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The Electric Field at the Ground Plane Near a Disk-Loaded Monopole

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In calculating ground losses for antennas with a ground-wire system, it is necessary to know the vertical electric-field strength and the tangential magnetic-field strength at the surface of the ground.

In this paper the vertical electric-field strength at the ground plane near the base of an electrically short vertical antenna with a top loading in the shape of a circular disk is calculated. Numerical computations are carried out to some extent.

1. Introduction

The losses in the ground around antennas with a ground-wire system may be considered to consist of two kinds of losses, the H-field losses and the E-field losses.

The H-field losses arise from the horizontally ground currents, which are numerically equal to the tangential magnetic-field strength. H-field losses have been treated, for example, by Abbott [1952] and by Knudsen [1959].

The existence of the E-field losses was pointed out by Wait [1958], who showed that transverse currents proportional to the vertical electric-field strength just above the ground gave rise to additional losses.

Usually in investigations of the losses around antennas with a ground-wire system the E-field losses are omitted because they are considered negligible as compared to the H-field losses. However, as was pointed out by Wait [1958], this may not be justified for monopole antennas with a top loading. It might therefore be of interest to examine and compare the two kinds of losses for a top-loaded antenna.

In order to make such an investigation of the magnitude of the E- and H-field losses it is necessary to know the tangential magnetic and the vertical electric-field strengths at the ground plane around the antenna in consideration.

These field strengths have been calculated for an electrically short vertical monopole with a top loading consisting of one, two, or four horizontal wires by Knudsen [1959] and by Knudsen and Larsen [1960].

For the disk-loaded monopole, which has the advantage over the wire-loaded antennas of rotational symmetry, only the magnetic-field strength has been calculated [Wait, 1959].

It is the purpose of this note to supplement the above-mentioned calculations with a calculation of the vertical electric-field strength at the ground plane around an electrically small disk-loaded monopole, so that the necessary material for an investigation of the losses around top-loaded antennas of the mentioned types is available.

2. Geometry and Current Distribution

The disk-loaded antenna is shown in figure 1. It consists of a vertical wire of length h and a top loading in the shape of a horizontal circular disk of radius a.

We introduce a cylindrical coordinate system (ρ,ϕ,z) with the z-axis pointing vertically downwards. The vertical wire extends from the point $z=h$ on the z-axis to the origin O which is also the center of the top loading. The plane $z=h$ coincides with the ground plane. Further, we introduce a spherical coordinate system (r,θ,ϕ) with the origin O and the axis $\theta=0$ pointing vertically downwards. Finally we introduce the polar coordinates (r_1, ϕ_1) to the variable point S on the disk.

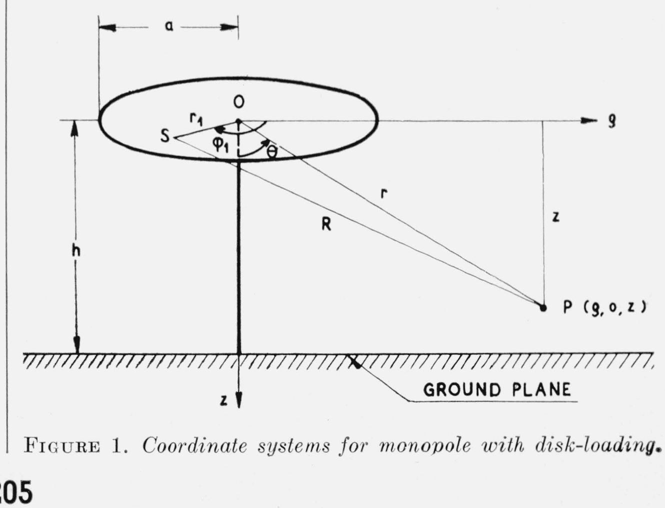

FIGURE 1. *Coordinate systems for monopole with disk-loading.*

We shall follow Wait [1959] in making the following assumptions regarding the current distribution.

1. The current distribution on the disk is quadratic, i.e., the surface current density

$$\overline{K}=\overline{K}(r_1)=K(r_1)\hat{r}_1,$$

where $\hat{r}_1$ denotes a unit vector coparallel to the vector $\overline{OS}$, is given by

$$K(r_1)=\frac{I_0}{2\pi r_1}\left[1-\left(\frac{r_1}{a}\right)^2\right],$$

where I_0 is the current at O.

