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Fields of Electric Dipoles in Sea Water—the Earth-Atmosphere-Ionosphere Problem[1]

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The theory of extremely low frequency radio wave propagation from vertical and horizontal electric dipoles in a half space, separated by an infinite slab from another half space, is discussed and application is made to the specific case of the sea water-atmosphere-ionosphere problem, with dipoles located in the sea water. Each of the media is assumed homogeneous and isotropic. When attention is restricted to the frequency range 1 to 1,000 c/s, integration in the complex plane leads to consideration of the pole corresponding to the TEM mode of transmission and two branch cut integrals. One of these (that giving rise to propagation of energy along and in the ionosphere) is found to be important in the case of the horizontal dipole.

1. Introduction

The "mode theory" of VLF and ELF radio wave propagation has been well developed by several investigators (such as for example Wait [1960b] and Brekhovskikh [1960] who also give extensive bibliographies). An integral representing the Hertz potential in any layer of a multilayered medium due to a source in the same or any other of the layers, is readily obtainable, the most common representation being as a superposition of cylindrical wave functions. Typical integrals of this sort are not easily evaluated, however, except in the very simplest cases. For this reason, most of the analytical effort in the past has been confined to models associated with rather common physical situations, as for example the propagation of seismic waves in the earth [Pekeris, 1948], or the electromagnetic wave radiation from lightning strokes in the atmosphere [Schumann, 1952].

In this paper attention will be directed to the particular three-layer model which simulates, in plane parallel geometry, the sea water-atmosphere-ionosphere problem of radiation from electric dipoles submerged in the sea water. The situation is illustrated in figure 1. It is expected that the results will approximate those of a spherical model reasonably well out to distances from the antenna on the order of the earth's radius [Wait and Carter, 1960]. The equivalent two layer problem (dipoles submerged in sea water with the atmosphere assumed homogeneous and semi-infinite) was first investigated by R. K. Moore [1951] and later by Baños and Wesley [1953, 1954].

As is customary in such problems, integrals expresssing the Hertz potentials are deformed into the complex plane and their determination is reduced to the evaluation of residues at poles, plus integrations around appropriate branch cuts. In the frequency range 1 to 1,000 c/s only one of the poles—that corresponding to the TEM mode of transmission—is of importance at appreciable distances. Asymptotic estimates of the branch line integrals are made, and it is found that the contribution from one of these, in the case of the horizontal dipole and the lower portion of the frequency range, is not generally negligible in comparison with the mode solution. This corresponds physically to the fact that appreciable energy is propagated along and within the ionosphere.

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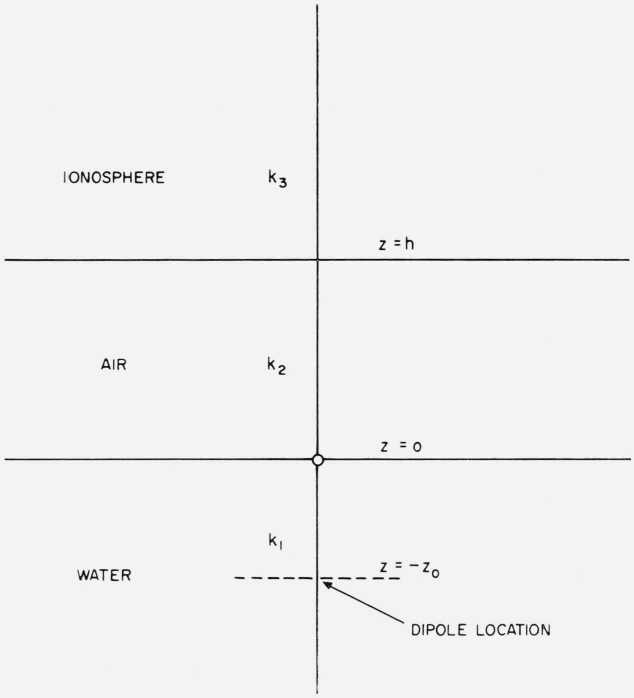

FIGURE 1. *Geometry of the problem.*

2. Development of the Hertz Potentials

2.1. Vertical Electric Dipole

The development in this and succeeding sections is given in terms of the Hertz potential, designated by the vector $\vec{\pi}$. The boundary conditions can be satisified, for the vertical dipole, with a single component, which will be considered a z-component, corresponding to dipole orientation along the z-axis of a cylindrical coordinate system. Subscripts 1, 2, and 3 will later be identified with sea water, atmosphere, and ionosphere in that order. Each of the three media is assumed homogeneous and isotropic. The time dependence is taken to be as $e^{i\omega t}$ and is thereafter suppressed.

