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Estimation of Dispersion Parameters

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This paper deals with a topic in multivariate analysis. Consider that a sample of size $n+1$ has been collected from a p-variate normal distribution having dispersion matrix $(\sigma_{jj'})$. Let $a_{jj'}/n$ denote the usual unbiased estimate of $\sigma_{jj'}$. Further, let $0<l<u$ be constants such that all characteristic roots of a matrix having the Wishart distribution lie in the interval $[l, u]$ with probability $1-\alpha$. A theorem of Roy, Bose, and Gnanadesikan [Ann. Math. Stat. **24,** 513–536 (1953); Biometrika **44,** 399–410 (1957)] may be stated as follows: The probability is $1-\alpha$ that every principal minor determinant of $l^{-1}(a_{jj'})-(\sigma_{jj'})$ and of $(\sigma_{jj'})-u^{-1}(a_{jj'})$ is nonnegative. The previous result may be used to prove the main theorem of the present paper. *Theorem:* The probability is at least $1-\alpha$ that the following system of relations hold simultaneously: $u^{-1}a_{jj}\leq\sigma_{jj}\leq l^{-1}a_{jj}$; $j=1, \ldots, p$ and $|\sigma_{jj'}-\frac{1}{2}(u^{-1}+l^{-1})a_{jj'}|\leq\frac{1}{2}(l^{-1}-u^{-1})(a_{jj}a_{j'j'})^{1/2}$, $j\neq j'$.

1. Notation

Throughout $Y_i=(y_{i1}, \ldots, y_{ip})'$ for $i=1, \ldots, N$ will be a sample of size $N=n+1$ from a p-variate normal distribution with mean vector $\xi=(\xi_1 \ldots \xi_p)'$ and dispersion matrix Σ. In short, the Y_i are independent and $N(\xi, \Sigma)$. Except for α and $\sum$, Greek letters will always refer to parameters. We will frequently write $\Sigma=(\sigma_{jj'})$ when we mean that $\sigma_{jj'}$ is the element in the jth row and j'th column of Σ. In the same spirit $\Sigma^{-1}=(\sigma^{jj'})$, and $A=(a_{jj'})$ will be common notations. Here

$$a_{jj'}=\sum_{i=1}^{N} (y_{ij}-\bar{y}_{ij})(y_{ij'}-\bar{y}_{ij'}). \tag{1.1}$$

2. Simultaneous Confidence Intervals

Let $0<l<u$ be constants such that all characteristic roots of a $p\times p$ matrix having the distribution $W(I, n)$ (see [1][2] for this notation) are in the interval $[l, u]$ with probability $1-\alpha$. The following theorem is, for our purposes, a more convenient statement of some results due to Roy, Bose, and Gnandesikan [8, 9].

THEOREM 1. The probability is $1-\alpha$ that $l^{-1}A-\Sigma$ and $\Sigma-u^{-1}A$ are positive semi-definite.

A proof is included since the original verification demonstrates many other interesting results as well, and consequently is very indirect. There exists a nonsingular triangular matrix C such that $C\Sigma C'=I$ [1, p. 156]. Making the transformation $Z_i=CY_i (i=1, \ldots, N)$ then Z_i has the distribution $N(C\xi, I)$ and $B\equiv \sum_{i=1}^{N} (Z_i-\bar{Z})(Z_i-\bar{Z})'=CAC'$ has the distribution $W(I,n)$. The theorem will be proved when it is shown that the following three statements are equivalent. (i) all roots of B are in the interval $[l, u]$. (ii) $B-lI$ and $uI-B$ are p.s.d. (positive semi-definite). (iii) $A-l\Sigma$ and $u\Sigma-A$ are p.s.d. (i) and (ii) are clearly equivalent. That $B-lI$ and $A-l\Sigma$ are p.s.d. together may be shown by considering their quadratic forms. Making the transformation $Q=CR$, where Q and R are column vectors, we have $Q'(A-l\Sigma)Q=R'(C'AC-lC'\Sigma C)R=R'(B-lI)R$. Similarly the matrices $uI-B$ and $u\Sigma-A$ are p.s.d. together.

