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The E-Field and H-Field Losses Around Antennas With a Radial Ground Wire System

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This paper describes an investigation of the ratio between the E-field and the H-field losses per unit area, and the absolute value of these losses around a half-wavelength monopole, a quarter-wavelength monopole, and around electrically short monopoles with as well as without top-loading all of them with a radial ground wire system.

1. Introduction

In calculations of the losses in the ground around antennas with a ground wire system usually only the H-field losses are taken into account. These losses have been calculated, e.g., by Monteath [1952] and by Abbott [1952]. The value of the H-field losses per unit area is given by

$$p_H = q_H |H_t|^2, \tag{1}$$

where q_H is a quantity only dependent on the ground and the ground wire system, and where H_t is the total tangential magnetic field strength at the surface of the ground.

However, Wait [1958] has shown that additional losses may occur due to currents flowing normally to the ground surface. The value of these losses per unit area may be written

$$p_E = q_E |E_z|^2, \tag{2}$$

where q_E is a quantity only dependent on the ground and the ground wire system, and where E_z is the total vertical electric field strength at the surface of the ground. These losses are termed the E-field losses. They will often turn out to be negligible as compared to the H-field losses [see, e.g., Knudsen and Larsen, 1960].

It is the purpose of this note to describe an investigation of the ratio of the E-field losses to the H-field losses and the absolute value of these losses around various antennas, namely a half-wavelength monopole, a quarter-wavelength monopole, and an electrically short monopole with and without a disk-loading. These antennas are chosen because of their rotational symmetry, which will lead to a simple radial ground wire system.

2. Formulas for $\frac{p_E}{p_H}$

With the values of the E-field losses and the H-field losses mentioned in the introduction we find the following expression for the ratio between the two sorts of losses

$$M = \frac{p_E}{p_H} = \frac{q_E}{q_H} \zeta_0^2 \left| \frac{E_z}{\zeta_0 H_t} \right|^2 = M_0 M'. \tag{3}$$

Here M_0 is the dimensionless ratio

$$M_0 = \frac{q_E}{q_H}\,\zeta_0^2, \tag{4}$$

where ζ_0 is the characteristic impedance of free space

$$\zeta_0 = \frac{1}{\eta_0} = \sqrt{\frac{\mu_0}{\epsilon_0}}; \tag{5}$$

μ_0 and ϵ_0 are the permeability and the dielectric constant of free space, respectively.

The quantity q_H is given by

$$q_H = \mathrm{Re}\left(\frac{1}{Y_j}\right)\left|\frac{1}{1+\dfrac{Y_g}{Y_j}}\right|^2, \tag{6}$$

where Y_j is the equivalent surface admittance of the ground and Y_g is the equivalent surface admittance of the ground wire system, which is supposed to be part of a plane parallel-wire grid:

$$Y_j = \eta_0 \sqrt{\epsilon_r\left(1+i\,\frac{\sigma_2}{\omega\epsilon_2}\right)} \tag{7}$$

$$Y_g = i\,\frac{\lambda\eta_0}{d \ln \dfrac{d}{2\pi a}}; \tag{8}$$

σ_2 and ϵ_2 are the conductivity and the dielectric constant of the ground ($\epsilon_r = \epsilon_2/\epsilon_0$), ω is the angular frequency, λ the wavelength, d the distance between adjacent wires of the grid, and a is the radius of the grid wires. The time factor is $e^{-i\omega t}$.

The quantity q_E is given by [Wait, 1959]

$$q_E = \frac{\bar{h}_{\mathrm{re}}\sigma_2 - \bar{h}_{\mathrm{im}}\omega\epsilon_2}{\sigma_2^2 + (\omega\epsilon_2)^2}\,(\epsilon_0\omega)^2, \tag{9}$$

where $\bar{h}_{\mathrm{re}}$ and $\bar{h}_{\mathrm{im}}$ are real and imaginary parts of an equivalent burying depth of the wire system, the real burying depth being h

$$\bar{h}_{\mathrm{re}} = h + \frac{d}{2\pi}\left\{\ln\frac{d}{2\pi a} - \frac{[\sigma_2^2 + (\omega\epsilon_0)^2(\epsilon_r^2 - 1)]\ln\left(1 - e^{-\frac{4\pi h}{d}}\right)}{\sigma_2^2 + (\omega\epsilon_0)^2(\epsilon_r + 1)^2}\right\} \tag{10}$$

$$\bar{h}_{\mathrm{im}} = \frac{d}{2\pi}\ln\left(1 - e^{-\frac{4\pi h}{d}}\right)\frac{2\sigma_2(\omega\epsilon_0)}{\sigma_2^2 + (\omega\epsilon_0)^2(\epsilon_r + 1)^2}. \tag{11}$$

The ratio M_0 is only dependent on the ground and the ground wire system; it is investigated numerically by Larsen [1960].

The dimensionless ratio M' is given by

$$M' = \left|\frac{E_z}{\zeta_0 H_t}\right|^2, \tag{12}$$

and it is only dependent on the antenna and the coordinates of the field point (the field strengths E_z and H_t are calculated under the assumption that the ground is perfectly conducting).

In what follows M' will be calculated for the types of antennas mentioned in the introduction.

3. Calculation of M'

3.1. Monopole With Sinusoidal Current Distribution

At first we will consider a simple vertical monopole of height l with a sinusoidal current distribution as shown in figure 1. The maximum current is called I_0, and the time factor is $e^{-i\omega t}$. A cylindrical coordinate system (r,ϕ,z) is introduced with the unit vectors denoted by $\hat{r},\hat{\phi},\hat{z}$. The tangential magnetic field strength around such an antenna is given, e.g., by Abbott [1952]; with the notation of figure 1 this field strength at a point of the ground surface at the distance r from the antenna base is given by

$$\overline{H}=\hat{\phi}i\frac{I_0}{2\pi r}\{e^{ikr}\cos kl-e^{iks}\}, \tag{13}$$

where

$$s=\sqrt{r^2+l^2}, \tag{14}$$

and where $k=\frac{2\pi}{\lambda}$ is the propagation constant.

The vertical electrical field strength at the ground plane is derived in appendix 1; with the same notation as above we find

$$\overline{E}=\hat{z}i\frac{\zeta_0 I_0}{2\pi}\left\{\frac{e^{iks}}{s}-\cos kl\frac{e^{ikr}}{r}\right\}. \tag{15}$$

We now find for the ratio M'

$$M'=\left|\frac{E_z}{\zeta_0 H_t}\right|^2=\frac{\left(\frac{r}{s}\cos ks-\cos kl\cos kr\right)^2+\left(\frac{r}{s}\sin ks-\cos kl\sin kr\right)^2}{(\sin ks-\cos kl\sin kr)^2+(\cos kl\cos kr-\cos ks)^2}. \tag{16}$$

For large values of r, r and s will be nearly equal, and the ratio M' will approach unity, i.e., the value for a plane wave.

For small values of r, s approaches l, and M' approaches $\cot^2 kl$.