2. The current flowing in the vertical wire is constant and equal to I_0.

3. Field of the Disk Loading

In this section we shall calculate the vertical component $E^d_{z\,(z=h)}$ of the electric field strength at the ground plane due to the current flowing on the disk loading.

So far, we do not take into consideration the influence of the ground plane. Using as time factor $e^{-i\omega t}$, we may express the Hertz vector at a point $P(\rho,0,z)$ due to the currents on the disk as

$$\overline{\Pi}=\frac{-1}{4\pi i\omega\epsilon_0}\int_{\text{disk}}\frac{e^{ikR}}{R}\overline{K}ds.$$

where $k=2\pi/\text{wavelength}$, $\epsilon_0=8.854\cdot 10^{-12}\ F/m$, and where R denotes the distance between S and P, i.e., between the current element $\overline{K}ds$ and the field point.

We now make the assumption that the greatest dimension of the antenna system can be considered small as compared to the wavelength.

For field points in the neighborhood of the disk we then have $e^{ikR}\approx 1$. Hence,

$$\overline{\Pi}=\frac{-1}{4\pi i\omega\epsilon_0}\int_{\text{disk}}\frac{1}{R}\overline{K}ds.$$

From the symmetry, it can be seen that the Hertz vector at P $\overline{\Pi}=\Pi_\rho\hat{\rho}+\Pi_\phi\hat{\phi}+\Pi_z\hat{z}$, where $\hat{\rho},\hat{\phi},\hat{z}$ denote unit vectors, has vanishing ϕ- and z-components. We then have

$$\Pi_\rho=\frac{-1}{4\pi i\omega\epsilon_0}\int_{\text{disk}}\frac{1}{R}\cos\phi_1\,K(r_1)ds$$

$$=\frac{-I_0}{4\pi i\omega\epsilon_0}\int_0^{2\pi}\int_0^a\frac{1}{R}\cos\phi_1\left[1-\left(\frac{r_1}{a}\right)^2\right]r_1\,d\phi_1\,dr_1,$$

where

$$R=\sqrt{r^2+r_1^2-2rr_1\cos\gamma},$$

with $\cos\gamma=\cos\theta\cos\theta_1+\sin\theta\sin\theta_1\cos\theta_1$, where $\theta_1=\frac{\pi}{2}$.

We now introduce the expansion

$$\frac{1}{R}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}(2-\delta^0_m)\frac{(n-m)!}{(n+m)!}P^m_n(\cos\theta)P^m_n(\cos\theta_1)$$

$$\cos m\phi_1\begin{cases}\dfrac{r_1^n}{r^{n+1}}, & r\geqq r_1\\[2ex] \dfrac{r^n}{r_1^{n+1}}, & r_1\geqq r\end{cases},$$

where $\delta^0_m=1$ if $m=0$ and $\delta^0_m=0$ if $m\neq 0$, and where P^m_n are associated Legendre polynomials.

We hereby obtain

$$\Pi_\rho=\frac{-I_0}{4\pi i\omega\epsilon_0}\int_0^{2\pi}\int_0^a\left[1-\left(\frac{r_1}{a}\right)^2\right]\sum_{n=0}^{\infty}\sum_{m=0}^{n}(2-\delta^0_m)\frac{(n-m)!}{(n+m)!}$$

$$P^m_n(\cos\theta)P^m_n(0)\cos m\phi_1\cos\phi_1\,r_1\begin{Bmatrix}\dfrac{r_1^n}{r^{n+1}} & r\geqq r_1\\[2ex] \dfrac{r^n}{r_1^{n+1}} & r_1\geqq r\end{Bmatrix}d\phi_1 dr_1$$

$$=\frac{-I_0}{4\pi i\omega\epsilon_0}\int_0^a\left[1-\left(\frac{r_1}{a}\right)^2\right]\sum_{n=0}^{\infty}\frac{P^1_n(0)}{n(n+1)}$$

$$P^1_n(\cos\theta)\begin{Bmatrix}\dfrac{r_1^n}{r^{n+1}} & r\geqq r_1\\[2ex] \dfrac{r^n}{r_1^{n+1}} & r_1\geqq r\end{Bmatrix}dr_1 \qquad (1)$$

