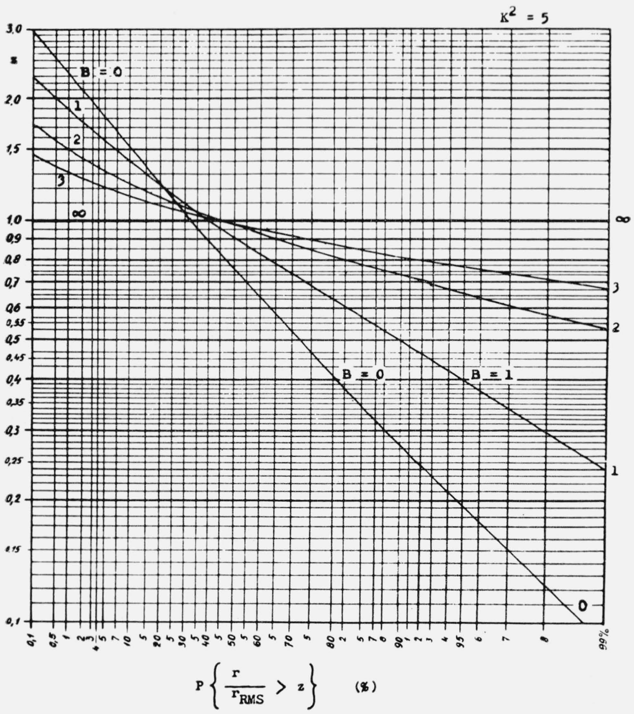

FIGURE 18. *Distribution curves of* $P\{r/r_{RMS}>z\}$ *for* $K^2=5$ *plotted on Rayleigh paper.*

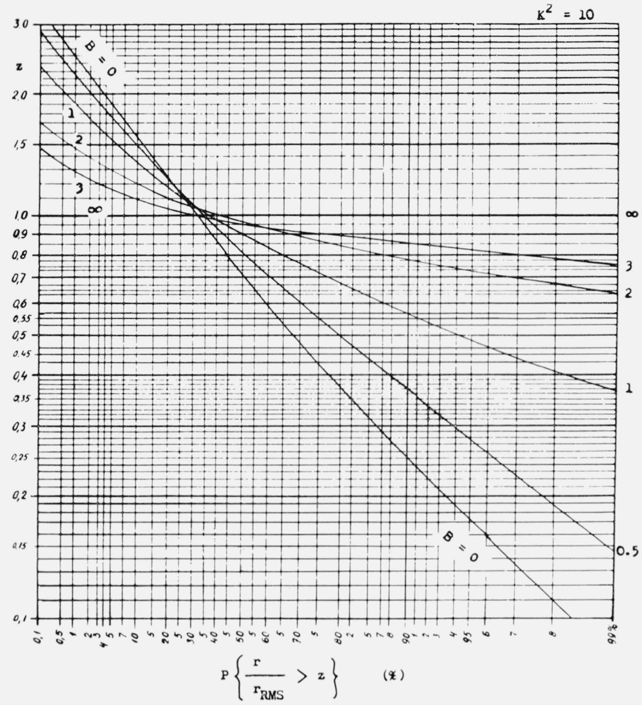

FIGURE 19. *Distribution curves of* $P\{r/r_{RMS}>z\}$ *for* $K^2=10$, *plotted on Rayleigh paper.*

where the meaning of θ', α' and β' is evident from figure 3. This integral may again be evaluated as an infinite series of Bessel functions (cf. appendix), but for practical purposes it is usually more simple to perform the numerical integration of (31) directly.

3. Special Cases

We now consider some special cases of the distribution (22). In most cases the expression (22) will formally remain unchanged, but the values of α, s_1, and s_2 will vary.

Let the elementary scattered waves be all of the same kind, so that the amplitude and phase distributions are the same for each wave. We first assume the amplitudes constant and equal to unity. Let the probability density of the phases be symmetrical about zero and equal to $w(\phi)$. Then from (4), (8), and (9) we have

$$\alpha = n\int w(\phi)\cos\phi\, d\phi \tag{32}$$

$$s_1 = n\int w(\phi)\cos^2\phi\, d\phi - \frac{\alpha^2}{n} \tag{33}$$

$$s_2 = n\int w(\phi)\sin^2\phi\, d\phi. \tag{34}$$

Table 1 gives the values of α, s_1, and s_2 as calcul d from (32) to (34), and also of B, K, and ρ in accordance with (19) to (21) for the normal, uniform, and Simpson-distributions; the symbol sinc $a=(\sin a)/a$ is used in the table.

TABLE 1.