An equivalent statement of theorem 1 is

THEOREM 1′. The probability is $1-\alpha$ that every principal minor determinant of $l^{-1}A-\Sigma$ and of $\Sigma-u^{-1}A$ is nonnegative.

Proof. The event whose probability is being calculated in 1′ is identical with that of theorem 1 [3, theorem 46.4].

Clearly the previous theorem provides the means of determining a simultaneous confidence region for the elements of the dispersion matrix Σ. However this region, call it $\mathscr{R}$, may or may not be interesting depending on its shape. We now begin an investigation of the shape of $\mathscr{R}$. Let $\mathscr{S}=(\Sigma{:}l^{-1}A-\Sigma$ is p.s.d.) and $\mathscr{T}=(\Sigma{:}\Sigma-u^{-1}A$ is p.s.d.). Then $\mathscr{R}=\mathscr{S}\cap\mathscr{T}$, the intersection or common part of $\mathscr{S}$ and $\mathscr{T}$.

THEOREM 2. $\mathscr{S}$ and $\mathscr{T}$ are convex cones with vertices $\Sigma=l^{-1}A$ and $\Sigma=u^{-1}A$, respectively.

Proof. Assume $\Sigma_1\epsilon\mathscr{S}$, $\Sigma_2\epsilon\mathscr{S}$ and $c\geq0$. Clearly $l^{-1}A\epsilon\mathscr{S}$. Further $c\Sigma_1+(1-c)l^{-1}A\epsilon\mathscr{S}$ since $l^{-1}A-[c\Sigma_1+(1-c)l^{-1}A]=c(l^{-1}A-\Sigma_1)$. Hence $\mathscr{S}$ is a cone with vertex $l^{-1}A$. To show $\mathscr{S}$ is convex we must demonstrate that $c\Sigma_1+(1-c)\Sigma_2\epsilon\mathscr{S}$ whenever $c\leq1$ But the quadratic form $Q'[l^{-1}A-c\Sigma_1-(1-c)\Sigma_2]Q=cQ'(l^{-1}A-\Sigma_1)Q+(1-c)Q'(l^{-1}A-\Sigma_2)Q$ is nonnegative and $\mathscr{S}$ is a convex cone. The proof for $\mathscr{T}$ is similar.

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[2] Figures in brackets indicate the literature references at the end of this paper.

According to theorem 2, $\mathscr{R}$ is the intersection of two convex cones. Further on in the paper (theorem 3) we shall prove a result which implies that $\mathscr{R}$ is bounded. Hence the confidence region $\mathscr{R}$ is convex and bounded.

Now, in the case $p=2$, define

$$w_1=\sigma_{11}-u^{-1}a_{11},\ w_2=\sigma_{12}-u^{-1}a_{12},\ w_3=\sigma_{22}-u^{-1}a_{22},$$
$$d_1=(l^{-1}-u^{-1})a_{11},\ d_2=(l^{-1}-u^{-1})a_{12},$$
$$d_3=(l^{-1}-u^{-1})\,a_{22}, \tag{2.1}$$

$$W=\begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix} \text{ and } D=\begin{pmatrix} d_1 & d_2 \\ d_2 & d_3 \end{pmatrix}.$$

$\mathscr{T}$ and $\mathscr{S}$ become $[w_1, w_2, w_3 : w_1\geq 0,\ w_3\geq 0$ and $w_2^2\leq w_1 w_3]$ and $[w_1, w_2, w_3 : w_1\leq d_1,\ w_3\leq d_3$ and $(w_2-d_2)^2\leq (w_1-d_1)(w_3-d_3)]$, respectively. Solving simultaneously we find that the intersection of the boundaries of $\mathscr{S}$ and $\mathscr{T}$ takes place on the plane $d_1w_3+d_3w_1-2d_2w_2=d_1d_3-d_2^2=|D|$. On this plane the equation of the required intersection is $d_3w_2^2=(|D|+2d_2w_2-d_1w_3)w_3$ or

$$d_3w_2^2-2d_2w_2w_3+d_1w_3^2-|D|w_3=0 \tag{2.2}$$

which will be an ellipse since $D=(l^{-1}-u^{-1})A$ is positive definite with probability one.