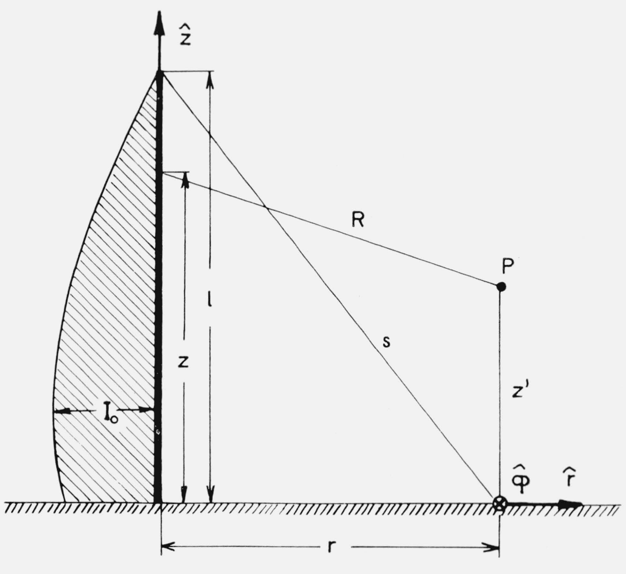

FIGURE 1. *Vertical monopole with sinusoidal current distribution.*

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In figure 2 is shown the value of M' as a function of the relative distance $\frac{r}{l}$ from the antenna base for vertical monopoles of different lengths with sinusoidal current distribution. It is seen as was seen from the formula that for all the antennas the ratio M' approaches unity, i.e., the value for a plane wave, when the distance from the antenna base increases, and for small values of this distance the ratio M' will assume large values for very small antennas and for antennas about half a wavelength long, but very small values for antennas about a quarter wavelength long.

3.2. Small Monopole Without Top-Loading

The current distribution on a monopole the height of which is small as compared to the wavelength may be described by the nearly linear end of a sine-curve, i.e., it may be approximated by a linear current distribution as shown in figure 3. The electric and magnetic field strengths arising from this current distribution are calculated in appendix 2. The near zone field strengths at the ground plane are with the notation of figure 3 given by

$$\overline{E}=-\hat{z}\frac{iI_0\zeta_0}{2\pi k}\frac{s-r}{lsr}, \tag{17}$$

$$\overline{H}=\hat{\phi}\frac{I_0}{2\pi}\frac{s-r}{rl}. \tag{18}$$

From these expressions we find

$$M'=\left|\frac{E_z}{\zeta_0 H_t}\right|^2=\frac{1}{(kl)^2}\left(\frac{l}{s}\right)^2. \tag{19}$$

Of course, the more exact expression valid for a sinusoidal current distribution may be used also in the case of a small monopole. However, for large values of $\frac{r}{l}$ it may be necessary to use some series expressions in the formula for M' as the numerical result will otherwise be too uncertain. This approximation is more thoroughly discussed in appendix 3.

In figure 4 is shown M' as a function of the relative distance from the antenna base for a small monopole with sinusoidal current distribution and linear current distribution ($kl=0.1$ and 0.04) and with a constant current distribution ($kl=0.1$) (this case will be discussed in the next section). It is seen that for small values of the distance from the antenna base the sinusoidal and the linear current distribution gives nearly the same value for M', while for greater values of $\frac{r}{l}$ there is a pronounced difference. This could be expected, as the formulas for the linear and constant current distributions are valid only for $kr<<1$ (in fact the simple formulas are not valid in the entire range of $\frac{r}{l}$ represented in fig. 4).

In the following discussion of the losses the simple formulas for the field strengths of the small monopoles have been used, as they give the same results as the more rigorous ones in the area near the antenna, where the losses are significant.

3.3. Small Monopole With Top-Loading

The main purpose of a top-loading on a vertical monopole antenna is to increase the current on the vertical member. Very often the top-loading itself is neglected in examinations of top-loaded antennas. However, as has been pointed out by Wait [1958] in some cases the top-loading may have a rather great influence on the E-field losses.

A top-loading, which will lead to a simple radial ground wire system is a plane disk-loading. The tangential magnetic field strength at the ground plane of a disk-loaded monopole was calculated by Wait [1959] and the vertical electric field strength at the ground plane was calculated by Hansen and Larsen [1960].

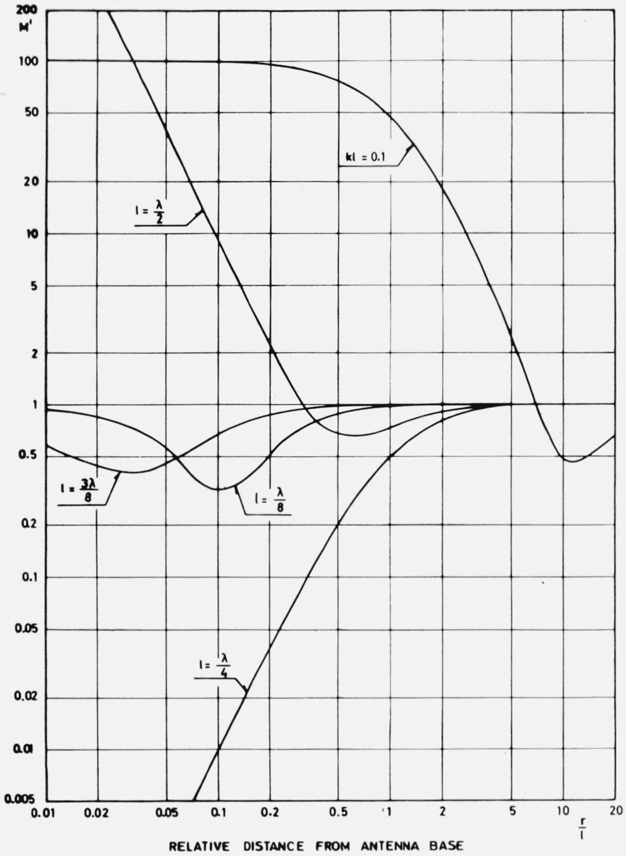


FIGURE 2. *Ratio* $M' = \left|\frac{E_z}{\zeta_0 H_t}\right|^2$ *between field strengths at ground plane around vertical monopoles with sinusoidal current distribution.*

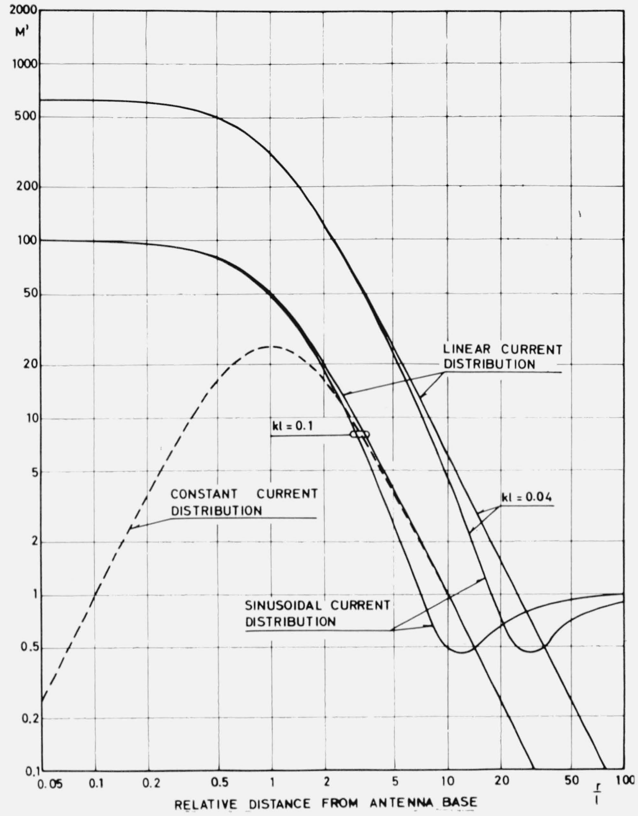


FIGURE 4. *Ratio* $M' = \left|\frac{E_z}{\zeta_0 H_t}\right|^2$ *between field strengths at ground plane around an electrically small monopole.*