The parameter k_n is defined by

$$k_n^2=\omega^2\mu_n\epsilon_n-i\omega\mu_n\sigma_n, \qquad n=1, 2, 3$$

with the understanding that the square root having negative imaginary part is to be taken. This may also be written

$$k_n^2=\omega^2\mu_n\bar{\epsilon}_n, \qquad n=1, 2, 3$$

where $\bar{\epsilon}_n=\epsilon_n+\frac{\sigma_n}{i\omega}$

is the so-called "complex permittivity." The quantities ϵ_n, μ_n, and σ_n are permittivity, permeability, and conductivity, respectively. M. K. S. units are used throughout.

The water-air boundary is designated by the plane $z=0$, and the air-ionosphere boundary by $z=h$. The dipole location is at $z=-z_0$ (see fig. 1). Representations for the Hertz potentials are made as superpositions of cylindrical wave functions, and upon application of the boundary conditions we find for regions 1 and 2 (where we are mainly concerned with the fields.)

$$\pi_1 = M\int_0^\infty \left[e^{i\beta_1|z+z_0|} + \frac{K_1 - Z_2}{K_1 + Z_2} e^{i\beta_1(z-z_0)}\right] \frac{J_0(\lambda\rho)}{i\beta_1} \lambda d\lambda \tag{2.1}$$

$$\pi_2 = 2M\int_0^\infty \frac{1}{i\beta_2}\left[\frac{K_2 \cosh(i\beta_2 z) - Z_2 \sinh(i\beta_2 z)}{K_1 + Z_2}\right] e^{-i\beta_1 z} J_0(\lambda\rho)\lambda d\lambda \tag{2.2}$$

where

$$M = \frac{Idl}{4\pi i\omega\bar{\epsilon}_1}$$

$$Z_2 = K_2 \frac{K_3 + K_2 \tanh(i\beta_2 h)}{K_2 + K_3 \tanh(i\beta_2 h)}$$

in which $K_n(\lambda) = \frac{i\beta_n}{P_n}$, $n = 1,2,3$

with β_n a separation parameter defined by

$$\beta_n^2 = k_n^2 - \lambda^2$$

and $P_n = \sigma_n + i\omega\epsilon_n$. The imaginary part of β_n is taken as negative.

The identity

$$J_0(\lambda\rho) = \frac{1}{2}[H_0^{(1)}(\lambda\rho) + H_0^{(2)}(\lambda\rho)]$$

allows the integrals for π_1 and π_2 to be written as the sum of two integrals of which the first is deformed into the first quadrant of the complex λ-plane; the second into the fourth quadrant, as shown in figure 2. It can be shown that the only contributions arise from integrations around branch cuts 2 and 3, and the residues at poles. At appreciable distances the contribution from branch cut 3 will be small, so that it can be omitted from consideration.

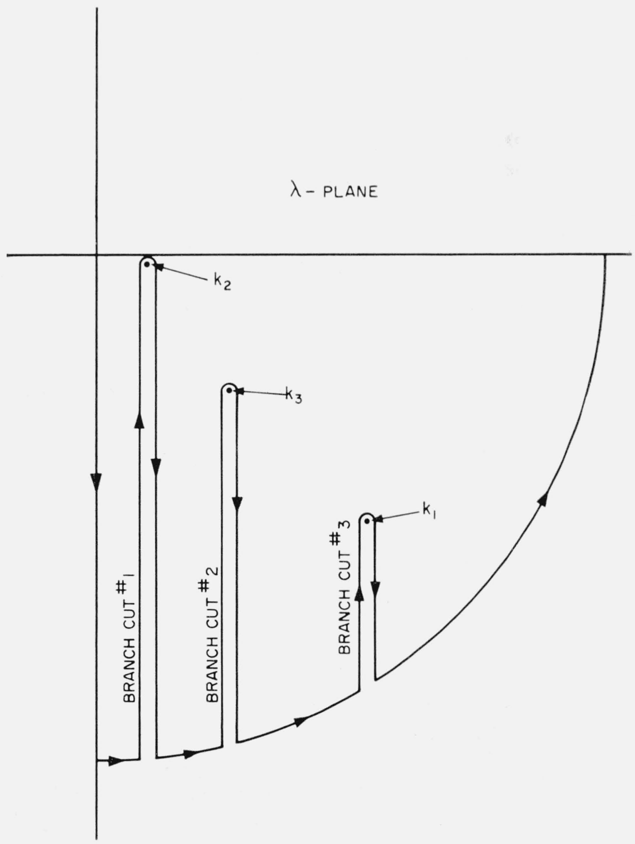

FIGURE 2. *Contour integration in the 4th quadrant of the complex λ-plane.*

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The formal expression for the residues at the poles is, for region 1,

$$-\frac{2i\pi M}{P_1}\sum_j H_0^{(2)}(\lambda_j\rho)\lambda_j\frac{e^{i\beta_1^{(j)}(z-z_0)}}{(K_1+Z_2)'_{\lambda=\lambda_j}} \tag{2.3}$$

where $\beta_1^{(j)}=(k_1^2-\lambda_j^2)^{1/2}$, and the λ_j are the zeros of (K_1+Z_2).