Since the vertices of $\mathscr{S}$ and $\mathscr{T}$ satisfy (2.2) the extent of w_2 and w_3 in $\mathscr{R}$ will be the same as their extent in the ellipse. To find the extent of w_2 in (2.2) consider w_2 to be fixed and solve for w_3. (2.2) is then a quadratic equation with discriminant $(|D|+2d_2w_2)^2-4d_1d_3w_2^2$. If w_3 is to be real, then the discriminant must be nonnegative; hence the extent of w_2 is given by the relation $(|D+2d_2w_2)^2\geq 4d_1d_3w_2^2$ or equivalently $|w_2-d_2/2|\leq \frac{1}{2}(d_1d_3)^{\frac{1}{2}}$. The extent of w_3 is even easier to calculate by the same methods. Solving (2.2) for w_2 the discriminant reduces to $4|D|w_3(d_3-w_3)$, which yields $0\leq w_3\leq d_3$. Similarly $0\leq w_1\leq d_1$. Using theorem 1′ and the eq (2.1) we summarize these computations as

LEMMA 1. In the case of $p=2$, the probability is at least $1-\alpha$ that the following three relations hold simultaneously

$$u^{-1}a_{11}\leq\sigma_{11}\leq l^{-1}a_{11},\ u^{-1}a_{22}\leq\sigma_{22}\leq l^{-1}a_{22},$$
$$\left|\sigma_{12}-\frac{u^{-1}+l^{-1}}{2}\,a_{12}\right|\leq\frac{l^{-1}-u^{-1}}{2}\,(a_{11}a_{22})^{\frac{1}{2}}.$$

Now consider the general case, where p is not necessarily equal to 2. In theorem 1′ we may choose to ignore all principal minor determinants of order greater than 2. If we do this, then the probability of the resulting event can only be increased. For each pair of variates, and by identical methods, we will obtain a system of relations just like those of lemma 1.

THEOREM 3. The probability is at least $1-\alpha$ that the following system of relations hold simultaneously

$$u^{-1}a_{jj}\leq\sigma_{jj}\leq l^{-1}a_{jj},\ j=1,\ldots,p$$

and

$$\left|\sigma_{jj'}-\frac{u^{-1}+l^{-1}}{2}\,a_{jj'}\right|\leq\frac{l^{-1}-u^{-1}}{2}\,(a_{jj}a_{j'j'})^{\frac{1}{2}},\ j\neq j'.$$

3. Precision of Instruments

A special multivariate model which frequently arises in connection with simultaneous measurement procedures [5, 11] requires that $\sigma_{jj}=\sigma^2+\sigma_j^2,\ j=1,\ldots, p$ and $\sigma_{jj'}=\sigma^2,\ j\neq j'$. For this application, p becomes the number of instruments used. The methods of the previous section yield interesting results when applied to this special case. Returning to eq (2.2) we calculate the extent of $t=w_3-w_2$; this will yield bounds for σ_2^2 in the two instrument case. Making the substitution $w_3=t+w_2$, eq (2.2) becomes

$$w_2^2(d_1+d_3-2d_2)+2tw_2(d_1-d_2)+d_1t^2-|D|(t+w_2)=0.$$

Computations similar to those made for w_2 show that the extent of t is $\frac{1}{2}(d_3-d_2)\pm\frac{1}{2}[d_3(d_1+d_3-2d_2)]^{\frac{1}{2}}$.

