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An Approximate Full Wave Solution for Low Frequency Electromagnetic Waves in an Unbounded Magneto-Ionic Medium

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Maxwell's equations in an anisotropic inhomogeneous medium are transformed by means of the Stratton-Chu formula into a vector integral equation which couples the various electric field components. In case the hypotheses of far-zone field and low frequency electromagnetic waves apply, this vector integral equation can be approximated by a system of uncoupled scalar integral equations. This implies an approximate equivalence between the original vector integral equation and a system of modified scalar inhomogeneous Helmholtz equations. The conditions under which the system of uncoupled scalar integral equations can be solved by Neumann series are discussed, and the first three terms of the Neumann series are given explicitly.

1. Introduction

Maxwell's equations for the electromagnetic field in an inhomogeneous anisotropic medium, such as the ionospheric magnetoplasma, can be written in the form [Turner, 1954]

$$\text{curl } \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \qquad \text{div } \mathbf{D} = 0$$

$$\text{curl } \mathbf{H} = \frac{\partial \check{\mathbf{D}}}{\partial t} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{div } \mathbf{H} = 0. \tag{1}$$

Assuming harmonic time dependence $e^{i\omega t}$ and that the ambient magnetic field $\mathbf{B}_0$ is directed along the z-axis, one obtains the following vector wave equation appropriate to an inhomogeneous anisotropic medium:

$$\text{curl curl } \mathbf{E} = \omega^2 \mu_0 \check{\mathbf{D}} = k_0^2 (\check{\kappa}) \cdot \mathbf{E} \tag{2}$$

where $k_0^2 = \omega^2 \mu_0 \epsilon_0$ and $(\check{\kappa})$ is the conductivity tensor [Turner, 1954] of a magnetoplasma in a uniform magnetic field:

$$(\check{\kappa}) = \begin{pmatrix} B & C & 0 \\ -C & B & 0 \\ 0 & 0 & A \end{pmatrix}. \tag{3}$$

In the case of the ionosphere* A, B, and C are given by

$$A = 1 - (X/U), \qquad B = 1 - UX/(U^2 - Y^2),$$
$$C = iXY/(U^2 - Y^2) \tag{4}$$

where X, Y, and $U = 1 - iZ$ are the usual definitions of magneto-ionic theory [Ratcliffe, 1959].

*For a cold magnetoplasma the collision frequency vanishes and definitions (4) reduce to

$$A = 1 - X,\ B = 1 - X/(1 - Y^2),\ C = iXY/(1 - Y^2) \tag{4'}$$

If the left-hand side of eq (2) is now expanded according to the usual vector identity, one obtains a term in grad div $\mathbf{E}$ = grad div $[(\check{\kappa})^{-1} \cdot \check{\mathbf{D}}]$. This term is a dyadic, and the various field components are thus inextricably coupled [Westfold, 1949] in an *inhomogeneous* anisotropic medium. In all but the simplest cases the resulting systems of differential equations are practically intractable, and those formulations [Keller, 1953; Clemmow and Heading, 1954; Budden and Clemmow, 1957] which do exist are extremely involved, even in rectangular coordinates. It is the purpose of this paper to show how the problem may be simplified, at low frequencies and in the far-zone field, by using the Stratton-Chu formula [Stratton, 1941; Silver, 1949] to transform to a vector integral equation [1] with uncoupled components.

2. Transformation to a Vector Integral Equation

In what follows, we shall assume that the medium is unbounded; i.e., consists of all of 3-space, so that there are no boundaries in the finite part of the medium. Silver-Müller type radiation conditions [Silver, 1949; Müller, 1957] will be imposed. The ionization will be assumed to be confined to a finite portion of space, so that at infinity $A=1$, $B=1$, and $C=0$. Thus at great distances, we have to do essentially with a free-space problem.

The Stratton-Chu formula [Stratton, 1941; Silver, 1949] reads as follows:

$$\int_V \{\mathbf{G} \cdot \nabla \wedge \nabla \wedge \mathbf{E} - \mathbf{E} \cdot \nabla \wedge \nabla \wedge \mathbf{G}\} dV = \int_{bdry\,V} \{\mathbf{E} \wedge \nabla \wedge \mathbf{G} - \mathbf{G} \wedge \nabla \wedge \mathbf{E}\} \cdot \mathbf{n}\, d\Sigma. \tag{5}$$

[1] A somewhat similar formulation has recently been published by A. T. Villeneuve [1961]

