JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 66B, No. 3, July–September 1962

# Mill's Ratio for Multivariate Normal Distributions[1]

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(May 23, 1962)

Two easily applied inequalities are given for the "tail probabilities" of multivariate normal distributions.

A basic integral arising in statistics is of the form:

$$F(\mathbf{a},\mathbf{M})=\frac{|\mathbf{M}|^{1/2}}{(2\pi)^{n/2}}\int_0^\infty \cdots \int_0^\infty \exp\left[-\frac{1}{2}(\mathbf{x}+\mathbf{a})\mathbf{M}(\mathbf{x}+\mathbf{a})'\right] dx_1 \ldots dx_n \tag{1}$$

where $\mathbf{a}$ and $\mathbf{x}$ are $n$ component row vectors, $\mathbf{M}$ is an $n\times n$ positive definite symmetric matrix with determinant $|\mathbf{M}|$ and "$'$" means transpose. In particular (1) arises in evaluating $Pr(\mathbf{X}\geq \mathbf{b})$ where $\mathbf{X}$ is a random variable with a multivariate normal distribution, and $\mathbf{b}$ is a vector of constants. In this note inequalities (upper and lower bounds) are presented for $F(a,\mathbf{M})$.

Let

$$f(\mathbf{x},\mathbf{M})=\frac{|\mathbf{M}|^{1/2}}{(2\pi)^{n/2}}\exp\left(-\frac{1}{2}\mathbf{xMx}'\right),$$

then

$$F(\mathbf{a},\mathbf{M})=f(\mathbf{a},\mathbf{M})\int_0^\infty \cdots \int_0^\infty \exp(-\mathbf{aMx}')\times\exp\left(-\frac{1}{2}\mathbf{xMx}'\right) dx_1 \ldots dx_n. \tag{2}$$

Assume throughout the following,

$$\mathbf{\Delta}=\mathbf{aM}>0,\text{ i.e., } \Delta_i=\sum_{j=1}^n a_j m_{ji}>0 \text{ for } i=1,\ldots,n. \tag{3}$$

The following elementary inequalities are required

$$\sum_{k=0}^{2r+1}(-c)^k/k!<e^{-c}<\sum_{k=0}^{2r}(-c)^k/k! \text{ for } r=0,1,\ldots \tag{4}$$

and $c>0$. In particular, (4) will be used with $c=\frac{1}{2}\mathbf{xMx}'$, the last quantity being positive whenever $\mathbf{x}\neq 0$, i.e., at least one $x_i\neq 0$, since $\mathbf{M}$ is positive definite.

Now using the right-hand side of (4) with $r=0$ in (2) one obtains

$$F(\mathbf{a},\mathbf{M})<f(\mathbf{a},\mathbf{M})\int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^n \Delta_i x_i\right) dx_1,\ldots, dx_n. \tag{5}$$

After evaluating the integral in (5) one has

$$F(\mathbf{a},\mathbf{M})<f(\mathbf{a},\mathbf{M})\left(\prod_{i=1}^n \Delta_i\right)^{-1}. \tag{I}$$

Now using the left-hand side of (4) with $r=0$ in (2) one obtains

$$F(\mathbf{a},\mathbf{M})>f(\mathbf{a},\mathbf{M})\int_0^\infty \cdots \int_0^\infty \left(1-\frac{1}{2}\sum_1^n\sum_1^n x_i m_{ij} x_j\right)\times\exp\left(-\sum_{i=1}^n \Delta_i x_i\right) dx_1,\ldots, dx_n. \tag{6}$$

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[1] Work supported in part by the Office of Naval Research.

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After evaluating the integral in (6) one obtains

$$F(\mathbf{a},\mathbf{M})>f(\mathbf{a},\mathbf{M})\left(\prod_{i=1}^{n}\Delta_i\right)^{-1}\left[1-\frac{1}{2}\sum_{1}^{n}\sum_{1}^{n}m_{ij}(1+\delta_{ij})/(\Delta_i\Delta_j)\right]. \tag{II}$$

where $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$.