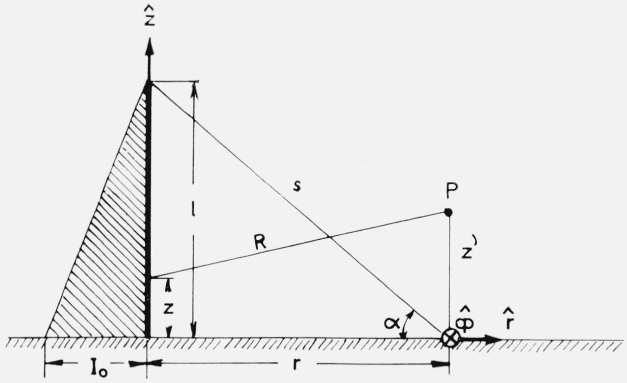


FIGURE 3. *Electrically small vertical monopole.*

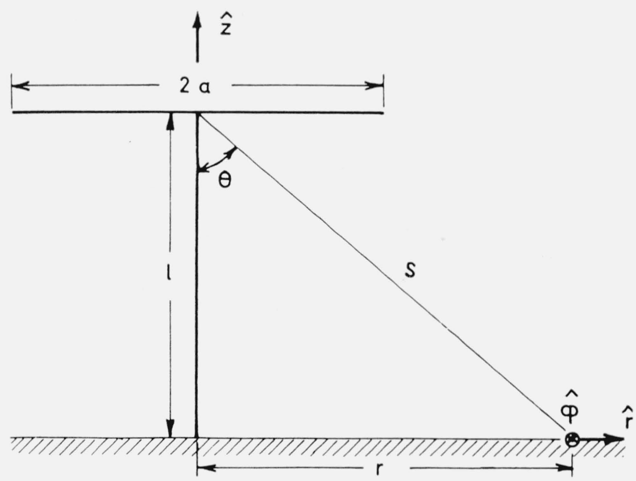


FIGURE 5. *Disk-loaded monopole.*

With the notation of figure 5 and with the current distribution on the vertical member being a constant current I_0, and the current on the disk varying so that the current on an element in the distance r_1 from the center of the disk of the width $r_1 d\phi$ is given by

$$I=\frac{I_0}{2\pi}d\phi\left(1-\left(\frac{r_1}{a}\right)^2\right), \tag{20}$$

we have the following values of the near zone electric and magnetic field strengths at the ground plane:

a. Vertical member

$$\overline{E}^v=-\hat{z}\frac{iI_0\zeta_0}{2\pi k}\frac{l}{s^3}, \tag{21}$$

$$\overline{H}^v=\hat{\phi}\frac{I_0}{2\pi}\frac{l}{rs}, \tag{22}$$

which gives the following value of M', when the only effect of the top-loading is assumed to be to make the vertical current constant

$$M'=\left|\frac{E_z}{\zeta_0 H_t}\right|^2=\frac{1}{(kl)^2}\left(\frac{r}{s}\frac{l}{s}\right)^2. \tag{23}$$

b. Disk-loading in the case $a\leqq l$

$$\overline{E}^d=\hat{z}\frac{iI_0\zeta_0}{k\pi}\frac{1}{l^2}\sum_{n=1}^{\infty}\left(\frac{a}{l}\right)^{n+1}\frac{(n+2)}{(n+1)(n+3)}\cos^{n+3}\theta\, P_n(0)P^1_{n+2}(\cos\theta)=\hat{z}\frac{iI_0\zeta_0}{k\pi}\frac{1}{l^2}\sum_{n=1}^{\infty}A_n, \tag{24}$$

$$\overline{H}^d=\hat{\phi}\frac{I_0}{\pi l}\sum_{n=1}^{\infty}\frac{P^1_n(0)}{(n+1)^2(n+3)}\left(\frac{a}{l}\right)^{n+1}\cos^{n+2}\theta_0 P^1_{n+1}(\cos\theta_0)=\hat{\phi}\frac{I_0}{\pi l}\sum_{n=1}^{\infty}B_n, \tag{25}$$

where $P_n(\cos\theta)$ and $P_n^m(\cos\theta)$ are Legendre polynomials and associated Legendre polynomials, respectively.

We now get for the top-loaded monopole when the field strengths of the top-loading as well as of the vertical member are considered:

$$M'=\left|\frac{E_z}{\zeta_0 H_t}\right|^2=\frac{1}{(kl)^2}\left|\frac{\left(\frac{l}{s}\right)^3-2\sum_{n=1}^{\infty}A_n}{\frac{l}{r}\frac{l}{s}+2\sum_{n=1}^{\infty}B_n}\right|^2. \tag{26}$$

These formulas are only valid in the range $a\leqq l$. For the case $a>l$ other more involved formulas should be used. However, the expression for E_z in this case is very slowly convergent; for this reason only the case $a\leqq l$ has been considered here.

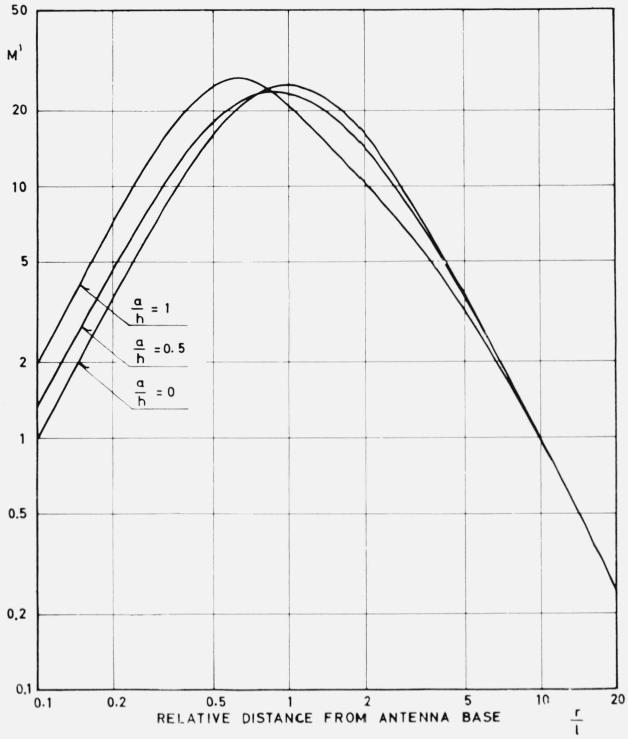
In figure 6 is shown M' as a function of $\frac{r}{l}$ for a disk-loaded monopole a) when no account is taken of the field from the top-loading, b) for $a/l=0.5$ and c) for $a/l=1$. It is seen that the top-loading will make M' increase near the antenna and decrease in some distance from the antenna, the influence being greatest near the antenna, where M' is increased to two times the value without top-loading when the radius of the disk is equal to the height of the vertical member. In great distance from the antenna base there is no difference between the curves of the top-loaded and the not top-loaded antenna.

4. Calculation of the Ratio Between E-Field and H-Field Losses

With the numerical results for M' of the foregoing section and the numerical result for M_0 of the report by Larsen [1960] the ratio $M=M'M_0$ between the E-field losses and the H-field losses around the antennas investigated in this note may now be computed.

M will be calculated as a function of the relative distance from the antenna base for two sorts of ground, both with the relative dielectric constant $\epsilon_r=10$, but with the conductivity

FIGURE 6. *Ratio* $M' = \left|\frac{E_z}{\zeta_0 H_t}\right|^2$ *between field strengths at ground plane around a monopole with disk-loading,* $kl = 0.1$.