In this expression $\mathbf{n}$ is the exterior normal to a region V of 3-space V_3, and $\mathbf{G}$ and $\mathbf{E}$ are twice-differentiable vector functions of position over the closure of V. The vector $\mathbf{E}$ will of course be taken to be the total electric field, so that the left-hand side of the vector wave eq (2) may be introduced into (5). As mentioned above, Silver-Müller type radiation conditions will be imposed upon $\mathbf{E}$ and $\mathbf{H}$, so that the right hand side of (5) vanishes if the boundary of V is at infinity; i.e., V is all of 3-space. Following Stratton and Chu, we shall take $\mathbf{G}$ to be the free-space Green's function e^{-ik_0R}/R times a constant but otherwise arbitrary vector $\mathbf{a}$. Thus

$$\mathbf{G}=g(P,Q)\,\mathbf{a}=\mathbf{a}e^{-ik_0R_{PQ}}/R_{PQ}. \tag{6}$$

For later use we note that

$$\text{curl curl }\mathbf{G}=\mathbf{a}k_0^2g+\mathbf{a}4\pi\delta(P-Q)+(\mathbf{a}\cdot\nabla)\nabla g. \tag{7}$$

The result of substituting (2) and (7) into (5), and taking account of the radiation conditions, is the vector integral equation in the total field $\mathbf{E}$:

$$\mathbf{E}(P)-\frac{1}{4\pi}\int_{V_3}\left\{k_0^2g(P,Q)\,(M)\cdot\mathbf{E}(Q)+(\nabla\cdot\mathbf{E})\,\nabla g\right\}dQ=0 \tag{8}$$

where

$$(M)=(\check{\kappa})-(I)=(\sigma)/(i\omega\epsilon_0), \tag{9}$$

and (I) is the unit dyadic. Since the operator ∇ applies to the source point Q, we may also write the integral equation (8) as

$$\mathbf{E}(P)=\frac{1}{4\pi}\int_{V_3}g(P,Q)\left\{k_0^2(M)\cdot\mathbf{E}(Q)+\left(ik_0+\frac{1}{R}\right)\hat{\mathbf{R}}(\nabla\cdot\mathbf{E})\right\}dQ, \tag{10}$$

where $\hat{\mathbf{R}}$ denotes a unit vector in the $\boldsymbol{r}_P-\boldsymbol{r}_Q$ direction. If we write the total field as the sum of incident and scattered fields:

$$\mathbf{E}(P)=\mathbf{E}^i(P)+\mathbf{E}^s(P), \tag{11}$$

then the integral equation (10) may be written as the inhomogeneous integral equation in the unknown scattered field:

$$\mathbf{E}^s(P)=\mathbf{F}^i(P)+\frac{1}{4\pi}\int_{V_3}g(P,Q)\left\{k_0^2(M)\cdot\mathbf{E}^s(Q)+\left(ik_0+\frac{1}{R}\right)\hat{\mathbf{R}}(\nabla\cdot\mathbf{E}^s)\right\}dQ, \tag{12}$$

where $\mathbf{F}^i(P)$ is a known vector function of position defined in terms of the incident field by the formula:

$$\mathbf{F}^i(P)=\frac{1}{4\pi}\int_{V_3}g(P,Q)\left\{k_0^2(M)\cdot\mathbf{E}^i(Q)+\left(ik_0+\frac{1}{R}\right)\hat{\mathbf{R}}(\nabla\cdot\mathbf{E}^i)\right\}dQ-\mathbf{E}^i(P). \tag{13}$$

3. The Low-Frequency, Far-Zone Approximate Form of the Integral Equation

In the sequel attention will be restricted to the case of low frequency electromagnetic waves, i.e., those fields for which the free space wave number is such that

$$k_0<<1. \tag{14}$$

However, this hypothesis does not imply that the first term of the integrand in (10) or (12)) is necessarily small, for even in a weakly ionized magneto-ionic medium the product $||k_0^2(M)||=\omega\mu_0||(\sigma)||$ can be large whenever $\omega<<\omega_H<<\omega_e^2/\omega$.