For region 2 the residues at poles are

$$2\pi M\sum_j H_0^{(2)}(\lambda_j\rho)\lambda_j\frac{e^{-i\beta_1^{(j)}z_0}}{\beta_2^{(j)}}\left[\frac{K_2\cos\beta_2 z-iK_1\sin\beta_2 z}{K_1+Z_2)'}\right]\lambda=\lambda_j. \tag{2.4}$$

In evaluating the branch line 2 integrals it will be necessary to make some approximations. For the frequency range 1 to 1,000 c/s it is permissible to state

$$k_1^2\approx -i\omega\mu_0\sigma_1$$

$$k_2^2\approx \omega^2\mu_0\epsilon_0$$

$$k_3^2\approx -i\omega\mu_0\sigma_3. \tag{2.5}$$

Designating $k_3=a(1-i)$, defining a new parameter of integration, t, by

$$\lambda=a(1-i-it)$$

and replacing the Hankel function by the first term of its asymptotic series, the branch line 2 integral for the Hertz potential in region 1 becomes

$$I_{BL2}^{(1)}\approx -2Ma^{5/2}e^{i\pi/4}\delta_1\delta_3\sqrt{\frac{2}{\pi\rho}}\,e^{-a\rho(1+i)}\int_0^\infty t^{1/2}F(t)e^{-a\rho t}dt \tag{2.6}$$

where

$$F(t)=\frac{[t+2(1+i)]^{1/2}(1-i-it)^{1/2}\beta_2^2 e^{i\beta_1(z-z_0)}}{D(\lambda(t))}$$

in which the β's are expressed in terms of t, and

$$D(\lambda)=\delta_3^2\beta_2^2(\beta_1 c+\delta_1\beta_2 s)^2-\beta_3^2(\beta_1 s-\delta_1\beta_2 c)^2.$$

The definitions $\delta_1=\sigma_1/\omega\epsilon_0$, $\delta_3=\sigma_3/\omega\epsilon_0$, $s=\sin\beta_2 h$, and $c=\cos\beta_2 h$ are used.

The first term of an asymptotic expansion, obtained by integrating terms resulting from a Maclaurin series expansion of $F(t)$, gives

$$I_{BL2}^{(1)}\approx 2\sqrt{2}Me^{-i3\pi/4}\frac{\delta_1 a e^{ib_1(z-z_0)}}{\delta_3(b_1\bar{c}+\delta_1 b_2\bar{s})^2}\frac{e^{-a\rho(1+i)}}{\rho^2} \tag{2.7}$$

where $b_n=(k_n{}^2+2ia^2)^{1/2}$, $n=1$, 2, and $\bar{s}=\sin b_2\mathrm{h}$, $\bar{c}=\cos b_2 h$. Since $|k_1{}^2|\gg|2ia^2|$ it is quite accurate to take $b_1\approx k_1$.

Succeeding terms in the series would go as ρ^{-3}, ρ^{-4}, etc. The factor $\exp[-a\rho(1+i)]$, or $\exp[-ik_3\rho]$, indicates the physical process to be one of propagation along and within the ionosphere.

Similarly the branch line 2 contribution to the Hertz potential in air is found to be

$$I_{BL2}^{(2)}\approx\frac{M2^{3/2}a\delta_1 e^{-i\pi/4}e^{-ik_1z_0}}{\delta_3 b_2(b_1\bar{c}+\delta_1 b_2\bar{s})^2}(b_1\sin b_2 z-\delta_1 b_2\cos b_2 z)\frac{e^{-a\rho(1+i)}}{\rho^2}. \tag{2.8}$$

Evaluation of the mode solutions requires knowledge of the poles, λ_j. Taking reasonable values for the physical constants involved, it can be shown that the pole for the lowest order mode is given approximately by

$$\lambda_0=k_2\left[1+\frac{1-i}{h\sqrt{2\omega\mu_0\sigma_3}}\right]^{1/2}. \tag{2.9}$$

Approximate formulas for poles of arbitrary order are available (see, for example, Wait [1960a and b]). For the frequency range under consideration here only one pole, that which for perfectly conducting boundaries would be related to the TEM mode of transmission, is very close to the real axis of λ. The others are located near and to the right of the negative imaginary axis, and their relatively great imaginary part insures that they are heavily damped at appreciable distances from the source.