Equations (2.1) and theorem 1′ now yield

LEMMA 2. In the two-instrument case, the probability is at least $1-\alpha$ that the following three relations hold simultaneously

$$\left|\sigma^2-a_{12}\,\frac{l^{-1}+u^{-1}}{2}\right|\leq\frac{l^{-1}-u^{-1}}{2}\,(a_{11}a_{22})^{\frac{1}{2}},$$

$$\left|\sigma_1^2-(a_{11}-a_{12})\,\frac{l^{-1}+u^{-1}}{2}\right|\leq\frac{l^{-1}-u^{-1}}{2}\,[a_{11}(a_{11}+a_{22}-2a_{12})]^{\frac{1}{2}},$$

$$\left|\sigma_2^2-(a_{22}-a_{12})\,\frac{l^{-1}+u^{-1}}{2}\right|\leq\frac{l^{-1}-u^{-1}}{2}\,[a_{22}(a_{11}+a_{22}-2a_{12})]^{\frac{1}{2}}.$$

For more than two instruments we may again choose to ignore all principal minor determinants of order in excess of 2. Theorem 1′ then gives

THEOREM 4. The probability is at least $1-\alpha$ that the following relations hold simultaneously

$$\max_{j\neq j'}\left[a_{jj'}\,\frac{l^{-1}+u^{-1}}{2}-\frac{l^{-1}-u^{-1}}{2}\,(a_{jj}a_{j'j'})^{\frac{1}{2}}\right]$$
$$\leq\sigma^2\leq\min_{j\neq j'}\left[a_{jj'}\,\frac{l^{-1}+u^{-1}}{2}+\frac{l^{-1}-u^{-1}}{2}\,(a_{jj}a_{j'j'})^{\frac{1}{2}}\right],$$

$$\max_{j\neq 1}\left\{(a_{11}-a_{1j})\,\frac{l^{-1}+u^{-1}}{2}-\frac{l^{-1}-u^{-1}}{2}\,[a_{11}(a_{11}+a_{jj}-2a_{1j})]^{\frac{1}{2}}\right\}$$
$$\leq\sigma_1^2\leq\min_{j\neq 1}\left\{(a_{11}-a_{1j})\,\frac{l^{-1}+u^{-1}}{2}+\frac{l^{-1}-u^{-1}}{2}\,[a_{11}(a_{11}+a_{jj}-2a_{1j})]^{\frac{1}{2}}\right\}.$$

plus similar inequalities involving $\sigma_2^2, \ldots \sigma_?^2$.

4. Choosing the Bounds

The joint distribution of the characteristic roots of a Wishart matrix is well known, see for example [1, theorem 13.3.2]. Also, there is much theoretical work on the distributions of the extreme roots [10]. Tables and charts have been prepared by Pillai [7] and Heck [6] from which critical values of the largest root may be determined for certain combinations of n, p, and α. However, critical values of the largest root are of no value in this application without some knowledge of the smallest root. The tabulations of Pillai and Heck do not treat large values of their parameter m and consequently, even though there is a relation between the distribution of the largest and smallest root, their results cannot be used to determine critical values of the smallest root in the present instance. The only result which seems to be ready for use in determining the bounds is an approximation due to P. L. Hsu (for an exposition see [2]). In our terminology, Hsu's result states that $P(l \leq \chi^2_{n-p+2})$ is, for large n, an approximation to the probability that the minimum root is at least as large as l.