We further require that the field point P be in the far-zone with respect to the distribution of ionization, the latter being assumed to be confined to a finite volume V_c of space. Thus we can impose the hypothesis that

$$R_{PQ}>>\lambda\!\!\!^{-}_0=\lambda_0/(2\pi)\quad\text{when}\quad Q\epsilon V_c. \tag{15}$$

The second term in the integrand of (10) (and (12)) contains the quantity div $\mathbf{E}$, which since we assume quasi-electrostatic equilibrium, can be identified as proportional to the polarization charge density:

$$\nabla\cdot\mathbf{E}=\rho_P/\epsilon_0$$

But the equation of continuity implies, since $\mathbf{J}=\dfrac{\partial\mathbf{P}}{\partial t}$, that

$$\nabla\cdot\mathbf{E}=-\frac{\nabla\cdot\mathbf{J}}{i\omega\epsilon_0} \tag{16}$$

where $\mathbf{J}$ is a conduction current occurring only in the ionized portion of space:

$$\mathbf{J}=(\sigma)\cdot\mathbf{E}. \tag{17}$$

That is

$$\nabla\cdot\mathbf{E}=\begin{cases}-\nabla\cdot((M)\cdot\mathbf{E})\text{ in }V_c\\ 0,\text{ in }V_3-V_c.\end{cases} \tag{18}$$

For those source points Q in V_c, the presence of ionization implies that $(\sigma)\neq(0)$, and so $(M)\neq(0)$ and $\nabla\cdot\mathbf{E}\neq0$; for all other Q, $(\sigma)=(0)$, and so (M) and $\nabla\cdot\mathbf{E}$ vanish. Therefore the only contribution to the integrals in (10) (and (12)) can arise from those Q which are both in V_c and such that $R>>\lambda_0$. If we now impose the hypothesis that

$$|\nabla\cdot((M)\cdot\mathbf{E})|\leq|(M)\cdot\mathbf{E}|, \tag{19}$$

(i.e., a slowly-varying distribution of ionization), then (19) together with (14) and (15) implies that the second term in the integrand of (10) (and (12)) is of higher order than the first. We may therefore approximate (10) by the polar integral equation

$$\mathbf{E}(P)=\frac{k_0^2}{4\pi}\int_{V_3}g(P,Q)(M)\cdot\mathbf{E}(Q)dQ. \tag{20}$$

We state this result as a theorem:

Theorem 1. Under hypotheses (14), (15), and (19) the vector integral equation reduces to the approximate form

$$\mathbf{E}(P)=\frac{k_0^2}{4\pi}\int_{V_3} g(P,Q)(M)\cdot\mathbf{E}(Q)dQ. \tag{20}$$

Corollary 1.1. Under the same hypotheses the integral equation (13) in the scattered field $\mathbf{E}^s$ reduces to the approximate form

$$\mathbf{E}^s(P)=\mathbf{F}^i(P)+\frac{k_0^2}{4\pi}\int_{V_3} g(P,Q)\,(M)\cdot\mathbf{E}^s(Q)\,dQ, \tag{21}$$

where $\mathbf{F}^i(P)$ is given by (13).

Corollary 1.2. For a cold plasma ($\nu=0$) hypothesis (19) may be replaced by the hypothesis that $X\gg1$, $Y\gg1$ over V_c.

Proof. Since

$$||M||=\left\|\begin{pmatrix}\frac{X}{1-Y^2} & i\frac{XY}{1-Y^2} & 0\\ -i\frac{XY}{1-Y^2} & -\frac{X}{1-Y^2} & 0\\ 0 & 0 & -X\end{pmatrix}\right\|$$

$$=X\left(2\frac{1+Y}{|1-Y^2|}+1\right)=X\left(\frac{2}{|1-Y|}+1\right) \tag{22}$$

the conclusion follows immediately.

4. Transformation to a System of Uncoupled Scalar Integral Equations

In an appropriate orthogonal basis ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) (which may, in particular, be the base vectors for cartesian or cylindrical coordinates),

$$(M)\cdot\mathbf{E}=[(B-1)E_1+CE_2]\mathbf{e}_1+[(B-1)E_2-CE_1]\mathbf{e}_2+(A-1)E_3\mathbf{e}_3. \tag{23}$$

The vector integral eq(20) can then be written as the coupled system of scalar integral equations

$$\left.\begin{aligned}E_1(P)&=\frac{k_0^2}{4\pi}\int_{V_3} g(P,Q)[(B-1)E_1(Q)+CE_2(Q)]dQ\\ E_2(P)&=\frac{k_0^2}{4\pi}\int_{V_3} g(P,Q)[-CE_1(Q)+(B-1)E_2]dQ\\ E_3(P)&=\frac{k_0^2}{4\pi}\int_{V_3} g(P,Q)(A-1)E_3(Q)dQ\end{aligned}\right\} \tag{24}$$

with a corresponding form for eq(21).