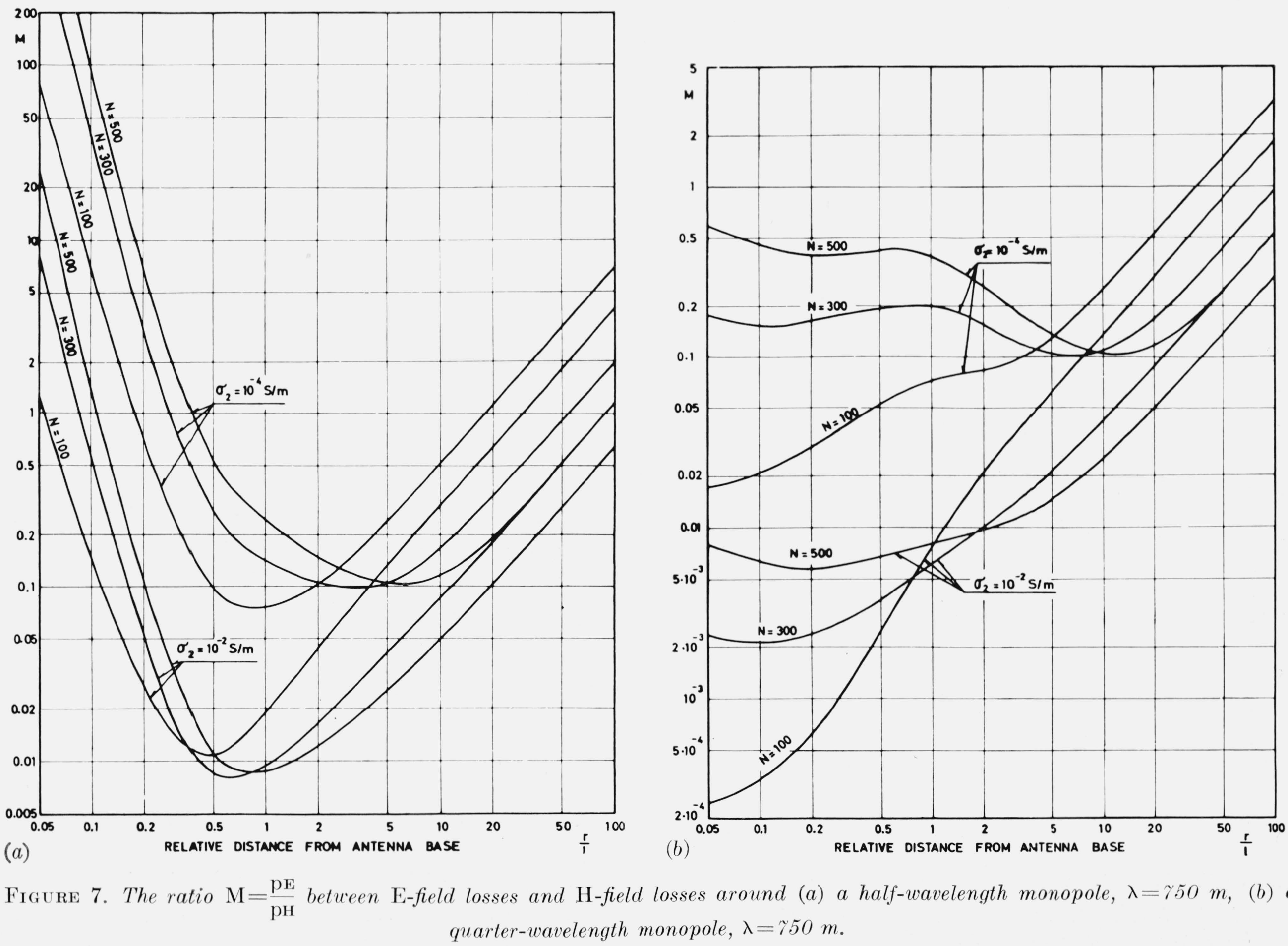


FIGURE 7. *The ratio* $M = \frac{p_E}{p_H}$ *between E-field losses and H-field losses around (a) a half-wavelength monopole,* $\lambda = 750$ *m, (b) a quarter-wavelength monopole,* $\lambda = 750$ *m.*

$\sigma_2=10^{-2} S/m$ and $\sigma_2=10^{-4} S/m$, respectively. It is assumed that the ground wire system consists of N radial wires of the same length, N being 100, 300, and 500, respectively. The wavelength is chosen to $\lambda=1{,}500$ m for the short monopoles and to $\lambda=750$ m for the quarter- and half-wavelength monopoles.

With this choice of antenna height and number of wires in the ground wire system the distance d between adjacent wires will in some cases become very small, so small that the expression for M_0 is not valid any longer, as it is evaluated under the assumption that $d \gg a$, a being the radius of the wires. In cases like this the curves are shown dotted.

The burying depth of the ground wire system is assumed to be 0.5 m.

In figure 7 is shown the ratio $M=p_E/p_H$ as a function of r/l in the six parameter cases mentioned above, namely, $\sigma_2=10^{-2} S/m$, $N=100$, 300, or 500 wires and $\sigma_2=10^{-4} S/m$, $N=100$, 300, or 500 wires for the following antenna types: Figure 7a, Half-wavelength monopole; figure 7b, Quarter-wavelength monopole; figure 7c, Small monopole with linear current distribution; figure 7d, Small monopole with constant current distribution, with and without disk-loading. The last mentioned case is only shown for $N=100$ and 500.

It is seen that for the half-wavelength monopole the E-field losses will be of the same order of magnitude as the H-field losses for the poorly conducting ground over the whole area around the antenna, while for the better conducting ground the E-field losses will be almost negligible