Here we present an exact treatment of the two variate case ($p=2$). Denote the roots by r and s where $r \leq s$. Our task is to determine constants l and u such that $P(l \leq r \leq s \leq u) = 1-\alpha$.

$$P = P(l \leq r \leq s \leq u)$$

$$= c \int_{s=l}^{u} \int_{r=l}^{s} (rs)^{\frac{n-3}{2}} \exp\left[-\frac{r+s}{2}\right] (s-r)\, dr ds,$$

where $c = \sqrt{\pi}\left[2^n \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n-1}{2}\right)\right]^{-1}$. Making the transformation $r=t-v$, $s=t+v$ then $P/4c$ may be written as

$$\int_{t=l}^{\frac{l+u}{2}} e^{-t} \int_{v=0}^{t-l} v(t^2-v^2)^{\frac{n-3}{2}}\, dv dt$$

$$+ \int_{t=\frac{l+u}{2}}^{u} e^{-t} \int_{v=0}^{u-t} v(t^2-v^2)^{\frac{n-3}{2}}\, dv dt.$$

Now, integrating with respect to v and transforming the resultant expressions, we obtain $(n-1)P/2c$ in the form

$$\int_{2l}^{2u} (x/2)^{n-1} e^{-x/2} dx$$

$$-\left(l^{\frac{n-1}{2}} e^{-\frac{l}{2}} + u^{\frac{n-1}{2}} e^{-\frac{u}{2}}\right) \int_{l}^{u} x^{\frac{n-1}{2}} e^{-\frac{x}{2}} dx.$$

Finally remembering the value of c and using the expression $\sqrt{\pi}\,\Gamma(n) = 2^{n-1}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)$ we obtain

$$P(l \leq r \leq s \leq u) \tag{4.1}$$

$$= P(2l \leq \chi^2_{2n} \leq 2u)$$

$$-\frac{\sqrt{\pi}}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n}{2}\right)} \left(l^{\frac{n-1}{2}} e^{-\frac{l}{2}} + u^{\frac{n-1}{2}} e^{-\frac{u}{2}}\right) P(l \leq \chi^2_{n+1} \leq u),$$

where χ^2_{2n} is a chi-square variate with $2n$ degrees of freedom.

As is usual in such problems, there is considerable freedom in choosing l and u so that $P=1-\alpha$. In table 1, u is taken to be $+\infty$ and in table 2, l equals 0. The bounds of table 3 are chosen as follows: Determine l so that $P(l \leq r) = 1-\alpha$, then holding l fixed at this value determine u_2 so that $P(l \leq r \leq s \leq u_2) = 1-2\alpha$. Note that the value of u_2 obtained in this way agrees very closely with the corresponding entry of table 2. It appears that the probabilistic dependence of the greatest and least root may be ignored for the purpose of this paper.

Table 1 was used to check the accuracy of Hsu's approximation in the bivariate case. The approximation runs from about 15 percent too large for $n=60$ to about 11 percent too large for $n=100$. Presumably the approximation would be equally poor in general, except for very large sample sizes.

TABLE 1. *Percentage points l of the smallest characteristic root of a bivariate Wishart matrix having sample size* n+1