Now define the following quantities:

$$\mathscr{E}_{1,2}(P)=E_1\pm iE_2 \tag{25}$$

$$\mathscr{E}_3(P)=E_3 \tag{26}$$

$$\mathscr{F}_{1,2}(P)=F_1\pm iF_2 \tag{27}$$

$$\mathscr{B}_{1,2}(P)=B-1\mp iC \tag{28}$$

$$\mathscr{A}(P)=A-1 \tag{29}$$

We note that $\mathscr{E}_1$ and $\mathscr{E}_2$ represent counter-rotating circularly polarized components about the x_3-axis, while $\mathscr{E}_3$ is the longitudinal field component directed along the x_3-axis. Further, under the hypothesis ($X\gg1$, $Y\gg1$), of corollary 2

$$\mathscr{B}_{1,2}\doteq X/Y=\mp\frac{\omega_e^2}{\omega\omega_H}, \tag{30}$$

and

$$\mathscr{A}\doteq -X=-\omega_e^2/\omega^2. \tag{31}$$

Then the system (24) can be written as the following *uncoupled* system of scalar integral equations in $\mathscr{E}_1$, $\mathscr{E}_2$, $\mathscr{E}_3$:

$$\left\{\begin{aligned}\mathscr{E}_{1,2}(P)&=\frac{k_0^2}{4\pi}\int_{V_3} g(P,Q)\mathscr{B}_{1,2}(Q)\mathscr{E}_{1,2}(Q)dQ\\ \mathscr{E}_3(P)&=\frac{k_0^2}{4\pi}\int_{V_3} g(P,Q)\mathscr{A}(Q)\mathscr{E}_3(Q)dQ\end{aligned}\right. \tag{32}$$

with a corresponding uncoupled form for the component form of eq(21):

$$\left\{\begin{aligned}\mathscr{E}^s_{1,2}(P)&=\mathscr{F}^i_{1,2}(P)+\frac{k_0^2}{4\pi}\int_{V_3} g(P,Q)\mathscr{B}_{1,2}(Q)\mathscr{E}^s_{1,2}(Q)dQ\\ \mathscr{E}^s_3(P)&=F^i_3(P)+\frac{k_0^2}{4\pi}\int_{V_3} g(P,Q)\mathscr{A}(Q)\mathscr{E}^s_3(Q)dQ.\end{aligned}\right. \tag{33}$$

5. Equivalences Between the Vector Wave Equation and Approximate Systems of Uncoupled Scalar Helmholtz Equations

The following equivalences can now be stated:

Theorem 2. Under hypotheses (14), (15), and (19) above, the vector wave eq(2) can be approximated by the uncoupled system of scalar Helmholtz equations:

$$\nabla^2\begin{pmatrix}E_1+iE_2\\ E_1-iE_2\\ E_3\end{pmatrix}+k_0^2\begin{pmatrix}B-iC & 0 & 0\\ 0 & B+iC & 0\\ 0 & 0 & A\end{pmatrix}\begin{pmatrix}E_1+iE_2\\ E_1-iE_2\\ E_3\end{pmatrix}=\mathbf{0}. \tag{34}$$

Proof. This follows immediately from applying to the system (32) the equivalence between the integral equation $u(P)-\Lambda\int g(P,Q)p(Q)u(Q)dQ=0$ and the boundary value problem consisting of $\nabla^2 u+\Lambda[1+p(P)]\,u(P)=0$ together with homogeneous Dirichlet or Neumann boundary conditions and/or a radiation condition, and at the same time taking into account definitions (25), (26), (28), and (29).

Theorem 3. Under hypotheses (14), (15), and (19) above, and the assignment of an $\mathbf{E}^i(P)$ such that there exists an $\mathscr{f}^i(P)$ satisfying

$$\mathscr{F}^i(P)=\frac{1}{4\pi}\int_{V_3} g(P,Q)\,\mathscr{f}^i(Q)dQ, \tag{35}$$

the vector wave equation curl curl $\mathbf{E}^s-k_0^2(\check{\kappa})\cdot\mathbf{E}^s=\mathbf{f}^i(P)$ is essentially equivalent to the uncoupled system of scalar inhomogeneous Helmholtz equations:

$$\nabla^2\begin{pmatrix}E_1^s+iE_2^s\\ E_2^s-iE_2^s\\ E_3^s\end{pmatrix}+k_0^2\begin{pmatrix}B-iC & 0 & 0\\ 0 & B+iC & 0\\ 0 & 0 & A\end{pmatrix}\cdot\begin{pmatrix}E_1^s+iE_2^s\\ E_1^s-iE_2^s\\ E_3^s\end{pmatrix}=\begin{pmatrix}f_1^i+if_2^i\\ f_1^i-if_2^i\\ f_3^i\end{pmatrix}. \tag{36}$$

Proof. The theorem follows in the same way as theorem 2 from the equivalence between the integral equation

$$u(P)-\Lambda\int g(P,Q)p(Q)u(Q)dQ=\int g(P,Q)q(Q)dQ$$

and the inhomogeneous Helmholtz equation $\nabla^2 u+\Lambda[1+p(P)]\,u(P)=q(P)$ with homogeneous boundary conditions and/or a radiation condition.