Thus the important mode solution for the 1 to 1,000 c/s frequency range is, in region 1,

$$I_m^{(1)} = -\frac{2i\pi M}{\sigma_1} \lambda_0 H_0^{(2)}(\lambda_0\rho) \frac{e^{i\beta_1^{(0)}(z-z_0)}}{(K_1+Z_2)'_{\lambda=\lambda_0}} \tag{2.10}$$

and in region 2

$$I_m^{(2)} = \frac{2\pi M\lambda_0}{\sigma_1} H_0^{(2)}(\lambda_0\rho) \frac{e^{-i\beta_1^{(0)}z_0}}{\beta_2^{(0)}} \frac{\delta_1\beta_2^{(0)} \cos \beta_2^{(0)}z + \beta_1^{(0)} \sin \beta_2^{(0)}z}{(K_1+Z_2)'_{\lambda=\lambda_0}}. \tag{2.11}$$

2.2. Horizontal Electric Dipole

The development in this case is similar except that two components, π_x and π_z, of the Hertz vector are required to match the boundary conditions, for a dipole oriented along the x-axis. Lack of symmetry in azimuth gives rise to Bessel functions of order 1 in the formulation of the π_z component. It is found that for the x-component

$$\pi_{x1} = M\int_0^\infty [e^{-i\beta_1|z+z_0|} + W_1(\lambda)\, e^{i\beta_1(z-z_0)}] \frac{\lambda}{i\beta_1} J_0(\lambda\rho)\, d\lambda \tag{2.12}$$

$$\pi_{x2} = M\int_0^\infty [V_2(\lambda)\, e^{-i\beta_2 z} + W_2(\lambda)\, e^{i\beta_2 z}] \frac{\lambda}{i\beta_1} e^{-i\beta_1 z_0} J_0(\lambda\rho) d\lambda \tag{2.13}$$

$$\pi_{x3} = M\int_0^\infty V_3(\lambda) e^{-i\beta_3 z} \frac{\lambda}{i\beta_1} e^{-i\beta_1 z_0} J_0(\lambda\rho) d\lambda \tag{2.14}$$

where $W_1(\lambda) = \dfrac{\beta_1 - U_2}{\beta_1 + U_2}$

with $U_2 = \beta_2 \dfrac{\beta_3 + i\beta_2 \tan \beta_2 h}{\beta_2 + i\beta_3 \tan \beta_2 h}$

and $V_2(\lambda) = -i\delta_1 \dfrac{\beta_1}{\beta_3} \dfrac{\beta_2 + U_2}{\beta_1 + U_2}$,

$W_2(\lambda) = -i\delta_1 \dfrac{\beta_1}{\beta_3} \dfrac{\beta_2 - U_2}{\beta_1 + U_2}$.

Region 3 will not be further considered.

For the z-component we have in regions 1 and 2

$$\pi_{z1} = M \cos \phi \int_0^\infty \phi_1(\lambda) e^{i\beta_1(z-z_0)} \frac{J_1(\lambda\rho)}{i\beta_1} \lambda^2 d\lambda \tag{2.15}$$

$$\pi_{z2} = M \cos \phi \int_0^\infty \left[\psi_2(\lambda) e^{-i\beta_2 z} + \phi_2(\lambda) e^{i\beta_2 z}\right] J_1(\lambda\rho) \frac{e^{-i\beta_1 z_0}}{i\beta_1} \lambda^2 d\lambda \tag{2.16}$$

where

$$\phi_1(\lambda) = \frac{2\beta_1[(i-\delta_1)(\beta_3 s - \delta_3\beta_2 c)(\beta_2 c + i\beta_3 s) - \delta_1(i+\delta_3)\beta_2^2]}{(\beta_2 c + i\beta_3 s)(\delta_3\beta_2 c - \beta_3 s)(\beta_1 + U_2)(\beta_1 + X_2)}$$

$$\psi_2(\lambda) = -\frac{1}{2\beta_2}\left[(\beta_1 + i\delta_1\beta_2)\phi_1 + \frac{2(i-\delta_1)\beta_1}{\beta_1 + U_2}\right]$$

$$\phi_2(\lambda) = \frac{1}{2\beta_2}\left[(\beta_1 - i\delta_1\beta_2)\phi_1 + \frac{2(i-\delta_1)\beta_1}{\beta_1 + U_2}\right]$$

in which $X_2=-i\sigma_2 Z_2$, and s and c are defined as before. Contour integration as in the case of the vertical electric dipole gives, for the important contributions in region 1,

$$I_{BL2}{}^{(x1)}\approx 2\sqrt{2}Mae^{-i\pi/4}G(0)\,e\,\frac{-a\rho(1+i)}{\rho^2} \tag{2.17}$$

where $G(0)=\dfrac{e^{ib_1(z-z_0)}}{(b_1\bar{c}+ib_2\bar{s})^2}$

with b_n, $n=1, 2$, defined as before. In addition

$$I_{BL2}{}^{(z1)}\approx M\cos\phi\,\frac{4a^2}{k_1b_2^2\bar{c}^2}\,e^{ik_1(z-z_0)}\,\frac{e^{-a\rho(1+i)}}{\rho^2}. \tag{2.18}$$