$w(\phi)$	Normal, $\langle\phi\rangle=0$, standard deviation σ	Uniform from $-a$ to $+a$	Simpson-distributed from $-2a$ to $+2a$
α	$ne^{-1/2\sigma^2}$	$n \operatorname{sinc} a$	$n \operatorname{sinc}^2 a$
s_1	$\frac{n}{2}(1-e^{-\sigma^2})^2$	$\frac{n}{2}(1+\operatorname{sinc} 2a-2\operatorname{sinc}^2 a)$	$\frac{n}{2}(1+\operatorname{sinc}^2 2a+\operatorname{sinc}^4 a)$
s_2	$\frac{n}{2}(1-e^{-2\sigma^2})$	$\frac{n}{2}(1-\operatorname{sinc} 2a)$	$\frac{n}{2}(1-\operatorname{sinc}^2 2a)$
B^2	$n\frac{e^{-\sigma^2}}{1-e^{-\sigma^2}}$	$n\frac{\operatorname{sinc}^2 a}{1-\operatorname{sinc}^2 a}$	$n\frac{\operatorname{sinc}^4 a}{1-\operatorname{sinc}^4 a}$
K^2	$\coth\frac{\sigma^2}{2}$	$\frac{1-\operatorname{sinc} 2a}{1+\operatorname{sinc} 2a-2\operatorname{sinc}^2 a}$	$\frac{1-\operatorname{sinc}^2 2a}{1+\operatorname{sinc}^2 2a+\operatorname{sinc}^4 a}$
ρ^2	$\frac{r^2}{n(1-e^{-\sigma^2})}$	$\frac{r^2}{n(1-\operatorname{sinc}^2 a)}$	$\frac{r^2}{n(1-\operatorname{sinc}^4 a)}$

If the phases of the elementary waves are distributed uniformly over the interval $(-\pi,\pi)$, we obtain for $a=\pi$ from table 1 $\alpha=0$, $s_1=s_2=n/2$, or $B=0$, $K=1$, $\rho=r/\sqrt{n}$; substituting these values in (22) we obtain, as was to be expected, the normalized Rayleigh distribution

$$p(\rho)=2\rho e^{-\rho^2}. \tag{35}$$

The Rayleigh distribution is also obtained if ϕ is distributed uniformly over any interval of length $2k\pi(k=1,2,\ldots)$ or if the variance of ϕ is much greater than π^2, in which case the distribution of ϕ is (with some very unrealistic exceptions) arbitrary. We shall call a vector whose elementary components $A_j \exp(i\phi)$ have their phases uniformly distributed over the interval $(-\pi,\pi)$ a "Rayleigh vector."

The distribution of the amplitude of the sum of a constant vector and a Rayleigh vector was found by Rice [1944, 1945] and detailedly analyzed by Norton, Vogler, Mansfield, and Short [1955] and Zuhrt [1957]. Directing the constant vector **V** along the x-axis, we immediately have $\alpha=V$, $s_1=s_2=n/2$, hence by (19) to (21) $B=\alpha/\sqrt{n}$, $\phi=r/\sqrt{n}$, $K=1$ and on substituting these values in (22) we find the normalized Rice distribution

$$p(\rho)=2\rho e^{-(B^2+\rho^2)}I_0(2B\rho). \tag{36}$$

The distribution of a vector whose x and y components are distributed normally with mean values zero and unequal variances s_1 and s_2 was found by Hoyt [1947]. Here we have $\alpha=B=0$; on substituting in (22) we obtain the normalized Hoyt distribution

$$p(\rho)=\frac{K^2+1}{K}\,\rho\,\exp\left[-\left(\frac{1+K^2}{2K}\right)^2\rho^2\right]I_0\left(\frac{K^4-1}{4K^2}\,\rho^2\right). \tag{37}$$

It is evident from (12) and figure 2 that the sum (1) may always be represented as the sum of a constant vector $\vec{\alpha}$ and a Hoyt vector **H**.

We next consider the case in which the amplitudes A_j of the elementary waves are random and governed by the (same) probability density $w_A(A)$. If A_j and ϕ_j are independent then it follows from the general formulas (4), (8), and (9) that

$$\alpha=n\langle A\rangle\int w_\phi(\phi)\cos\phi\,d\phi, \tag{38}$$

$$s_1=n\langle A^2\rangle\int w_\phi(\phi)\cos^2\phi\,d\phi-\frac{\alpha^2}{n} \tag{39}$$

$$s_2=n\langle A^2\rangle\int w_\phi(\phi)\sin^2\phi\,d\phi. \tag{40}$$

If, for example, the phases ϕ are uniformly distributed over an interval of length 2π, we obtain $\alpha=0$, $s_1=s_2=\frac{1}{2}n\langle A^2\rangle$; substituting in (17) we find

$$p(r)=\frac{2r}{n\langle A^2\rangle}e^{-r^2/n\langle A^2\rangle}. \tag{41}$$

Comparing this result with (2) this will be recognized as a Rayleigh distribution, in which the number of components n is multiplied by the mean power [6] of each component.