6. Solution of the Uncoupled Integral Equations by means of Neumann Series

Each of eqs (33) is of the form of a singular Fredholm integral equation with symmetrizable kernel. Since the domain V_3 has been taken to be all of 3-space, and so is infinite, the integral equations will be singular. However, Neumann series solutions of the integral equations can still be defined [Schmeidler, 1955] in the usual form and subsequently shown to actually converge to the solutions of the respective integral equations, provided the kernels $K_{1,2}(P,Q)=g(P,Q)\mathscr{B}_{1,2}(Q)$ and $K_3(P,Q)=g(P,Q)\mathscr{A}(Q)$ are integrable over V_3 with respect to every function $d\xi\,(P)$ in L_2 and with respect to every function $d\eta(Q)$ in L_2, and both of the following iterated integrals exist and are equal:

$$\int_{V_3}\xi'(P)\left(\int_{V_3}K_h(P,Q)\eta'(Q)dQ\right)dP$$

$$=\int_{V_3}\eta'(Q)\left(\int_{V_3}K_h(P,Q)\xi'(P)dP\right)dQ,\ (h=1,2,3)$$

The way to proceed is described by the following theorem [Schmeidler, 1955]: For a kernel $K(P,Q)$ which satisfies the three requirements just stated and whose associated bilinear form is absolutely bounded:

$$\left|\int_{V_3\times V_3}K(P,Q)\xi'^*(P)\eta'(Q)dPdQ\right|<M,$$

The Fredholm equation of the second kind

$$y(P)-\Lambda\int_{V_3}K(P,Q)y(Q)dQ=F(P),\quad y\epsilon L_2$$

has a unique solution in the form of a mean-square convergent Neumann series

$$y(P)=l.i.m.\left\{F(P)+\Lambda\int K(P,Q)F(Q)dQ\right.$$

$$\left.+\Lambda^2\int K(PQ)\left(\int K(Q,Q')F(Q')dQ'\right)dQ+\cdots\right\}.$$

In the present application the hypotheses of this theorem are equivalent to

$$\left|\int_{V_3\times V_3}g(P,Q)\mathscr{B}_{1,2}(Q)\xi'^*(P)\eta'(Q)dPdQ\right|<M_{1,2} \tag{37}$$

and

$$\left|\int_{V_3\times V_3}g(P,Q)\mathscr{A}(Q)\xi'^*(P)\eta'(Q)dPdQ\right|<M_3 \tag{38}$$

where $\xi'(P)$ and $\eta'(P)$ are arbitrary L_2 functions on V_3, and

$$M_g<\pi\lambda_0^2.\qquad(g=1,2,3) \tag{39}$$

The zero-order terms in the Neumann series solution for $\mathscr{E}_1^s(P)$, $\mathscr{E}_2^s(P)$, and $\mathscr{E}_3^s(P)$ are simply $\mathscr{F}_1^i(P)$, $\mathscr{F}_2^i(P)$, and $F_3^i(P)$, respectively. The respective first-order terms (Born approximation) are given by the integrals:

$$\frac{k_0^2}{4\pi}\int_{V_3}g(P,Q)\mathscr{B}_{1,2}(Q)\mathscr{F}_{1,2}^i(Q)dQ \tag{40}$$

and

$$\frac{k_0^2}{4\pi}\int_{V_3}g(P,Q)\mathscr{A}(Q)F_3^i(Q)dQ. \tag{41}$$

The third terms in the corresponding Neumann series are, respectively,

$$\left(\frac{k_0^2}{4\pi}\right)^2\int_{V_3} dQ\, g(P,Q)\mathscr{B}_{1,2}(Q)\int_{V_3} g(Q,Q')\mathscr{B}_{1,2}(Q')\mathscr{F}^i_{1,2}(Q')dQ' \tag{42}$$

and

$$\left(\frac{k_0^2}{4\pi}\right)^2\int_{V_3} dQ\, g(P,Q)\mathscr{A}(Q)\int_{V_3} g(Q,Q')\mathscr{A}(Q')F^i_3(Q')dQ', \tag{43}$$

etc. Since $k_0^2/4\pi<<1$ in the low frequency case, the first two or three terms in the Neumann series will ordinarily suffice for numerical purposes.

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