The electric-field intensity $\overline{E}'^d$ may now be obtained by

$$\overline{E}'^d=\nabla\nabla\cdot\overline{\Pi}+k^2\overline{\Pi}.$$

Since the Hertz vector has vanishing ϕ- and z-components, the vertical component $E_z'^d$ of $\overline{E}'^d$ is given by

$$E_z'^d=\frac{1}{\rho}\frac{\partial\Pi_\rho}{\partial z}+\frac{\partial^2\Pi_\rho}{\partial\rho\partial z}$$

or, expressed in spherical coordinates,

$$E_z'^d=\frac{\cot\theta\cos^2\theta}{r}\frac{\partial\Pi_\rho}{\partial r}-\frac{2\cos^2\theta}{r^2}\frac{\partial\Pi_\rho}{\partial\theta}+\frac{\sin 2\theta}{2}\frac{\partial^2\Pi_\rho}{\partial r^2}$$

$$+\frac{\cos 2\theta}{r}\frac{\partial^2\Pi_\rho}{\partial r\partial\theta}-\frac{\sin 2\theta}{2r^2}\frac{\partial^2\Pi_\rho}{\partial\theta^2}. \qquad (2)$$

In the following calculations we shall treat separately the two cases:

1. The field point P is situated outside a sphere of radius a and center O.

2. The field point P is situated inside a sphere of radius a and center O.

1. $r \geqq a$

In this case we obtain from (1)

$$\Pi_\rho = \frac{-I_0}{4\pi i\omega\epsilon_0}\int_0^a \left[1-\left(\frac{r_1}{a}\right)^2\right]\sum_{n=0}^{\infty}\frac{P_n^1(0)}{n(n+1)}\,P^n(\cos\theta)\,\frac{r_1^n}{r^{n+1}}\,dr_1$$

$$=\frac{-I_0}{2\pi i\omega\epsilon_0}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{n+1}\frac{P_n^1(0)}{n(n+1)^2(n+3)}\,P_n^1(\cos\theta). \qquad (3)$$

Inserting (3) in (2) gives

$$E_z'^d = \frac{-I_0}{2\pi i\omega\epsilon_0 \sin\theta}\sum_{n=0}^{\infty}\frac{a^{n+1}}{r^{n+3}}\frac{P_n^1(0)}{(n+1)(n+3)}\left[P_{n+1}^1(\cos\theta)-\cos\theta\,P_{n+2}^1(\cos\theta)\right].$$

Using the formula

$$P_{n+1}^1(\cos\theta)-\cos\theta P_{n+2}^1(\cos\theta)=-\sin\theta(n+2)\,P_{n+2}(\cos\theta),$$

where P_{n+2} is the Legendre polynomial of order $n+2$, the above expression may be rewritten

$$E_z'^d=\frac{I_0}{2\pi i\omega\epsilon_0}\sum_{n=0}^{\infty}\frac{a^{n+1}}{r^{n+3}}\frac{n+2}{(n+1)(n+3)}P_n^1(0)P_{n+2}(\cos\theta). \qquad (4)$$

We are interested only in the value of $E_z'^d$ at $z=h$. Therefore, we insert in (4) $r=\frac{h}{\cos\theta}$. The influence of the earth, which is assumed to be perfectly conducting, may now be taken into consideration by adding to $(E_z'^d)_{z=h}$ the field $(E_z''^d)_{z=h}$ of the mirror image. We hereby obtain

$$(E_z^d)_{z=h}=(E_z'^d)_{z=h}+(E_z''^d)_{z=h}$$

$$=2(E_z'^d)_{z=h}$$

$$=\frac{I_0}{\pi i\omega\epsilon_0}\frac{1}{h^2}\sum_{n=0}^{\infty}\left(\frac{a}{h}\right)^{n+1}\frac{n+2}{(n+1)(n+3)}\cos^{n+3}\theta\,P_n^1(0)P_{n+2}(\cos\theta)2\ \ (r\geqq a). \qquad (5)$$