If a Rayleigh vector consists of components with different (constant or random) amplitudes A_j, then from (4), (8), and (9)

$$\alpha=0;\quad s_1=s_2=\frac{1}{2}\sum_{j=1}^{n}\langle A_j^2\rangle. \tag{42}$$

Substituting this in (17) and (27), certain properties of a Rayleigh vector with components of unequal but constant amplitudes postulated by Norton, Vogler, Mansfield, and Short [1955] are immediately proved as correct (1. $P\{r>z\}=\exp\left(-z^2\Big/\sum_{j=1}^{n}A_j^2\right)_j$; 2. θ distributed uniformly between 0 and 2π; 3. r and θ independent).

If the random amplitudes and phases of the elementary waves are correlated, formulas (4), (8), and (9) may not be simplified; however, it is still true that the mean power of the random component equals the sum of the mean powers of the individual (identically distributed) elementary components, for in this case we have from (8) and (9)

$$\begin{aligned} s_1+s_2&=n\int A^2\left[\int w(A,\phi)d\phi\right]dA-\frac{\alpha^2}{n}\\ &=n\int A^2w_A(A)dA-\frac{\alpha^2}{n}=n\langle A^2\rangle-\frac{\alpha^2}{n} \end{aligned} \tag{43}$$

[6] More precisely the mean square of the amplitude. This differs from the mean power by a constant factor F, which in MKS units equals 120π for propagation in free space. The distinction is immaterial for our present purposes and will be disregarded; if the reader objects to this procedure, he may consider (1) to present n sinusoidal voltages interfering across a one-ohm resistor, in which case F equals unity.

so that by (18)

$$\langle r^2\rangle = n\langle A^2\rangle + \alpha^2\left(1-\frac{1}{n}\right). \tag{44}$$

Since n is by assumption large, the second term will practically equal α^2, i.e., the power of the constant component. Thus the power of the random component equals $n\langle A^2\rangle$, or the sum of the mean powers of the individual components.

It is instructive to observe the transition from a purely coherent field (mean power equal to n^2) to a purely incoherent field (mean power equal to n). This depends on the phase distribution $w(\phi)$; if the phases are constant, i.e., $D\{\phi\}=0$, then $\langle r^2\rangle = n^2$, whereas for phase distributions with large variances, i.e., for $D\{\phi\}>>\pi^2$ the mean power $\langle r^2\rangle = n$. Thus for example from (18) and table 1 we find for normally distributed phases

$$\langle r^2\rangle = n^2 e^{-\sigma^2} + n\,(1-e^{-\sigma^2}) \tag{45}$$

yielding $\langle r^2\rangle = n^2$ for $\sigma=0$, but $\langle r^2\rangle = n$ for $\sigma\to\infty$ $(\sigma>>\pi)$.

Similarly, for uniformly distributed phases we find

$$\langle r^2\rangle = n^2 \operatorname{sinc}^2 a + n(1-\operatorname{sinc}^2 a) \tag{46}$$

again yielding $\langle r^2\rangle = n^2$ for $a=0$, but $r^2=n$ for $a\to\infty$ $(a>>\pi)$, and also for $a=2k\pi$.

4. Conclusion

The statistical distribution of the amplitude and phase of a multiply scattered electromagnetic field is equal to the statistical distribution of the sum of two-dimensional vectors with random amplitudes and phases. When these phases are distributed symmetrically, the amplitude distribution of the resulting vector is given by (17) or in the normalized form (22), or by the curves of figures 5 to 19; the phase distribution is given by (27). In the general case, which includes asymmetrical phase distributions, the resulting distribution is given by the integral (31). Various distribution laws of the amplitudes and phases of the elementary vectors change the values of α, s_1, and s_2, but not the general form of the above formulas. The distributions derived by Rayleigh [1896], Rice [1944–5], Hoyt [1947], and Beckmann [1959] are special cases of the above distribution.

The distribution derived here is met, among other cases, in the propagation of radio waves in irregular terrain and in tropospheric scatter propagation, since in both cases scattering from rough surfaces is involved. From the above derivation it is seen that the amplitude of a field consisting of very many elementary scattered waves is not necessarily Rayleigh-distributed (as is often erroneously assumed), but that the Rayleigh distribution, even in its most general form, is met only if the phases of the individual scattered waves are distributed uniformly over an interval of length 2π or in some equivalent way indicated after equation (35). In practice this will not be the case if, for example, the scatterers are distributed in space in such a way that the variance of path lengths between source and point of observation is smaller than one wavelength. Such a case is shown in figure 4, where a rough or turbulent layer is assumed to be normally distributed about a mean level (M) with variance σ and the condition

$$2\sigma \sin\gamma < \lambda \tag{47}$$

holds. This condition is very often satisfied in practice, especially for the longer wavelengths λ; experimental measurements of tropospheric propagation beyond the horizon in the meter band [Beckmann, 1960] have in fact shown distributions as in figures 15 to 19 more often than a pure Rayleigh distribution.