$P(l \leq \text{both roots}) = 1-\alpha$

n \ α	0.005	0.01	0.025	0.05
2	1.5815 (-5)*	6.2868 (-5)	3.8585 (-4)	1.4999 (-3)
3	5.0125 (-3)	1.0050 (-2)	2.5318 (-2)	5.1293 (-2)
4	4.0472 (-2)	6.4770 (-2)	1.2161 (-1)	1.9810 (-1)
5	1.2641 (-1)	1.8124 (-1)	2.9488 (-1)	4.3175 (-1)
6	2.6589 (-1)	3.5734 (-1)	5.3421 (-1)	7.3400 (-1)
7	4.5495 (-1)	5.8588 (-1)	8.2809 (-1)	1.0905
8	6.8802 (-1)	8.5956 (-1)	1.1672	1.4910
9	9.5975 (-1)	1.1721	1.5444	1.9277
10	1.2655	1.5183	1.9539	2.3949
11	1.6012	1,8940	2,3915	2,8883
12	1.9636	2.2957	2.8535	3.4045
13	2.3498	2.7204	3.3370	3.9406
14	2.7575	3.1659	3.8398	4.4944
15	3.1846	3.6300	4.3599	5.0639
16	3.6293	4.1108	4.8952	5.6474
17	4.0906	4.6074	5.4454	6.2446
18	4.5665	5.1181	6.0082	6.8533
19	5.0561	5.6418	6.5828	7.4728
20	5.5586	6.1777	7.1685	8.1023
22	6.5982	7.2822	8.3699	9.3884
24	7.6793	8.4257	9.6068	1.0707 (1)
26	8.7970	9.6038	1.0875 (1)	1.2053 (1)
28	9.9472	1.0813 (1)	1.2170 (1)	1.3424 (1)
30	1.1126 (1)	1.2049 (1)	1.3491 (1)	1.4817 (1)
35	1.4185 (1)	1.5243 (1)	1.6885 (1)	1.8382 (1)
40	1.7376 (1)	1.8561 (1)	2.0387 (1)	2.2049 (1)
45	2.0676 (1)	2.1982 (1)	2.3983 (1)	2.5797 (1)
50	2.4068 (1)	2.5482 (1)	2.7655 (1)	2.9613 (1)
60	3.1068 (1)	3.2698 (1)	3.5185 (1)	3.7413 (1)
70	3.8313 (1)	4.0139 (1)	4.2912 (1)	4.5388 (1)
80	4.5745 (1)	4.7755 (1)	5.0797 (1)	5.3504 (1)
90	5.3333 (1)	5.5515 (1)	5.8811 (1)	6.1735 (1)
100	6.1051 (1)	6.3397 (1)	6.6932 (1)	7.0062 (1)

*The symbol (i) means that the tabled entry is to be multiplied by 10^i.

TABLE 2. *Percentage points* u_1 *of the largest characteristic root of a bivariate Wishart matrix having sample size* $n+1$

$P(\text{both roots} \leq u_1) = 1-\alpha$

n \ α	0.005	0.01	0.025	0.05
2	1.3663 (1)*	1.2160 (1)	1.0147 (1)	8.5948
3	1.6162 (1)	1.4568 (1)	1.2416 (1)	1.0740 (1)
4	1.8399 (1)	1.6727 (1)	1.4457 (1)	1.2677 (1)
5	2.0476 (1)	1.8735 (1)	1.6360 (1)	1.4489 (1)
6	2.2444 (1)	2.0639 (1)	1.8170 (1)	1.6214 (1)
7	2.4331 (1)	2.2466 (1)	1.9909 (1)	1.7878 (1)
8	2.6153 (1)	2.4234 (1)	2.1594 (1)	1.9491 (1)
9	2.7924 (1)	2.5952 (1)	2.3235 (1)	2.1065 (1)
10	2.9654 (1)	2.7631 (1)	2.4840 (1)	2.2606 (1)
11	3.1344 (1)	2.9275 (1)	2.6415 (1)	2.4120 (1)
12	3.3004 (1)	3.0889 (1)	2.7962 (1)	2.5610 (1)
13	3.4636 (1)	3.2479 (1)	2.9487 (1)	2.7079 (1)
14	3.6246 (1)	3.4045 (1)	3.0991 (1)	2.8529 (1)
15	3.7832 (1)	3.5591 (1)	3.2477 (1)	2.9967 (1)
16	3.9400 (1)	3.7119 (1)	3.3948 (1)	3.1387 (1)
17	4.0949 (1)	3.8630 (1)	3.5400 (1)	3.2793 (1)
18	4.2482 (1)	4.0126 (1)	3.6842 (1)	3.4188 (1)
19	4.4001 (1)	4.1608 (1)	3.8270 (1)	3.5570 (1)
20	4.5505 (1)	4.3078 (1)	3.9694 (1)	3.6944 (1)
22	4.8479 (1)	4.5981 (1)	4.2498 (1)	3.9662 (1)
24	5.1408 (1)	4.8845 (1)	4.5266 (1)	4.2348 (1)
26	5.4300 (1)	5.1682 (1)	4.8005 (1)	4.5007 (1)
28	5.7157 (1)	5.4480 (1)	5.0712 ( )	4.7638 (1)
30	5.9991 (1)	5.7249 (1)	5.3394 (1)	5.0249 (1)
35	6.6949 (1)	6.4065 (1)	6.0008 (1)	5.6689 (1)
40	7.3767 (1)	7.0755 (1)	6.6509 (1)	6.3027 (1)
45	8.0476 (1)	7.7342 (1)	7.2922 (1)	6.9284 (1)
50	8.7096 (1)	8.3843 (1)	7.9253 (1)	7.5471 (1)
60	1.0009 (2)	9.6623 (1)	9.1689 (1)	8.7586 (1)
70	1.1286 (2)	1.0919 (2)	1.0397 (2)	9.9614 (1)
80	1.2543 (2)	1.2158 (2)	1.1609 (2)	1.1150 (2)
90	1.3785 (2)	1.3382 (2)	1.2807 (2)	1.2327 (2)
100	1.5014 (2)	1.4594 (2)	1.3996 (2)	1.3495 (2)