2. $r \leqq a$

In this case we obtain from (1)

$$\Pi_\rho=\frac{-I_0}{4\pi i\omega\epsilon_0}\sum_{n=0}^{\infty}\frac{P_n^1(0)}{n(n+1)}P_n^1(\cos\theta)\left\{\int_0^r\frac{r_1^n}{r^{n+1}}\left[1-\left(\frac{r_1}{a}\right)^2\right]dr_1+\int_r^a\frac{r^n}{r_1^{n+1}}\left[1-\left(\frac{r_1}{a}\right)^2\right]dr_1\right\} \qquad (6)$$

$$=\frac{-I_0}{4\pi i\omega\epsilon_0}\sum_{\substack{n=1\\ n\neq 2}}^{\infty}\frac{P_n^1(0)}{n(n+1)}P_n^1(\cos\theta)\left\{\left(\frac{r}{a}\right)^n\frac{2}{n(n-2)}-\left(\frac{r}{a}\right)^2\frac{2n+1}{(n+3)(n-2)}+\frac{2n+1}{n(n+1)}\right\}. \qquad (7)$$

The term $n=2$ in (6) vanishes as the two integrals are finite, and $P_2^1(0)=0$. The term $n=0$ in (6) vanishes as the two integrals are finite, and

$$\frac{P_n^1(0)P_n^1(\cos\theta)}{n}\rightarrow 0 \text{ as } n\rightarrow 0.^*$$

Inserting (7) in (2) gives

$$E_z'^d=\frac{-I_0}{4\pi i\omega\epsilon_0}\cot\theta\sum_{\substack{n=1\\ n\neq 2}}^{\infty}P_n^1(0)\left\{P_n^1(\cos\theta)\left[\frac{2n+1}{n^2(n+1)(n-2)}\left(\cos^2\theta\left[2(1-2n)\frac{r^{n-2}}{a^n}+n(n+1)\frac{1}{a^2}+(1-n)(n-2)\frac{1}{r^2}\right]\right.\right.\right.$$

$$\left.\left.+\left[2(n-1)\frac{r^{n-2}}{a^n}-\frac{n}{a^2}-\frac{n-2}{r^2}\right]\right)\right]+P_{n+1}^1(\cos\theta)\left[\frac{1}{n+1}\left(\cos\theta(2n+1)\left[\frac{4}{n(n-2)}\frac{r^{n-2}}{a^n}-\frac{1}{a^2}\frac{2}{n-2}+\frac{1}{r^2}\frac{2}{n}\right]\right.\right.$$

$$\left.\left.\left.-\frac{1}{\cos\theta}\frac{2}{n-2}\left[\frac{r^{n-2}}{a^n}-\frac{1}{a^2}\frac{2n+1}{n+3}\right]\right)\right]-P_{n+2}^1(\cos\theta)\left[\frac{r^{n-2}}{a^n}\frac{2}{n(n-2)}-\frac{1}{a^2}\frac{2n+1}{(n+3)(n-2)}+\frac{1}{r^2}\frac{2n+1}{n(n+1)}\right]\right\}. \qquad (8)$$

We are interested only in the value of $E_z'^d$ at $z=h$. Therefore, we insert in (8) $r=\frac{h}{\cos\theta}$. The influence of the earth, which is assumed to be perfectly conducting, may now be taken into consideration by adding to $(E_z'^d)_{z=h}$ the field $(E_z''^d)_{z=h}$ of the

*See appendix.

mirror image. We hereby obtain

$$(E_z^d)_{z=h}=(E_z'^d)_{z=h}+(E_z''^d)_{z=h}$$

$$=2(E_z'^d)_{z=h}$$

$$=\frac{-I_0}{2\pi i\omega\epsilon_0}\frac{\cot\theta}{h^2}\sum_{\substack{n=1\\n\neq 2}}^{\infty}\frac{P_n^1(0)}{n(n+1)(n-2)}\left\{P_n^1(\cos\theta)\,\frac{2n+1}{n}\left[\left(\frac{h}{a}\right)^n\cos^{-n+2}\theta\,2[(1-2n)\cos^2\theta+n-1]\right.\right.$$

$$\left.+\left(\frac{h}{a}\right)^2 n\,[(n+1)\cos^2\theta-1]-\cos^2\theta(n-2)[(n-1)\cos^2\theta+1]\right]+\frac{1}{n+3}\left[P_{n+1}^1(\cos\theta)\,\frac{1}{\cos\theta}\left[\left(\frac{h}{a}\right)^n\right.\right.$$

$$\left.\left.\cos^{-n+2}\theta\,(n+3)[2(2n+1)\cos^2\theta-n]+\left(\frac{h}{a}\right)^2 2n(2n+1)[1-(n+3)\cos^2\theta]+\cos^4\theta\,2(2n+1)(n+3)(n-2)\right]\right.$$

$$\left.\left.-P_{n+2}^1(\cos\theta)\left[\left(\frac{h}{a}\right)^n\cos^{-n+2}\theta\,2(n+1)(n+3)-\left(\frac{h}{a}\right)^2 n(n+1)(2n+1)+\cos^2\theta(n+3)(n-2)(2n+1)\right]\right]\right\}. \tag{9}$$

$(r\leqq a)$.