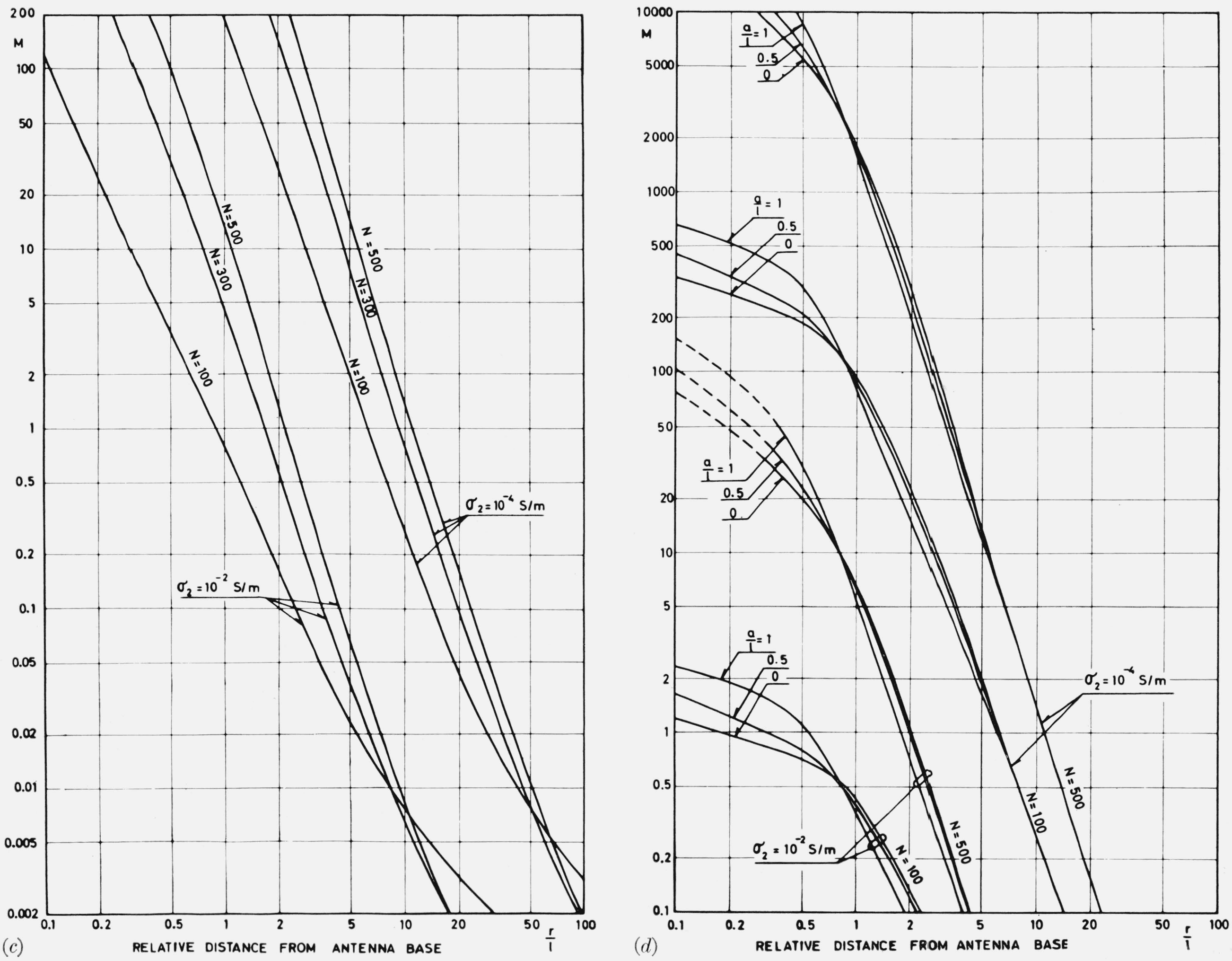


FIGURE 7. *The ratio* $M=\frac{p_E}{p_H}$ *between E-field losses and H-field losses around (c) an electrically small monopole with linear current distribution, $\lambda=1{,}500$ m, $kl=0.1$, (d) an electrically small monopole with constant current distribution with and without disk-loading, $\lambda=1{,}500$ m, $kl=0.1$.*]

as compared to the H-field losses except in a small area very close to the antenna and in so large a distance from the antenna base, that the field strengths are very small.

For the quarter-wavelength monopole the E-field losses will not exceed the H-field losses except in so large a distance from the antenna base, that the losses are very small. It is seen that for this type of antenna it would cause no considerable deviation not to take into account the E-field losses.

For all the small monopoles the E-field losses far exceed the H-field losses in the part of the area around the antennas where the losses are significant. The ratio M between the E-field losses and the H-field losses assumes the largest value in the case of the monopole with the linear current distribution. The disk increases the ratio M as compared to the value for the monopole with constant current distribution and no disk-loading, but not as much that the values for the monopole with the linear current distribution are obtained.

5. Calculation of the H-Field Losses

In order to find the absolute values of the losses around the antennas investigated, the absolute value of the H-field losses per unit area, p_H, will be calculated. The absolute value of the E-field losses per unit area, p_E, may then easily be found from the ratio M as

$$p_E = p_H \cdot M, \tag{27}$$

and the total losses per unit area p_{tot} may be found from

$$p_{\text{tot}} = p_H(1+M). \tag{28}$$

The H-field losses per unit area are given by (1)

$$p_H = q_H |H_t|^2.$$

In figure 8 is shown q_H as a function of the distance d in the parameter cases $\lambda = 750$ m, $\sigma_2 = 10^{-2}$, 10^{-4}, and 10^{-5} S/m and $\lambda = 1{,}500$ m, $\sigma_2 = 10^{-2}$, 10^{-4}, and 10^{-5} S/m. These curves have been used for calculating the absolute values of the H-field losses around the antennas examined in this paper.

In calculating $|H_t|^2$ for the various antennas we have put the reference current I_0 equal to 1 amp.

In figure 9 is shown the absolute value of p_H as a function of the relative distance from the antenna base in the following six parameter cases, $\sigma_2 = 10^{-2}$ S/m, $N = 100$, 300, and 500 wires and $\sigma_2 = 10^{-4}$ S/m, $N = 100$, 300, and 500 wires for the following antenna types: Figure 9a, Half-wavelength monopole; figure 9b, Quarter-wavelength monopole; figure 9c, Small monopole with linear current distribution; figure 9d, Small monopole with constant current distribution and small monopole with disk-loading; the curves of the disk-loaded monopole are shown only for $N = 100$ and 500.

It is seen that for all the antennas the H-field losses decrease when the distance from the antenna base approaches zero and when it approaches infinity, the first mentioned effect being due to the small distance between adjacent wires in the ground wire system near the antenna and the last mentioned effect to the decrease in field strengths far from the antenna.

6. Comparison of Absolute Values of Losses for Various Antennas

A sketch of the antennas investigated in this note is given in figure 10. The current distributions are shown with all the reference currents made equal. Further are given the current-areas A_c, defined as

$$A_c = \int_0^l I dx, \tag{29}$$

these values relative to the value $A_{c\pi}$ of the half-wavelength monopole, and the value of the

