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Two Matrix Eigenvalue Inequalities

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A lower bound is given for the quantity λ_1/λ_n, and an upper bound for the quantity $\lambda_1-\lambda_n$, where λ_1 and λ_n are respectively the greatest and least characteristic roots of a matrix with positive roots. The bounds involve the first and second coefficients of the characteristic equation of the matrix.

Suppose $A=(a_{ij})$ is a nonsingular $n\times n$ matrix with characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_n$, so ordered that $|\lambda_1|\geq|\lambda_2|\geq \ldots \geq|\lambda_n|$. The quantity $|\lambda_1|/|\lambda_n|$ provides a rough measure of the probable error in the computation of the inverse of A; it has been called by J. Todd [1, 2, 3] [1] the P-condition number of A and may be denoted $P(A)$. Von Neumann and Goldstine [4] have shown that if A is symmetric and positive definite (in which case the λ_i are all positive), then the error in the inverse of A computed by a certain elimination method, can be bounded by a quantity proportional to $P(A)$; if A is not symmetric positive definite, the error can be bounded by a quantity proportional to $P(AA')$.

We shall restrict our consideration to matrices whose roots are all positive. For these, P. J. Davis, E. V. Haynsworth, and M. Marcus [5] obtained bounds on P involving det A and one other symmetric function of the roots of A. If the characteristic polynomial of A is $p(x)=x^n-C_1x^{n-1}+C_2x^{n-2}+\ldots+(-1)^nC_n$ and we set $D_1=(n^nC_n)/C_1^n$, they showed that

$$\frac{1}{D_1^{\frac{1}{n-1}}}\leq P\leq\frac{1+\sqrt{1-D_1}}{1-\sqrt{1-D_1}}; \tag{1}$$

and they found similar inequalities involving C_n $(=\det A)$ and any other one of the C_i.

In many cases, however, C_n is not known and it is in general difficult to calculate. $C_1(=\text{trace } A)$ is easy to calculate, and C_2 can in general be calculated more easily than can C_n since it is the sum of $[n(n-1)]/2$ determinants of order two. In this paper we present (in Theorems 1 and 1′) a lower bound for P in terms of C_1 and C_2; an attempt to obtain a corresponding upper bound fails, but leads to an inequality (Theorem 2) on $\lambda_1-\lambda_n$, the "spread" of the roots of A. Finally we apply the method of proof of Theorem 1 to obtain an improvement of the lower bound in (1).

[1] Figures in brackets indicate the literature references at the end of this paper.

If we set $C_K=\binom{n}{K}S_K(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then $S_K^{1/K}$ is the K'th symmetric mean of $\lambda_1, \ldots, \lambda_n$. The S_K satisfy [6] the inequalities

$$S_1\geq S_2^{1/2}\geq S_3^{1/3}\geq \ldots \geq S_n^{1/n}. \tag{2}$$

Setting $\mu_K=\frac{\lambda_K}{\lambda_n}$, we have $P=\mu_1\geq\mu_2\geq\mu_3\geq \ldots \geq\mu_n=1$,

and

$$\frac{S_1^2(\mu_1, \mu_2, \ldots, \mu_n)}{S_2(\mu_1, \mu_2, \ldots, \mu_n)}=\frac{S_1^2(\lambda_1, \ldots, \lambda_n)}{S_2(\lambda_1, \ldots, \lambda_n)}.$$

Let

$$R(x_2, x_3, \ldots, x_{n-1})=\frac{S_1^2(P, x_2, x_3, \ldots, x_{n-1}, 1)}{S_2(P, x_2, x_3, \ldots, x_{n-1}, 1)}$$

and

$$f_1(P)=\max_{1\leq x_i\leq P} R(x_2, x_3, \ldots, x_{n-1})$$

and

$$f_2(P)=\min_{1\leq x_i\leq P} R(x_2, \ldots, x_{n-1}).$$

Then f_1 and f_2 can be seen to be increasing in P, and

$$f_2(P)\leq\frac{S_1^2}{S_2}\leq f_1(P). \tag{3}$$

Thus the right half of (3) should provide a lower bound for P, while the left half should provide an upper bound. We first calculate $f_2(P)$:

By direct calculation we can show that $(\partial^2R)/\partial x_i^2$ is nonnegative at all points for each i. Therefore R attains its maximum at a point where each x_i is either 1 or P. Letting $R_K(P)$ denote the value of R when $K-1$ of $x_2, x_3, \ldots, x_{n-1}$ are equal to P and the remainder are equal to one, we find that R_K is equal to:

$$\frac{n-1}{n}\,\frac{(KP+n-K)^2}{(KP+n-K)^2-(KP^2+n-K)}. \tag{4}$$

This rational function of K attains its maximum at $K=\frac{n}{P+1}$ and so we obtain

$$\frac{S_1^2}{S_2}\leq\frac{n-1}{n}\frac{1}{1-\frac{(P+1)^2}{4Pn}} \tag{5}$$

which is equivalent to:

THEOREM 1:

$$\frac{1+\sqrt{1-A}}{1-\sqrt{1-A}}\leq P, \textit{where } A=\frac{1}{n-\frac{S_2}{S_1^2}(n-1)}.$$