For $r=a$ the above expression checks with (5).

4. Field of the Vertical Wire

The vertical component $E^w_{z(z=h)}$ of the electric-field strength at the ground plane in the neighborhood of an electrically short vertical wire of length h carrying a constant current I_0 is given [Knudsen, 1960] by

$$E^w_{z(z=h)}=\frac{-I_0}{2\pi i\omega\epsilon_0}\frac{1}{h^2}\left[1+\left(\frac{\rho}{h}\right)^2\right]^{-\frac{3}{2}}. \tag{10}$$

5. Numerical Results

In order to make the field-strength expressions dimensionless we define

$$e_{z(z=h)}=\frac{\omega\epsilon_0}{I_0}\,a^2E_{z(z=h)}. \tag{11}$$

Numerical computations of the normalized vertical components $e_{z(z=h)}$ of the electric field at the ground plane due to the currents on the disk and on the vertical wire have been carried out in the following cases:

1. The length of the vertical wire is twice the radius of the disk, i.e., $\frac{h}{a}=2$.

2. The length of the vertical wire is equal to the radius of the disk, i.e., $\frac{h}{a}=1$.

In both cases $r\geqq a$. Hence, in the computation of $e^d_{z(z=h)}$, formula (5) in connection with (11) apply.

The two above-mentioned values of $\frac{h}{a}$ are the same as those used by Wait [1959] in his computations. In his paper Wait also gives curves for $\frac{h}{a}=0.6$ and $\frac{h}{a}=0.4$. Therefore, it might have been desirable in the present investigation to include computations in these cases, too. However, it turns out that for these values of $\frac{h}{a}$, $r\leqq a$ for some points on the ground, so that in the computation of $e^d_{z(z=h)}$ one should apply formula (9), which, unfortunately, for the parameter values in consideration, converges rather slowly. So the cases $\frac{h}{a}=0.6$ and $\frac{h}{a}$ 0.4 have been omitted. In figure 2 curves showing the imaginary unit times the vertical component of the normalized electric-field strength as a function of the relative horizontal distance $\frac{\rho}{a}$ from the base of the vertical antenna are plotted.

For $\frac{h}{a}=2$ there is a shift in sign for $e^d_{z(z=h)}$ at $\frac{\rho}{a}\approx 1.7$, and for $\frac{h}{a}=1$ there is a shift in sign for $e^d_{z(z=h)}$ at $\frac{\rho}{a}\approx 1.0$. Similar shifts in sign for the vertical component of the electric-field strength at the ground plane due to the current in a top loading consisting of one, two, or four horizontal wires was found by Knudsen and Larsen [1960].