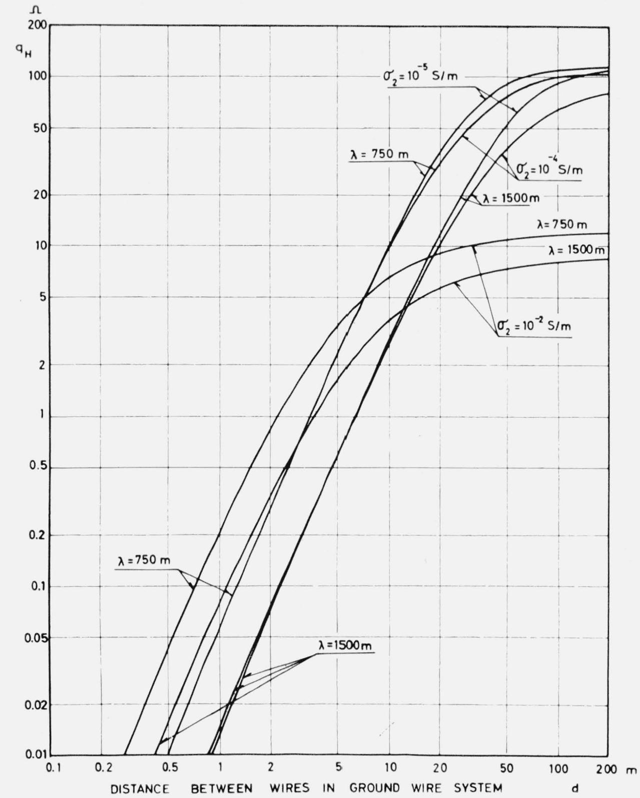


FIGURE 8. *Factor* q_H *used in calculations of* H-*field losses.*

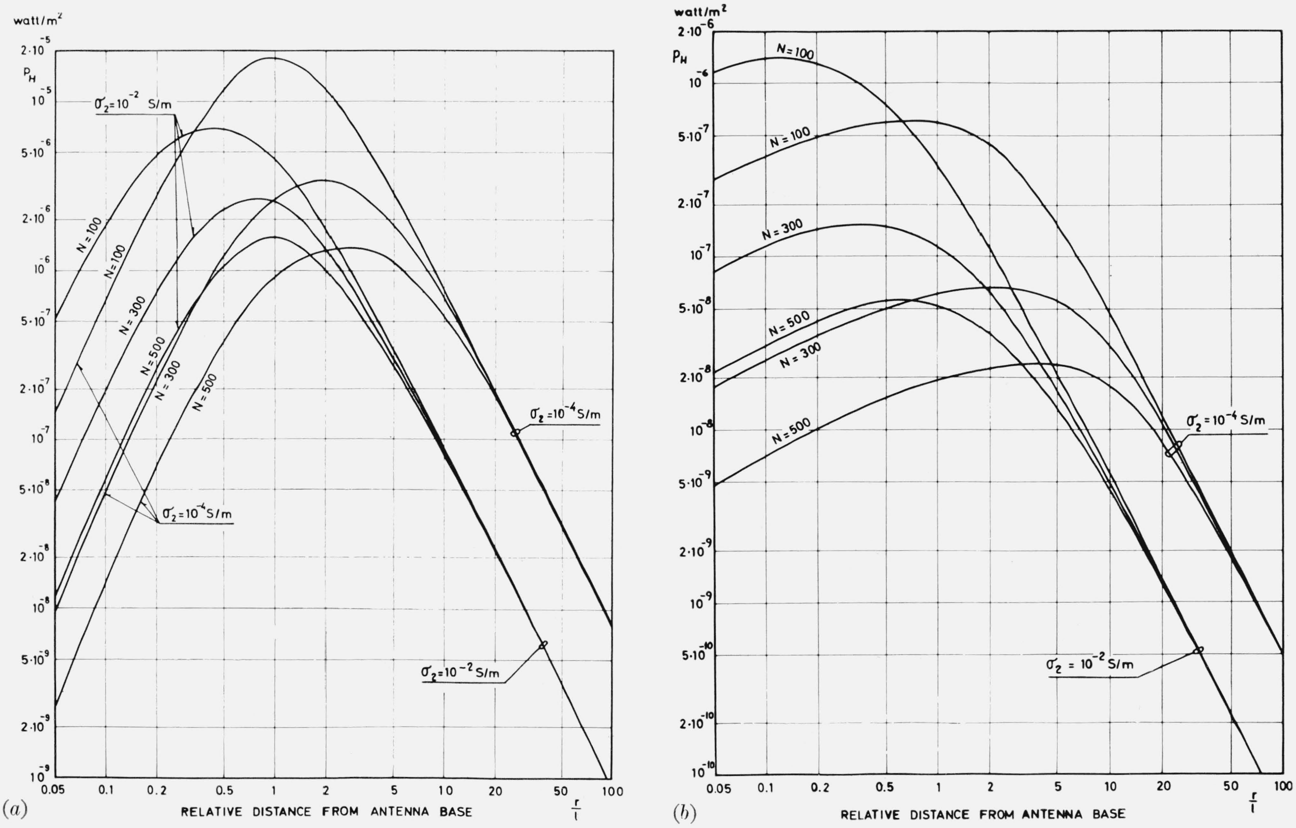


FIGURE 9. H-*field losses around* (*a*) *a half-wavelength monopole,* $\lambda = 750$ *m,* $I_0 = 1$ *amp,* (*b*) *a quarter-wavelength monopole,* $\lambda = 750$ *m,* $I_0 = 1$ *amp.*

reference currents, which will make all the current-areas equal to that of the half-wavelength monopole with $I_0=1$ amp. (The current-areas of the two top-loaded antennas are made equal to that of the small antenna with a constant current distribution.)

The last mentioned values of the currents I_0 have been used in making the survey diagrams in figure 11 and figure 12 of the absolute values of the losses around the antennas. Figure 11 is valid for the quarter-wavelength and the half-wavelength monopole and figure 12 applies to the small monopoles. Linear scales have been used both for the losses and for the distances from the antenna base, but different scales have been used in the two diagrams.

It is seen that the E-field losses form a small part of the total losses around the quarter-wavelength and the half-wavelength monopoles, whereas for the small antennas the E-field losses are important.

Considering only the antennas with sinusoidal current distribution we see that the total losses around the half-wavelength monopole are considerably larger than around the quarter-wavelength monopole, and that the E-field losses do contribute appreciably to the total losses for the half-wavelength antenna, while the E-field losses for the quarter-wavelength antenna are vanishingly small.

Considering the small antennas only we see that the largest total losses occur for the monopole with the linear current distribution. The top-loading will increase the E-field losses, but not to any great extent. The linear current distribution will cause larger E-field losses than a disk-loading, the radius of which is equal to or less than the height of the vertical member.

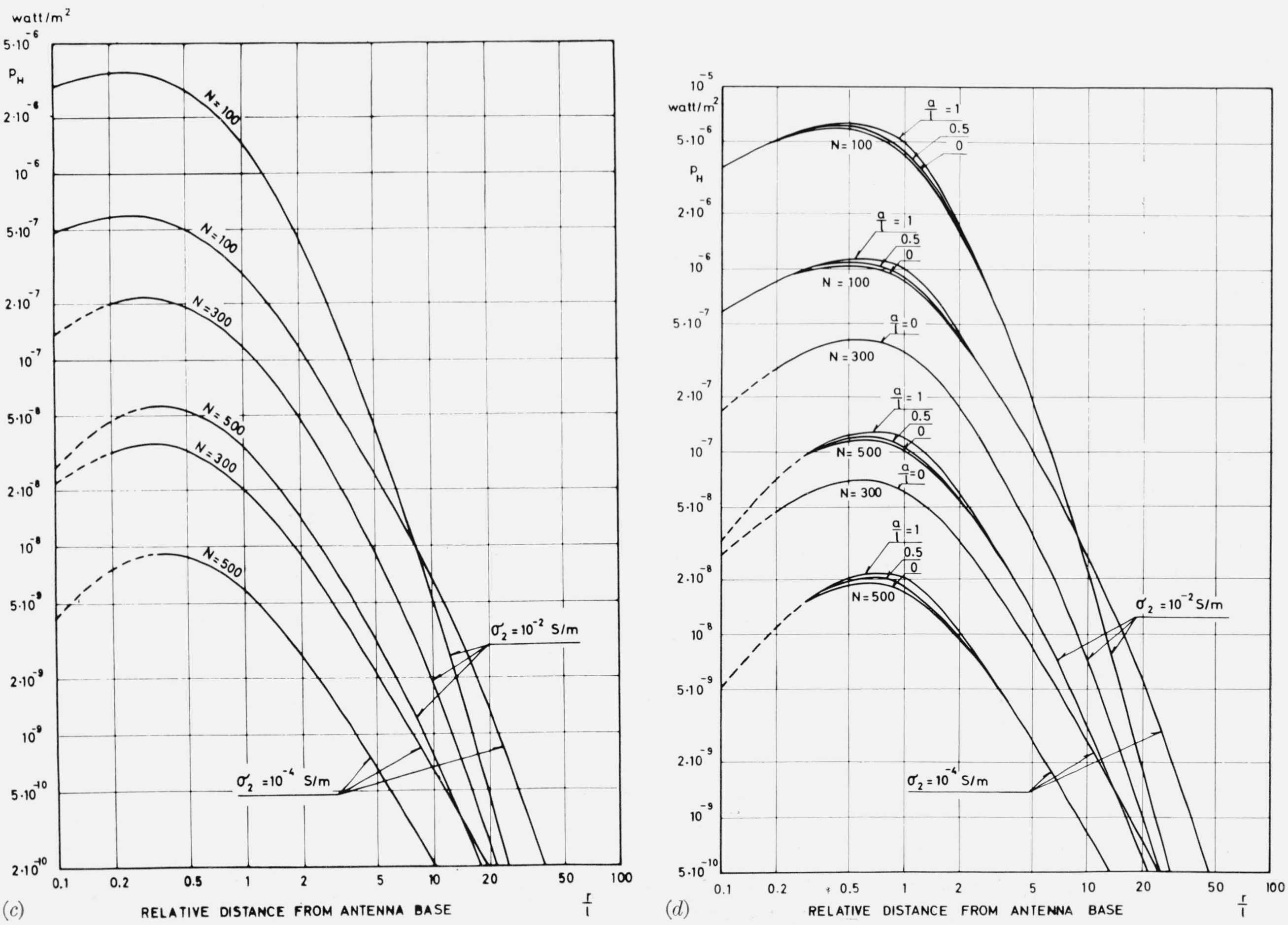


FIGURE 9. H-*field losses around* (*c*) *an electrically short monopole with linear distribution,* $\lambda=1{,}500$ *m,* $kl=0.1$, $I_0=1$ *amp.*