5. Appendix

To evaluate the integral (31), one may use a result derived by Chytil [1961], which after elementary modifications reads

$$u(P,Q,R) = \int_0^{2\pi} \exp\,(-P^2\cos^2\theta + Q\cos\theta + R\sin\theta)d\theta$$
$$= 2\pi e^{-\frac{P}{2}} \sum_{m=0}^{\infty} (-1)^m \epsilon_m I_m\left(\frac{P}{2}\right) I_{2m}(\sqrt{Q^2+R^2})$$
$$\cos\left[2m\left(\arctan\frac{R}{Q}\right)\right] \tag{48}$$

reducing to the formula (15) derived by Beckmann and Schmelovsky [1958] for $R=0$. For $R=0$ and large $Q(Q>P>>1)$ one obtains by saddle-point integration [Beckmann and Schmelovsky, 1958]

$$u(P,Q,0) \approx \sqrt{\frac{2\pi}{Q}}\, e^{Q-P} \sum_{m=0}^{\infty} A_m\left(\frac{2P}{Q}\right)^m \tag{49}$$

where

$$A_m = \frac{1.3.5\ldots(2m-1)}{2.4.6\ldots 2m};\quad A_0=1. \tag{50}$$

Using (49) to evaluate (14), we find after normalizing by (19) to (21) for $Q>2\mathrm{P}>>1$.

$$p\,(\rho) \approx \frac{1+K^2}{K}\,\frac{\exp\left[-\frac{1+K^2}{2}(B-\rho)^2\right]}{\sqrt{2\pi B}}$$
$$\sqrt{\rho}\sum_{m=0}^{\infty} A_m\left(\frac{K^2-1}{BK^2\rho}\right)^m. \tag{51}$$

Now if

$$B^2 >> \frac{K^2-1}{K^2} \tag{52}$$

this expression will obviously be negligibly small for all values of ρ except in the neighborhood of $\rho=B$, where the exponential factor will dominate, the terms with $\rho^{(1/2)-m}$ being either negligible or practically constant in this short interval. But the exponential is that of the normal distribution with mean value

$\langle\rho\rangle = B$ and variance $D\{\rho\} = 1/(1+K^2)$; hence for large B (52), the distribution of ρ becomes normal. As may be seen from figures 10 to 14, in practice this is the case for $B \geq 3$, $K^2 < 5$.

6. References

Beckmann, P., The probability distribution of the sum of n unit vectors with arbitrary phase distributions, Acta Technica ČSAV **4,** 323–335 (1959).

Beckmann, P., A generalized Rayleigh distribution and its application to tropospheric propagation, Electromagnetic Wave Propagation, pp. 445–449, Academic Press, London (1960).

Beckmann, P., and Schmelovsky, K. H., Über ein bei Schwunderscheinungsuntersuchungen auftretendes Integral, Wiss. Z. HFE Ilmenau **4,** 167–171 (1958).

Bernstein, S. N., Generalization of the central limit theorem of probability theory for sums of dependent random variables (in Russian), Usp. fiz. nauk, No. 10, 55–114 (1944).

Chytil, B., Depolarization by randomly spaced scatterers, Práce ÚRE—ČSAV, Inst. Rep. No. 20 (1961).

Gnedenko, B. V., and A. N. Kolmogorov, Limit distributions for sums of random variables. Russian original: Gostekhizdat, Moscow-Leningrad 1949. English translation: Wesley Publ. Co., Cambridge, Mass. (1954).

Hoyt, R. S., Probability functions for the modulus and angle of the normal complex variate, Bell Syst. Tech. J. **26,** 318–359 (1947).

Hristow, W. K., Grundlagen der Wahrscheinlichkeitsrechnung, p. 125, Verlag Bauwesen, Berlin (1961).

Norton, K. A., L. E. Vogler, W. V. Mansfield, and P. J. Short, The probability distribution of the amplitude of a constant vector plus a Rayleigh-distributed vector, Proc. I.R.E. **43,** 1354–1361 (1955).

Rayleigh, Lord, The Theory of Sound, Sec. 42a, 3d ed., London (1896).

Rice, S. O., Mathematical analysis of random noise, Bell Syst. Tech. J. **23,** 282–332 (1944); **24,** 46–156 (1945).

Richter, H., Wahrscheinlichkeitstheorie. Springer Verlag, Berlin-Göttingen-Heidelberg (1956).

Zuhrt, H., Die Summenhäufigkeitskurven der exzentrischen Rayleigh-Verteilung und ihre Anwendung auf Ausbreitungsmessungen, AEÜ, **11,** 478–484 (1957).

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