and the mode contribution (excited only by π_z) is

$$I_m^{(z1)}\approx 2\pi M\cos\phi\lambda_0 H_1^{(2)}(\lambda_0\rho)\,e^{i\beta_1^{(0)}(z-z_0)}\,\frac{\Gamma(\lambda_0)}{\left(-\dfrac{1}{\lambda_0}\right)(\beta_1+X_2)'_{\lambda=\lambda_0}} \tag{2.19}$$

where

$$\Gamma(\lambda)=\frac{(i-\delta_1)(\beta_3s-\delta_3\beta_2c)(\beta_2c+i\beta_3s)-\delta_1(i+\delta_3)\beta_2^2}{(\delta_3\beta_2c-\beta_3s)[\beta_1(\beta_2c+i\beta_3s)+\beta_2(\beta_3c+i\beta_2s)]}.$$

In region 2

$$I_{BL2}^{(x2)}\approx 2Me^{i\pi/4}a^{1/2}S(0)\,\frac{e^{-a\rho(1+i)}}{\rho^2} \tag{2.20}$$

where

$$S(0)=\frac{2a^{1/2}e^{-ib_1z_0}(b_1\cos b_2z+ib_2\sin b_2z)}{(b_1\bar{c}+ib_2\bar{s})^2}.$$

$$I_{BL2}^{(z2)}\approx 4M\cos\phi a^2\delta_1e^{i3\pi/4}e^{-ib_1z_0}\,\frac{(b_2\cos b_2z+ib_1\sin b_2z)}{b_1^2b_2^2\bar{c}^2}\,\frac{e^{-a\rho(1+i)}}{\rho^2} \tag{2.21}$$

and for the mode we have

$$I_m^{(z2)}=2i\pi M\cos\phi\lambda_0H_1^{(2)}(\lambda_0\rho)\,\frac{e^{-i\beta_1^{(0)}z_0}}{\beta_2^{(0)}}\,\frac{r(\lambda_0,z)}{(\beta_1+X_2)'_{\lambda=\lambda_0}} \tag{2.22}$$

where

$$r(\lambda,z)=\frac{[(i-\delta_1)(\beta_3s-\delta_3\beta_2c)(\beta_2c+i\beta_3s)-\delta_1(i+\delta_3)\beta_2^2](\beta_1\sin\beta_2z-\delta_1\beta_2\cos\beta_2z)}{\left(-\dfrac{1}{\lambda_0}\right)(\beta_2c+i\beta_3s)(\delta_3\beta_2c-\beta_3s)(\beta_1+U_2)}.$$

3. Field Components

3.1. Vertical Electric Dipole

The field components are found in the usual way by differentiation of the Hertz potentials. In the following the superscript (v,m) will be used to denote "vertical electric dipole, mode solution" (assuming that only the one mode contributes substantially), and (v,b) will denote "vertical electric dipole, branch line solution." It is noted that the term $(k_1^2-\beta_1^{(0)2})$ is equal to λ_0^2, but otherwise the approximation $\beta_1^{(0)}\approx k_1$ can be made throughout. The field components in the sea water are:

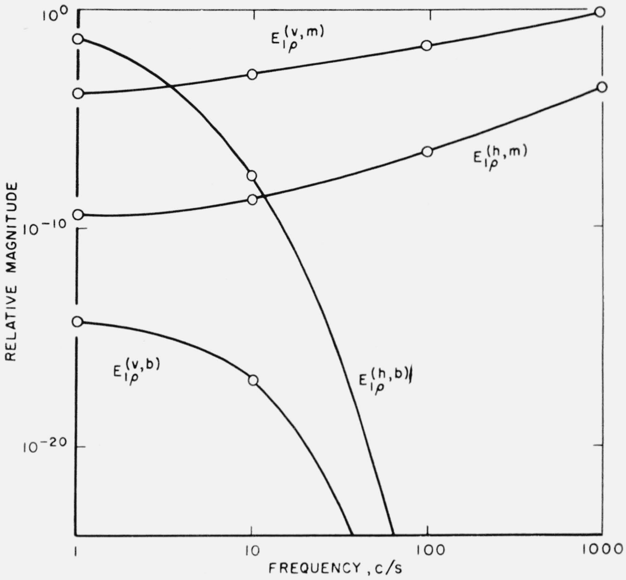
$$E_{1\rho}^{(v,m)}=\frac{2\pi M}{\sigma_1}\lambda_0^2k_1H_1^{(2)}(\lambda_0\rho)\,\frac{e^{ik_1(z-z_0)}}{(K_1+Z_2)'_{\lambda=\lambda_0}}, \tag{3.1}$$

$$E_{1z}^{(v,m)}=-\frac{2i\pi M}{\sigma_1}\lambda_0^3H_0^{(2}(\lambda_0\rho)\,\frac{e^{ik_1(z-z_0)}}{(K_1+Z_2)'_{\lambda=\lambda_0}}, \tag{3.2}$$

$$H_{1\phi}^{(v,m)}=2i\pi M\lambda_0^2H_1^{(2)}(\lambda_0\rho)\,\frac{e^{ik_1(z-z_0)}}{(K_1+Z_2)'_{\lambda=\lambda_0}}; \tag{3.3}$$