H-*field losses around* (*d*) *an electrically short monopole with and without top-loading,* $\lambda=1{,}500$ *m,* $kl=0.1$, $I_0=1$ *amp.*

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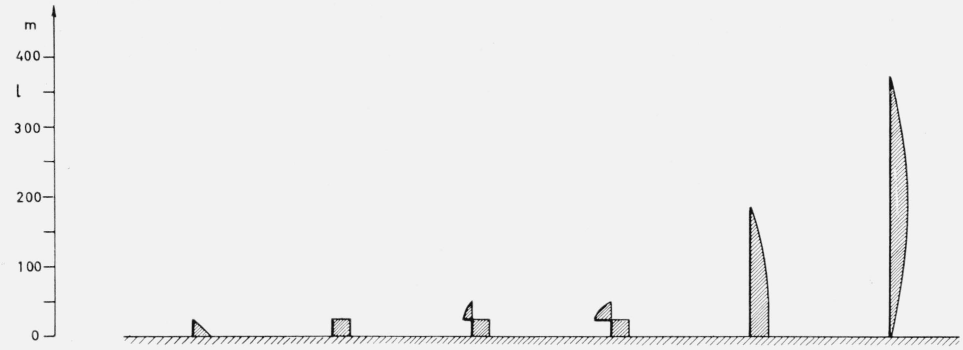

kl	0.1	0.1	0.1	0.1	$\frac{\pi}{2}$	π
λ m	1500	1500	1500	1500	750	750
A_c Amp·m	11.937 I_o	23.873 I_o	23.873 I_o	23.873 I_o	119.37 I_o	238.73 I_o
$A_c/A_{c\pi}$	0.05	0.1	0.1	0.1	0.5	1
I_o Amp	20	10	10	10	2	1

FIGURE 10. *Survey of antennas investigated.*

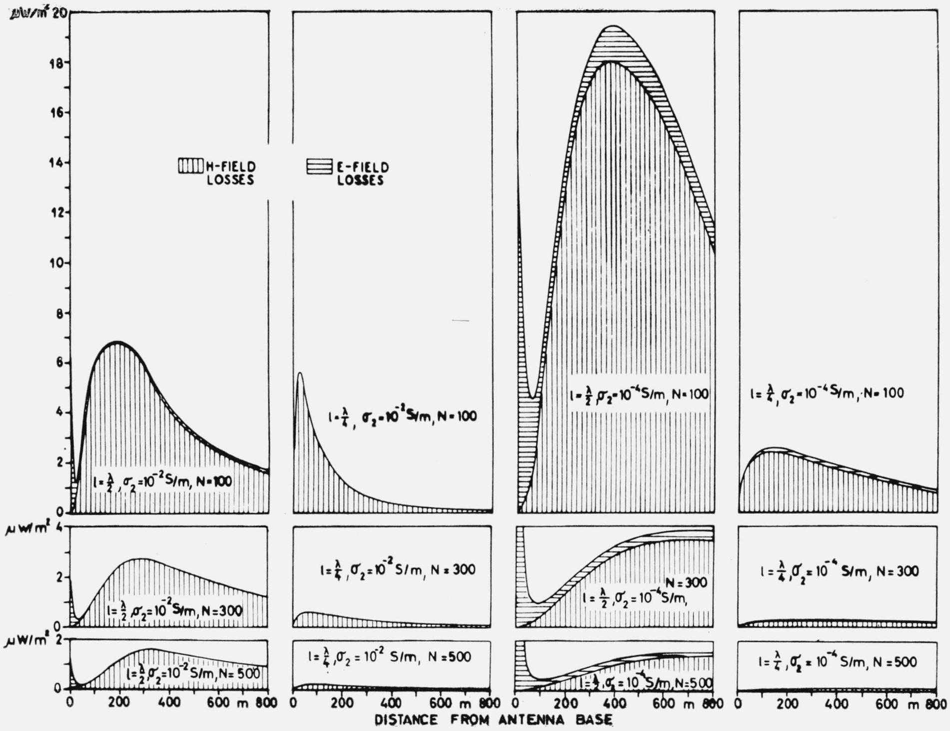

FIGURE 11. *Comparison of losses around a half-wavelength and a quarter-wavelength monopole.*

7. Conclusion

The ratio between the E-field losses and the H-field losses and the absolute value of these losses around various antennas with radial ground wire systems have been investigated, and a number of curves showing the variation of these losses with the distance from the antenna base in different parameter cases have been plotted. The antennas considered are vertical mono-

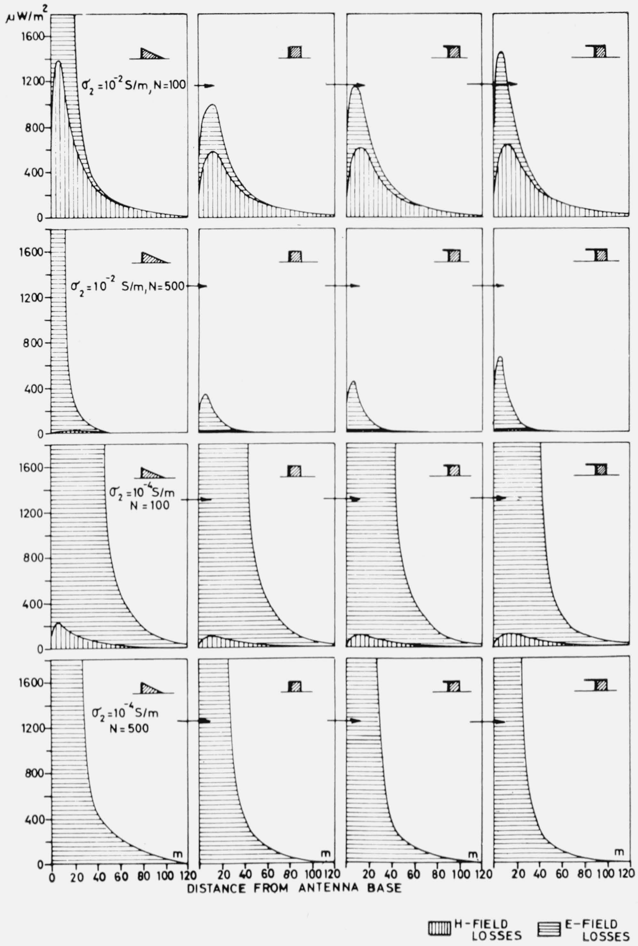

FIGURE 12. *Comparison of losses around small monopoles with and without top-loading.*

poles with sinusoidal current distribution and vertical monopoles the length of which is small as compared to the wavelength, with as well as without a top-loading. It is found that for the monopoles with sinusoidal current distribution the E-field losses are almost negligible as compared to the H-field losses, whereas for the small monopoles the E-field losses are large as compared to the H-field losses. The disk-loading on the small monopole is found to increase the E-field losses, but not to any great extent, the losses being mainly determined by the current distribution on the vertical member.

This investigation was carried out by means of a support from the Air Force Cambridge Research Center, United States Air Force.