The upper bound (5) can be sharpened, since in fact we need only consider integer values of K in (4). It can be shown that $R_1=\max_K R_K$ if $P^2\geq[(n-1)(n-2)]/2$. Thus if $P^2\geq[(n-1)(n-2)]/2$, $S_1^2/S_2\leq R_1(P)$; and so, setting $\rho=S_1^2/S_2$, we have:

$$[(\rho-1)+\sqrt{\rho(\rho-1)}]n+1\leq P. \tag{6}$$

Now if $P^2<[(n-1)(n-2)]/2$, then $\rho\leq\max_K R_K(P)=R_K^*(P)$, say; and since each $R_K(P)$ is a strictly increasing function of P, $\rho<R_K^*([(n-1)(n-2)]/2)\leq R_1([(n-1)(n-2)]/2)$ and so $\sqrt{\frac{(n-1)(n-2)}{2}}>R_1^{-1}(\rho)$. However it can easily be seen that if $\rho\geq(\sqrt{2}+\frac{3}{2})/(\sqrt{2}+1)$, then $R_1^{-1}(\rho)\geq\sqrt{\frac{(n-1)(n-2)}{2}}$; and so we conclude that:

THEOREM 1′: *If* $\rho\geq(\sqrt{2}+\frac{3}{2})/(\sqrt{2}+1)$, *then* $[(\rho-1)+\sqrt{\rho(\rho-1)}]n+1\leq P$.

This lower bound is better than the previous one, whenever it applies.

The attempt to derive an upper bound for P from the left half of (3) fails, because $f_2(P)$ approaches 1 uniformly in P as n increases, and so the inequality $f_2(P)\leq S_1^2/S_2$ will in most cases hold for all values of P. We can, however, by considering the minimum of $S_1^2(\lambda_1, y_2, y_3, \ldots, y_{n-1}, \lambda_n)-S_2(\lambda_1, y_2, \ldots, y_{n-1}, \lambda_n)$ subject to the condition $\lambda_1\geq y_2\geq y_3\geq\ldots\geq y_{n-1}\geq\lambda_n$, obtain an upper bound for the spread of the roots of A. Calling the above function of $y_2, \ldots, y_{n-1}$ D, we evaluate its minimum directly by observing that $\partial D/\partial y_i=-2/[n(n-1)](S_1-y_i)$ and $\partial^2D/\partial y_i\partial y_j=2/n^2$ if $i=j$ and $-2/[n^2(n-1)]$ if $i\neq j$; $i, j=2, 3, \ldots, n-1$. The $(n-2)\times(n-2)$ matrix (d_{ij}), with $d_{ij}=\partial^2D/\partial y_i\partial y_j$ is symmetric, and by Gerschgorin's theorem each of its eigenvalues lies in the circle $|Z-2/n^2|\leq 2/n^2\left(\frac{n-3}{n-1}\right)$ and so is positive. Thus (d_{ij}) is positive definite, and, setting each $\partial D/\partial y_i$ equal to zero, we find that D is at a minimum when each y_i is equal to $(\lambda_1+\lambda_n)/2$, and that the minimal value is $[(\lambda_1-\lambda_n)^2]/[2n(n-1)]$. We may then conclude:

THEOREM 2: $\lambda_1-\lambda_n\leq\sqrt{2n(n-1)(S_1^2-S_2)}$.

By the method of Theorem 1, it is possible to sharpen the left side of inequality (1). Following the notation of [5] we write

$$D_1(x_1, x_2, \ldots, x_n)=\frac{[S_n(x_1, \ldots, x_n)]}{[S_1^n(x_1, \ldots, x_n)]}=\frac{x_1\times x_2\times\ldots\times x_n}{\left(\frac{x_1+\ldots+x_n}{n}\right)^n},$$

and seek an upper bound for D_1 subject to the condition $P=x_1\geq x_2\geq\ldots\geq x_n=1$. As in the proof of Theorem 1, we show that $\partial^2D_1/\partial x_i^2\geq 0$ for $i=2, 3, \ldots, n-1$ and finally obtain the relation

$$D_1\leq\left[\frac{P-1}{\log P}e^{-1+\frac{\log P}{P-1}}\right]^n \tag{7}$$

which leads to the inequality:

THEOREM 3:

$$\frac{e}{D_1^{\frac{1}{n}}}\leq\frac{P-1}{\log P}e^{\frac{\log P}{P-1}}.$$

This inequality yields a lower bound for P which is always higher than that given by (1) (as may be seen by comparing the proof of Theorem 3 with the proof of (1) in [5]). However it is cumbersome. It can be simplified (and somewhat weakened) as follows: Since $xe^{1/x}=a$, $a\geq e$, implies that $x\geq a-e+1$, and $e/D_1^{\frac{1}{n}}\geq e$ by (2), we may conclude that $(P-1)/\log P\geq e/D_1^{\frac{1}{n}}-e+1$. Denoting this last quantity by A, we have $(P-1)/\log P\geq A$, which has as an immediate consequence:

THEOREM 3′: $P\geq A\log A-1$; $A=e/D_1^{\frac{1}{n}}-e+1$.

This lower bound is most often, though not always, better than that given in (1).

References

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