FIGURE 3. *Relative magnitude of $E_{1\rho}$ as it varies with frequency at 1,000 km distance from source. Attenuation due to antenna submergence is not included.*

Behavior of other field components is similar, except that the rate of increase of E_{1z} (mode) is greater than that of $E_{1\rho}$ (mode), as frequency increases.



$$E_{1\rho}^{(r,\,b)} \approx \frac{4Mk_1a^2}{\delta_1\delta_3b_2^2\bar{s}^2}\, e^{ik_1(z-z_0)}\, \frac{e^{-(1+i)a\rho}}{\rho^2}, \tag{3.4}$$

$$E_{1z}^{(v,\,b)} \approx \frac{4\sqrt{2}Ma^3e^{-i\pi/4}}{\delta_1\delta_3b_2^2\bar{s}^2}\, e^{ik_1(z-z_0)}\, \frac{e^{-a\rho(1+i)}}{\rho^2}, \tag{3.5}$$

$$H_{1\phi}^{(v,\,b)} \approx -\frac{2M\omega^3\mu_0\epsilon_0^2}{b_2^2\bar{s}^2}\, e^{ik_1(z-z_0)}\, \frac{e^{-a\rho(1+i)}}{\rho^2}, \tag{3.6}$$

In the atmosphere the fields are

$$E_{2\rho}^{(v,\,m)} = -\frac{2M\pi}{\sigma_1}\, \lambda_0 H_1^{(2)}(\lambda_0\rho)\, e^{-i\beta_1^{(0)}z_0}\, \frac{q(\lambda_0,z)}{(-1/\lambda_0)(K_1+Z_2)'_{\lambda=\lambda_0}} \tag{3.7}$$

$$E_{2z}^{(v,\,m)} = -\frac{2M\pi}{\sigma_1}\, \lambda_0^2 H_0^{(2)}(\lambda_0\rho)\, e^{-i\beta_1^{(0)}z_0}\, \frac{p(\lambda_0,z)}{(-1/\lambda_0)(K_1+Z_2)'_{\lambda=\lambda_0}} \tag{3.8}$$

$$H_{2\phi}^{(v,\,m)} = -\frac{2iM\pi}{\delta_1}\, \lambda_0 H_1^{(2)}(\lambda_0\rho)\, e^{-i\beta_1^{(0)}z_0}\, \frac{p(\lambda_0,z)}{(-1/\lambda_0)(K_1+Z_2)'_{\lambda=\lambda_0}} \tag{3.9}$$

$$E_{2\rho}^{(v,\,b)} \approx \frac{4Ma^2e^{+i3\pi/4}e^{-ik_1z_0}}{\delta_1\delta_3b_2^2\bar{s}^2}\, \frac{e^{-a\rho(1+i)}}{\rho^2}\,(b_1\cos b_2z+\delta_1b_2\sin b_2z) \tag{3.10}$$

$$E_{2z}^{(v,\,b)} \approx -\frac{4\sqrt{2}Ma^3e^{-ik_1z_0}}{\delta_1\delta_3b_2^3\bar{s}^2}\, \frac{e^{-a\rho(1+i)}}{\rho^2}\,(b_1\sin b_2z-\delta_1b_2\cos b_2z) \tag{3.11}$$

$$H_{2\phi}^{(v,\,b)} \approx \frac{4Mk_2^2a^2e^{i\pi/4}}{\omega\mu_0\delta_1\delta_3b_2^3\bar{s}^2}\, e^{-ik_1z_0}\, \frac{e^{-a\rho(1+i)}}{\rho^2}\,(b_1\sin b_2z-\delta_1b_2\cos b_2z) \tag{3.12}$$

where

$$q(\lambda,z)=\delta_1\beta_2\sin\beta_2z-\beta_1\cos\beta_2z$$

$$p(\lambda,z)=\delta_1\cos\beta_2z+\beta_1/\beta_2\sin\beta_2z.$$

3.2. Horizontal Electric Dipole

Using the superscript (h,m) for the mode solution in this case, and (h,b) for the branch line contribution, it is found that in the sea water,

$$E_{1\rho}^{(h,m)}=2Mik_1\pi\cos\phi\lambda_0^2[H_0^2(\lambda_0\rho)-H_2^2(\lambda_0\rho)]e^{ik_1(z-z_0)}\frac{\Gamma(\lambda_0)}{(-1/\lambda_0((\beta_1+X_2)'_{\lambda=\lambda_0}} \tag{3.13}$$