Inequalities similar to (I) and (II) can be obtained by using $r>0$ in (4). These inequalities would be complicated notationally. In particular, however, Feller [1, pp. 166, 179] [3] presents these inequalities (for general $r$) when $n=1$ and $m_{11}=1$. Pólya [2] has considered the case of $r>0$ for the integrand in (1) for a different region of integration. Pólya's sequence of inequalities has the desirable property of improving as $r$ increases until the sharpest form is obtained and then of becoming successively weaker. Ruben [3] has considered the special case where $m_{ii}=1$ for $i=1, \ldots, n$ and $m_{ij}=m$ for $i\neq j$. His results appear to be complicated.

If the case considered by Ruben is specialized in the following manner: $n=2$ and $a_1=a_2=a$, then one obtains from using (4) in (2) the following

$$F(\mathbf{a},\mathbf{M})\lesseqgtr f(\mathbf{a},\mathbf{M})\sum_{i=0}^{i^*}\Delta^2(-2\Delta^2)^{-i}\left[\sum{}^{(i)}\frac{(2u+v)!(2w+v)!(2m)^v}{u!v!w!}\right], \tag{III}$$

where $>$ ($<$) if $i^*$ is odd (even), $\Delta=1+m$, and $\Sigma^{(i)}$ is over all $u,v,w\geq 0$ such that $u+v+w=i$. It is not clear that the inequalities in (III) have the desirable feature of the Pólya bounds.

*Example 1.* Assume $\mathbf{X}$ has a bivariate normal distribution with mean vector zero, and variance covariance matrix $\mathbf{M}^{-1}=\begin{vmatrix}1 & 1/2\\ 1/2 & 1\end{vmatrix}$. Approximate $Pr(\mathbf{X}\geq(3,3))$. The integral expression for the desired probability is

$$\int_3^\infty\int_3^\infty f(\mathbf{x},\mathbf{M})dx_1dx_2.$$

In this integral make the following change of variables $y_1=x_1-3$ and $y_2=x_2-3$. The integral then becomes $F(\mathbf{a},\mathbf{M})$ where $\mathbf{a}=(3,3)$, $\mathbf{M}=\begin{vmatrix}4/3 & -2/3\\ -2/3 & 4/3\end{vmatrix}$, and $|\mathbf{M}|=4/3$. Note $\Delta_1=\Delta_2=2$. Now using (I) one obtains

$$Pr(\mathbf{X}\geq(3,3))<\frac{(4/3)^{1/2}e^{-6}}{8\pi}\doteq 0.000114,$$

and from (II) one obtains

$$Pr(\mathbf{X}\geq(3,3))>0.000114\left(1-\frac{16/3-4/3}{8}\right)\doteq 0.000057.$$

From [5] one finds the correct value to be 0.000082.

In this example the ratio of the upper to the lower bound is 2. If one considers the same problem, with positive correlations of 1/2 replaced by $-1/2$ the corresponding ratio is 54/49. Since the probabilities are smaller with negative than with positive correlations, this is not a surprising result.

*Example 2.* Assume $\mathbf{M}=\begin{vmatrix}4/3 & 2/3\\ 2/3 & 4/3\end{vmatrix}$ so that $|\mathbf{M}|=4/3$, also assume $\mathbf{a}=(2,2)$ so that $\boldsymbol{\Delta}=(4,4)$ then

$$0.0000039>F(\mathbf{a},\mathbf{M})>0.0000031.$$

[3] Figures in brackets indicate the literature references at the end of this paper.

To compare the accuracy of the results for the bivariate case with the univariate case, consider the following univariate example where the true value is about the same as in the above bivariate example. Let $n=1$, $m_{11}=1$, and $a=4.5$, then one obtains

$$0.00000355>F(4.5,\, 1)>0.00000337.$$

The correct value, from [4], is 0.000003398.