*The symbol (i) means that the tabled entry is to be multiplied by 10^i.

TABLE 3. *Percentage points* u_2 *to be used in conjunction with table 1 for obtaining simultaneous upper and lower bounds on the roots of a bivariate Wishart matrix*

$P(l \leq \text{both roots} \leq u_2) = 1-2\alpha$

n \ 2α	0.01	0.05
2	1.3648 (1)*	1.0066 (1)
3	1.6149 (1)	1.2344 (1)
4	1.8387 (1)	1.4388 (1)
5	2.0465 (1)	1.6293 (1)
6	2.2433 (1)	1.8103 (1)
7	2.4320 (1)	1.9842 (1)
8	2.6142 (1)	2.1528 (1)
9	2.7913 (1)	2.3169 (1)
10	2.9641 (1)	2.4773 (1)
11	3.1332 (1)	2.6347 (1)
12	3.2994 (1)	2.7894 (1)
13	3.4626 (1)	2.9418 (1)
14	3.6234 (1)	3.0921 (1)
15	3.7821 (1)	3.2407 (1)
16	3.9389 (1)	3.3877 (1)
17	4.0937 (1)	3.5329 (1)
18	4.2470 (1)	3.6770 (1)
19	4.3989 (1)	3.8198 (1)
20	4.5494 (1)	3.9620 (1)
22	4.8467 (1)	4.2424 (1)
24	5.1397 (1)	4.5191 (1)
26	5.4288 (1)	4.7926 (1)
28	5.7145 (1)	5.0635 (1)
30	5.9978 (1)	5.3316 (1)
35	6.6937 (1)	5.9927 (1)
40	7.3754 (1)	6.6425 (1)
45	8.0463 (1)	7.2831 (1)
50	8.7078 (1)	7.9164 (1)
60	1.0009 (2)	9.1633 (1)
70	1.1285 (2)	1.0391 (2)
80	1.2542 (2)	1.1602 (2)
90	1.3784 (2)	1.2801 (2)
100	1.5013 (2)	1.3989 (2)

*The symbol (i) means that the tabled entry is to be multiplied by 10^i.

Following a verbal presentation of the results of this paper, certain unpublished portions of R. Gnanadesikan's 1956 thesis [4] were kindly pointed out as being pertinent. Gnanadesikan has obtained general recursion formulae for calculating two-sided bounds on the characteristic roots of a Wishart matrix. In the bivariate case I have been able to verify, by partial integration that (4.1) is equivalent to Gnanadesikan's expression. Confidence intervals for the dispersion parameters were also obtained by Gnanadesikan. His result is different from that of theorem 3 and may provide an alternative approach to the problem.

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