$$E_{1z}^{(h,m)}=4M\pi\cos\phi H_1^{(2)}(\lambda_0\rho)\lambda_0^3e^{ik_1(z-z_0)}\frac{\Gamma(\lambda_0)}{(-1/\lambda_0)(\beta_1+X_2)'_{\lambda=\lambda_0}} \tag{3.14}$$

$$H_{1\phi}^{(h,m)}=4\pi M\sigma_1\cos\phi\lambda_0^2[H_0^{(2)}(\lambda_0\rho)-H_0^{(2)}(\lambda_0\rho)]e^{ik_1(z-z_0)}\frac{\Gamma(\lambda_0)}{(-1/\lambda_0)(\beta_1+X_2)'_{\lambda=\lambda_0}} \tag{3.15}$$

$$E_{1\rho}^{(h,b)}=\cos\phi(k_1^2+2ia^2)\pi_{x1}^{(b)}+(1-i)k_1a\pi_{z1}^{(b},\tag{3.16}$$

$$E_{1\phi}^{(h,b)}=-\sin\phi k_1^2\pi_{x1}^{(b)},\tag{3.17}$$

$$E_{1z}^{(h,b)}=\cos\phi k_1a(1-i)\pi_{x1}^{(b)},\tag{3.18}$$

$$H_{1\rho}^{(h,b)}=\frac{k_1^3}{\omega\mu_0}\sin\phi\pi_{x1}^{(b)},\tag{3.19}$$

$$H_{1\phi}^{(h,b)}=ik_1\cos\phi\pi_{x1}^{(b)}+(1+i)a\pi_{z1}^{(b)}\tag{3.20}$$

$$H_{1z}^{(h,b)}=\sin\phi(1+i)a\pi_{x1}^{(b)}\tag{3.21}$$

where

$$\pi_{x1}^{(b)}\approx-2\sqrt{2}aMe^{i3\pi/4}\frac{e^{ik_1(z-z_0)}}{(b_1\bar{c}+ib_2\bar{s})^2}\frac{e^{-a\rho(1+i)}}{\rho^2},$$

$$\pi_{z1}^{(b)}\approx\frac{4a^2M\cos\phi}{k_1b_2^2\bar{c}^2}e^{ik_1(z-z_0)}\frac{e^{-a\rho(1+i)}}{\rho^2}.$$

In the atmosphere the field components are:

$$E_{2\rho}^{(h,m)}=i\pi M\cos\phi\lambda_0^2[H_0^{(2)}(\lambda_0\rho)-H_2^{(2)}(\lambda_0\rho)]e^{-ik_1z_0}\frac{u(\lambda_0,z)}{(\beta_1+X_2)'_{\lambda=\lambda_0}}\tag{3.22}$$

$$E_{2z}^{(h,m)}=2i\pi M\cos\phi\lambda_0^3H_1^{(2)}(\lambda_0\rho)\frac{e^{-ik_1z_0}}{\beta_2^{(0)}}\frac{r(\lambda_0,z)}{(\beta_1+X_2)'_{\lambda=\lambda_0}}\tag{3.23}$$

$$H_{2\phi}^{(h,m)}=-\omega\epsilon_0\pi M\cos\phi\lambda_0^2[H_0^{(2)}(\lambda_0\rho)-H_2^{(2)}(\lambda_0\rho)]\frac{e^{-ik_1z_0}}{\beta_2^{(0)}}\frac{r(\lambda_0,z)}{(\beta_1+X_2)'_{\lambda=\lambda_0}}\tag{3.24}$$

where

$$u(\lambda,z)=\frac{[(i-\delta_1)(\beta_3s-\delta_3\beta_2c)(\beta_2c+i\beta_3s)-\delta_1(i+\delta_3)\beta_2^2](\beta_1\cos\beta_2z+\delta_1\beta_2\sin\beta_2z)}{(-1/\lambda_0)(\beta_2c+i\beta_3s)(\delta_3\beta_2c-\beta_3s)(\beta_1+U_2)}$$

$$r(\lambda,z)=\frac{[(i-\delta_1)(\beta_3s-\delta_3\beta_2c)(\beta_2c+i\beta_3s)-\delta_1(i+\delta_3)\beta_2^2](\beta_1\sin\beta_2z-\delta_1\beta_2\cos\beta_2z)}{(-1/\lambda_0)(\beta_2c+i\beta_3s)(\delta_3\beta_2c-\beta_3s)(\beta_1+U_2)}$$

and

$$E_{2\rho}^{(h,b)}=\cos\phi B_1(b_1\cos b_2z+ib_2\sin b_2z)\frac{e^{-a\rho(1+i)}}{\rho^2}\tag{3.25}$$

$$E_{2\phi}^{(h,b)}=-\sin\phi A_1k_2^2(b_1\cos b_2z+ib_2\sin b_2z)\frac{e^{-a\rho(1+i)}}{\rho^2}\tag{3.26}$$