8. Appendix 1. Electric Field Strength Around a Monopole With Sinusoidal Current Distribution

With the notation of figure 1 the current on the antenna is given by

$$I=I_0 \sin k(l-z).$$

The vector potential A at the point P at the distance r from the antenna and at the height z' above the ground plane will be given by

$$\bar{A}=\hat{z}\,\frac{\mu I_0}{4\pi}\int_0^l \frac{\sin k(l-z)e^{ikR}}{R}\,dz,$$

R being the distance from the varying point on the antenna to the field point P

$$R=\sqrt{r^2+(z-z')^2}.$$

The electric field strength $\bar{E}$ at P will be given by

$$\bar{E}=i\omega\left(\frac{1}{k^2}\nabla\nabla\cdot\bar{A}+\bar{A}\right).$$

As the vector potential has only a z-component the electric field strength will be in the z-direction, and it will be given by

$$E_z=i\omega\left[\frac{1}{k^2}\frac{\partial^2 A_z}{\partial z'^2}+A_z\right]$$

$$=\frac{i\omega\mu I_0}{4\pi}\left[\frac{1}{k^2}\int_0^l \sin k(l-z)\frac{\partial^2}{\partial z'^2}\left(\frac{e^{ikR}}{R}\right)dz+\int_0^l\frac{\sin k(l-z)e^{ikR}}{R}dz\right].$$

From the above equation for R it is seen that

$$\frac{\partial}{\partial z'}=-\frac{\partial}{\partial z},$$

and so

$$\frac{\partial^2}{\partial z'^2}=\frac{\partial^2}{\partial z^2}.$$

By partial integration we then find the following expression for E_z

$$E_z=\frac{i\omega\mu I_0}{4\pi}\left\{-\frac{1}{k^2}\sin kl\left[\frac{\partial}{\partial z}\left(\frac{e^{ikR}}{R}\right)\right]_{z=0}+\left[\frac{1}{k}\cos k(l-z)\frac{e^{ikR}}{R}\right]_0^l\right\}.$$

We have

$$\left[\frac{\partial}{\partial z}\left(\frac{e^{ikR}}{R}\right)\right]_{z=0}=\left(ik-\frac{1}{R}\right)\frac{e^{ikR}}{R}\frac{(-z')}{R}.$$

However, at the ground plane $z'=0$ this expression is equal to zero. Taking into account the image by introducing a factor 2 we therefore find the following expression for the vertical component E_z of the electric field strength at the ground plane

$$E_z=i\frac{\zeta_0 I_0}{2\pi}\left[\frac{e^{iks}}{s}-\cos kl\,\frac{e^{ikr}}{r}\right],$$

where we have put $R=s$ for $z=l$ and $R=r$ for $z=0$.

9. Appendix 2. Electric and Magnetic Field Strength Around an Electrically Small Monopole With Linear Current Distribution

With the notation of figure 3 the current distribution is given by

$$I=I_0\left(1-\frac{z}{l}\right).$$

The vector potential $\bar{A}$ at the point P at the distance r from the antenna and at the height z' above the ground plane is given by

$$\bar{A}=\hat{z}A_z=\hat{z}\frac{\mu I_0}{4\pi}\int_0^l\frac{e^{ikR}}{R}\left(1-\frac{z}{l}\right)dz,$$

where

$$R=\sqrt{r^2+(z-z')^2}.$$

As the antenna is assumed to be electrically short, i.e., $l<<\lambda$, and as an expression for the near zone field is wanted, we may put

$$e^{ikR}\simeq 1.$$

We then find

$$A_z=\frac{\mu I_0}{4\pi}\left[\left(1-\frac{z'}{l}\right)\log\frac{l-z'+\sqrt{r^2+(l-z')^2}}{-z'+\sqrt{r^2+z'^2}}-\frac{1}{l}\left(\sqrt{r^2+(l-z')^2}-\sqrt{r^2+z'^2}\right)\right].$$

We may now find the vertical electric and the tangential magnetic field strengths from the following expressions

$$E_z=\frac{i\omega}{k^2}\frac{\partial^2 A_z}{\partial z'^2}+i\omega A_z$$

$$H_r=\frac{1}{\mu}\frac{1}{r}\frac{\partial A_z}{\partial\phi}=0$$

$$H_\phi=-\frac{1}{\mu}\frac{\partial A_z}{\partial r}.$$

Performing the differentiations we obtain

$$\frac{\partial A_z}{\partial z'}=\frac{\mu I_0}{4\pi}\left[\frac{1}{\sqrt{r^2+z'^2}}-\frac{1}{l}\log\frac{l-z'+\sqrt{r^2+(l-z')^2}}{-z'+\sqrt{r^2+z'^2}}\right].$$

$$\frac{\partial^2 A_z}{\partial z'^2}=\frac{\mu I_0}{4\pi}\left[\frac{-z'}{\sqrt{(r^2+z'^2)^3}}-\frac{1}{l}\left(\frac{-1}{\sqrt{r^2+(l-z')^2}}+\frac{1}{\sqrt{r^2+z'^2}}\right)\right].$$

$$\frac{\partial A_z}{\partial r}=\frac{\mu I_0 r}{4\pi}\left[\left(1-\frac{z'}{l}\right)\left(\frac{1}{(l-z'+\sqrt{r^2+(l-z')^2})\sqrt{r^2+(l-z')^2}}-\frac{1}{(-z'+\sqrt{r^2+z'^2})\sqrt{r^2+z'^2}}\right)\right.$$
$$\left.-\frac{1}{l}\left(\frac{1}{\sqrt{r^2+(l-z')^2}}-\frac{1}{\sqrt{r^2+z'^2}}\right)\right].$$

We seek the field strengths at the ground plane, i.e., for $z'=0$. Putting

$$\sqrt{r^2+l^2}=s$$

we finally obtain, when regard is taken of the image,

$$E_z=\frac{-iI_0}{2\pi\omega\epsilon_0}\frac{s-r}{lsr},\qquad H_\phi=\frac{I_0}{2\pi}\frac{s-r}{rl}.$$

10. Appendix 3. M′ for an Electrically Small Monopole With Sinusoidal Current Distribution

In section 3.1 we found the following exact expression for the ratio $M'=\left|\frac{E_z}{\zeta_0 H_t}\right|^2$ for a monopole with sinusoidal current distribution

$$M'=\frac{\left(\frac{r}{s}\cos ks-\cos kl\cos kr\right)^2+\left(\frac{r}{s}\sin ks-\cos kl\sin kr\right)^2}{(\sin ks-\cos kl\sin kr)^2+(\cos kl\cos kr-\cos ks)^2}.$$

In order to utilize the fact that the antenna height is small as compared to the wavelength we rewrite this expression in the following way

$$M'=\frac{\left(1-\frac{r}{s}\right)^2-\sin^2 kl+2\frac{r}{s}\left[\sin^2\frac{k}{2}(l+s-r)+\sin^2\frac{k}{2}(l-s+r)\right]}{-\sin^2 kl+2\left[\sin^2\frac{k}{2}(l+s-r)+\sin^2\frac{k}{2}(l-s+r)\right]}.$$

If in this expression we use the first order approximation $\sin x\simeq x$ for $x<<1$ we get

$$M'\simeq\frac{1-(kl)^2\left(\frac{s}{l}\right)^2}{(kl)^2\left(\frac{s}{l}\right)^2},$$

which for small values of $\frac{r}{l}$ is very close to the expression for M' derived in section 3.2 for a linear current distribution

$$M'=\frac{1}{(kl)^2\left(\frac{s}{l}\right)^2}.$$

However, for large values of $\frac{r}{l}$ the approximate value for M' becomes negative. This means that a better approximation is needed. We therefore put

$$\sin x\simeq x-\frac{x^3}{6}$$

and

$$\sin^2 x\simeq x^2-\frac{x^4}{3}\quad \text{for } x<<1.$$

Inserting these expressions in the exact expression for M' and setting $kl=\beta$ and $tg\frac{\alpha}{2}=y$, α being the angle which the direction from the field point to the top of the antenna forms with the horizontal plane (fig. 3), we find

$$M'=\frac{48y^4-12\beta^2y^2(1+y^2)^2+\beta^4(3+2y^2+4y^4+6y^6+y^8)}{12\beta^2y^2(1+y^2)^2+\beta^4(3-10y^4-8y^6-y^8)}.$$

For values of $\beta=kl<<1$ this expression will be more suitable for numerical computations than the exact expressions.

11. References

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