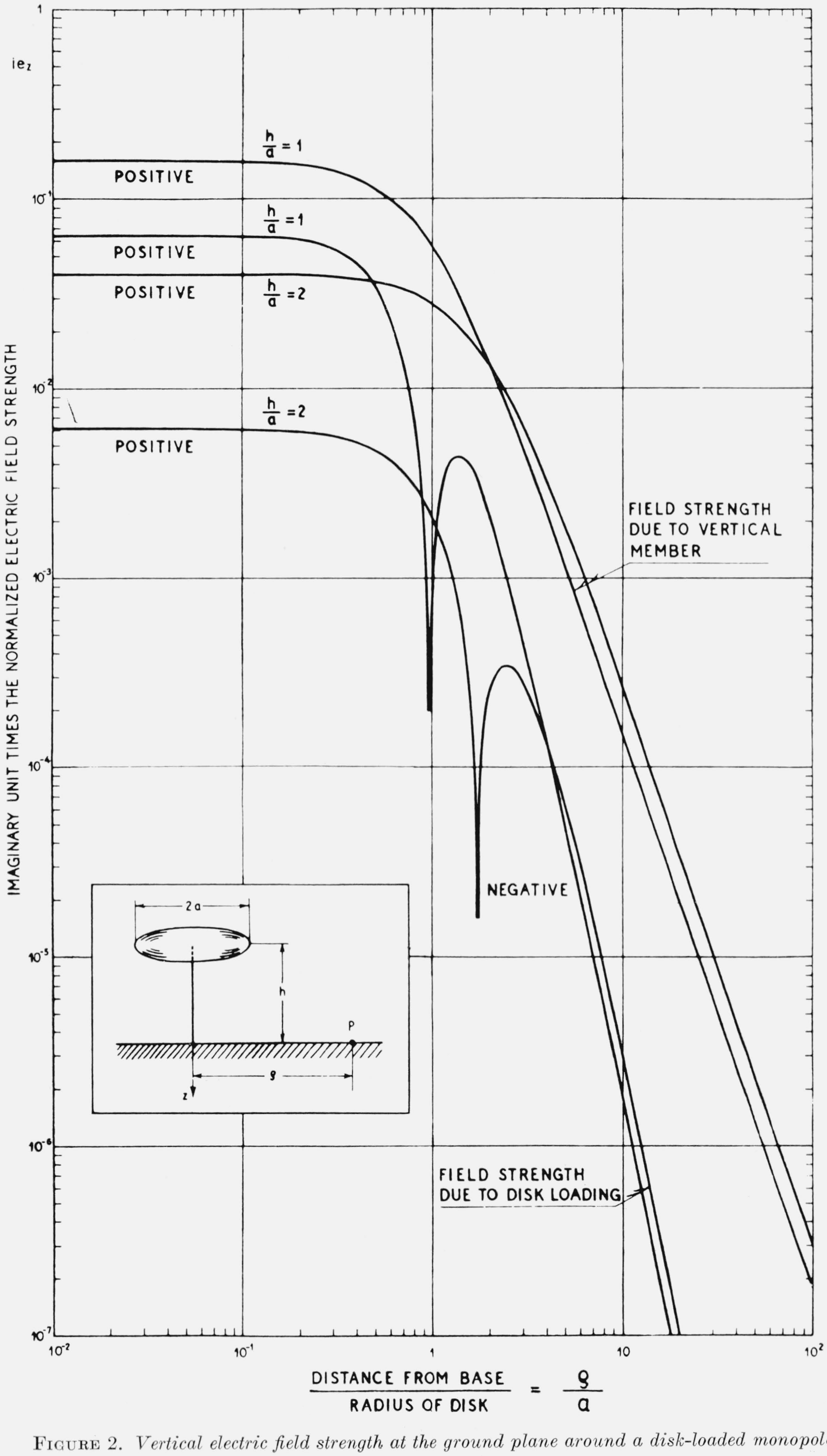

FIGURE 2. *Vertical electric field strength at the ground plane around a disk-loaded monopole.*

6. Conclusion

Formulas have been derived for the vertical electric-field strength at the ground plane near an electrically small disk-loaded monopole, and numerical computations have been carried out in the cases of the height of the vertical member being equal to the radius of the disk and the height of the vertical member being twice the radius of the disk, respectively.

In both cases the field strength due to the top loading is of the same order of magnitude, but smaller than the field strength due to the vertical member for points of the ground plane situated just below the disk, while for points at a greater distance from the antenna base the field from the top loading is negligible as compared to the field from the vertical member. As could be expected, the ratio between the numerical values of the field strength due to the disk and the field strength due to the vertical member is largest for the larger disk in the whole area around the antennas.

7. Appendix

We shall examine the expression $\frac{1}{n}P_n^1(0)P_n^1(\cos\theta)$ as $n\to 0$. Due to L'Hospital's rule we have

$$\lim_{n\to 0}\frac{1}{n}P_n^1(0)P_n^1(\cos\theta)$$

$$=\left[\frac{\frac{\partial}{\partial n}[P_n^1(0)P_n^1(\cos\theta)]}{\frac{\partial}{\partial n}(n)}\right]_{n=0}$$

$$=\left[P_n^1(0)\frac{\partial}{\partial n}P_n^1(\cos\theta)+P_n^1(\cos\theta)\frac{\partial}{\partial n}P_n^1(0)\right]_{n=0}=0,$$

since

$P_0^1(\cos\theta)=0$, and $\left[\frac{\partial}{\partial n}P_n^1(\cos\theta)\right]_{n=0}$ is limited [Tsu, 1960].

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