In the present notation, the general result for the univariate case given by Feller [1, p. 179] is

$$F(a,1)\gtrless a^{-1}f(a,1)\sum_{r=0}^{R}\frac{(-1)^r(2r)!}{2^r r! a^{2r}},$$

where $a>0$ and $>(<)$ if $R$ is odd (even).

*Example 3.* If $m_{ii}=1$ for $i=1, \ldots, n$ and $m_{ij}=m$ for $i\neq j$, and $\mathbf{a}=(a, \ldots, a)$, then $|\mathbf{M}|=(1-m)^{n-1}[1+(n-1)m]$ and $\Delta_i=a[1+(n-1)m]$. (To preserve the positive definite character of $\mathbf{M}$ it is necessary that $1+(n-1)m>0$, i.e., $m>-(n-1)^{-1}$ and $m<1$.) Now using (I) and (II), one obtains

$$F(\mathbf{a},\mathbf{M})<|\mathbf{M}|^{1/2}e^{-na\Delta/2}(2\pi)^{-n/2}\Delta^{-n}$$

and

$$F(\mathbf{a},\mathbf{M})>|\mathbf{M}|^{1/2}e^{-na\Delta/2}(2\pi)^{-n/2}\Delta^{-n}\left[1-\frac{n(1+\Delta)}{2\Delta^2}\right],$$

where $\Delta=1+(n-1)m$. Consider the case with $n=3$, $m=1/3$ and $a=5(3/2)^{1/2}$. (This numerical example corresponds to finding the probability that a trivariate normal variable with mean vector zero, and covariance matrix having 1's on the diagonal and 1/2's off the diagonal has each component larger than 5.) In this case the ratio of the upper bound to the lower bound of $F(\mathbf{a},\mathbf{M})$ is 25/13. If one considers $a=9(3/2)^{1/2}$ the corresponding ratio is 27/23.

With the previous notation and restrictions, consider a new integral, $G(\mathbf{a},\mathbf{M})$ obtained from modifying $F(\mathbf{a},\mathbf{M})$ by replacing the upper limits of integration by ones, i.e.,

$$G(\mathbf{a},\mathbf{M})=\int_0^1\cdots\int_0^1 f(\mathbf{x}+\mathbf{a},\mathbf{M})dx_1\ldots dx_n. \tag{7}$$

Then evaluating the integral analogous to (5), one obtains

$$G(\mathbf{a},\mathbf{M})<f(\mathbf{a},\mathbf{M})\left(\prod_{i=1}^{n}[1-e^{-\Delta_i}]/\Delta_i\right). \tag{IV}$$

From the analogue of (6), one obtains

$$G(\mathbf{a},\mathbf{M})>f(\mathbf{a},\mathbf{M})\left[\prod_{i=1}^{n}(1-e^{-\Delta_i})/\Delta_i\right]\left\{1-\frac{1}{2}\left[\sum_{i=1}^{n}\frac{m_{ii}}{\Delta_i}\left(1+\frac{1-(\Delta_i+1)^2e^{-\Delta_i}}{1-e^{-\Delta_i}}\right)\right.\right.$$
$$\left.\left.+\sum\sum_{i\neq j}\frac{m_{ij}}{\Delta_i\Delta_j}\left(1-\frac{\Delta_i e^{-\Delta_i}}{1-e^{-\Delta_i}}\right)\left(1-\frac{\Delta_j e^{-\Delta_j}}{1-e^{-\Delta_j}}\right)\right]\right\}. \tag{V}$$

Results (IV) and (V) were obtained by Professor Olkin.

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A first draft of this paper was prepared in 1954 while the author was at the National Bureau of Standards. That work was stimulated by Dr. Benjamin Epstein in conversations on the distribution of extreme values for dependent variables. Professor Ingram Olkin suggested changes in the original draft which have been incorporated at this time. During the summer of 1961, under a National Science Foundation Undergraduate Research Program, Thomas Kieran did some additional numerical analysis for the manuscript.

## References

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(Paper 66B3–77)