$$E_{2z}^{(h,b)}=\cos\phi B_2(b_2\cos b_2z+ib_1\sin b_2z)\frac{e^{-a\rho(1+i)}}{\rho^2}\tag{3.27}$$

$$H_{2\rho}^{(h,b)} = -\sin\phi \frac{A_1 k_2^2 b_2}{i\omega\mu_0} (b_1 \sin b_2 z - i b_2 \cos b_2 z) \frac{e^{-a\rho(1+i)}}{\rho^2} \tag{3.28}$$

$$H_{2\phi}^{(h,b)} = \cos\phi B_3 (b_2 \cos b_2 z + i b_1 \sin b_2 z) \frac{e^{-a\rho(1+i)}}{\rho^2} \tag{3.29}$$

$$H_{2z}^{(h,b)} = \sin\phi A_1 a(1+i)(b_1 \cos b_2 z + i b_2 \sin b_2 z) \frac{e^{-a\rho(1+i)}}{\rho^2}, \tag{3.30}$$

where

$$A_1 = \frac{2^{3/2} M e^{i\pi/4} a e^{-ib_1 z_0}}{(b_1 \overline{c} + i b_2 \overline{s})^2}$$

$$A_2 = \frac{4M e^{i3\pi/4} \delta_1 a^2 e^{-ib_1 z_0}}{b_1^2 b_2^2 \overline{c}^2}$$

$$B_1 = (k_2^2 + 2ia^2) A_1 + a(1+i) b_2 A_2,$$

$$B_2 = k_3^2 A_2 + a(1-i) b_2 A_1,$$

$$B_3 = a(1+i) A_2 + i b_2 A_1.$$

4. Conclusions

Not too many numerical results have been obtained from the formulas as yet. Some calculations have been carried out, however, for the fields in sea water at a distance of 1,000 km and frequencies of 1, 10, 100, and 1,000 c/s. Conductivities of 4 and 10^{-5} mhos/m were assumed for sea water and ionosphere, respectively. The ionosphere height was taken to be 90 km, a figure which has been used with some success in ELF studies [Wait and Carter, 1960].

The more important results can be summarized as follows: (See also fig. 3.)

1. The mode solution (corresponding approximately to the TEM mode in a parallel-plane waveguide) involved here is excited more efficiently by the vertical than by the horizontal electric dipole. This is not unexpected.

2. The branch line contribution (corresponding to energy transmitted along the air-ionosphere boundary, but within the ionosphere) in the case of the horizontal dipole is greater than the mode contribution at 1 and 10 c/s by as much as 10^8 depending on the component concerned.

3. At 1 c/s the field components due to the horizontal dipole are greater than those of a vertical dipole of the same moment by virtue of the branch line contribution. At 10 c/s and higher the field components of the vertical dipole are greater by virtue of the mode solution. At greater ranges, the vertical dipole mode solution would predominate even at 1 c/s, due to its much smaller attenuation with distance.

4. The system attenuation between half wavelength (in water) "coaxial antennas" [Moore, 1951] separated by 1,000 km is more than 250 decibels for any frequency in the range considered, not including attenuation in the z-direction, that is, attenuation due strictly to the sea water. In view of the fact that noise power is substantial in this frequency range, the difficulty of practical communication under such conditions would seem to be great.

5. At distances less than 1,000 km the branch line contribution will be important even in the higher portion of the 1 to 1,000 c/s frequency range, and should therefore be investigated more carefully. Since the asymptotic series expansion is usable only for large values of ρ, some method other than that of asymptotic expansion must be used. Perhaps numerical integration would be in order. At closer distances it is possible that substantial contributions from the more highly damped modes may occur; therefore these may also require investigation.

More details as to the actual figures obtained by computation are available in a report, by the writer, published by the University of New Mexico, Engineering Experiment Station, Technical Report EE–44, February, 1961.

The reader will note by reference to figure 3 that since the magnitude of the field components increases with frequency in the case of the mode solutions, and since attenuation due to travel in sea water increases monotonically with frequency, then at any combined depth of receiving and transmitting antennas there is evidently an optimum frequency. There is no optimum for the branch line solutions, however.

It should be mentioned that in view of the large system attenuation at distances of 1,000 km or greater, it is perhaps only of somewhat academic interest to consider refinements in the model which would take into account the anisotropy and inhomogeneity of the ionosphere. As pointed out by Wait [1960a] the assumption of a refractive index increasing exponentially with height causes an attenuation rate which varies more rapidly with frequency than that for a homogeneous ionosphere. The experimental data seems to require this more elaborate model. In the problem considered here, however, experimental testing of the theory is probably not feasible except at distances less than 1,000 km or so, and at such distances it is not expected that the results will be drastically changed by the different attenuation rate. The effect of the anisotropy caused by the earth's magnetic field is likewise expected to be relatively minor [Wait, 1960a] at such distances.

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