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The Nonsingular Embedding of Transition Processes Within a More General Framework of Coupled Variables

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Reflection and coupling processes exhibited by plane electromagnetic waves propagated in an inhomogeneous horizontally-stratified anisotropic ionosphere are associated with discrete transition points or with continuous coupling regions. These arise when the fourth order differential equations are written in first order coupled form, and many terms in these equations become infinite at the transition points. This procedure is rendered more precise by means of a special linear transformation that reformulates the equations in a new way, thereby exhibiting the manner in which local coupling processes are embedded in the more general background process of otherwise independently propagated characteristic waves. To exhibit the power of the matrix algebra involved, the case of an arbitrary number of characteristic waves is considered; moreover, Försterling-type coupled equations are produced in a more generalized form than hitherto considered, and a discussion of the equations governing continuous coupling completes the paper.

1. Introduction

The equations governing the propagation of plane electromagnetic waves in a horizontally-stratified inhomogeneous anisotropic ionosphere may be expressed in various ways, depending upon the particular investigation for which the equations are required. If the Oz-axis of a right-handed set of Cartesian axes Ox, Oy, Oz is directed vertically upwards, then it often proves convenient to use second or fourth order differential equations in the dependent variables E_x or E_y. Heading and Whipple [1952] and Heading [1955] have employed this representation when analytical solutions of the differential equations are under consideration. The reason is that all known analytical solutions of the field equations may be expressed in terms of generalized hypergeometric functions, hypergeometric functions, confluent hypergeometric functions, Bessel functions and so on, these functions naturally satisfying second or fourth order differential equations. Such functions provide the reflection and conversion coefficients for a particular anisotropic model ionosphere, but by inspecting the functional form of the electric field throughout the medium no immediate physical interpretation is possible, whereby the actual process of wavereflection and wavecoupling may be explicitly exhibited.

Sets of equations known as *coupled equations* have therefore been introduced, in order to enable approximate solutions for the electromagnetic field to be obtained, and in order to exhibit directly the important physical processes of wave reflection and wave coupling. Försterling [1942] considered a coupled system of two second order equations; these refer to an ionospheric model in which the external magnetic field is oblique but in which propagation is vertical. Gibbons and Nertney [1951, 1952] have used these coupled equations to obtain approximate numerical solutions for various ionospheric models, while Budden [1952] has used them in discussing limiting polarization. Heading [1953] and Budden and Clemmow [1957] have given similar coupled equations governing the case when incidence is oblique upon an ionosphere in which the magnetic field is vertical.

When both the direction of propagation and of the magnetic field are oblique, four coupled equations are used, each of the first order. These were first introduced by Bremmer [1949], but more systematically using matric techniques by Clemmow and Heading [1954]. The most up-to-date account of these first order coupled equations is found in chapter 18 of the recent text by Budden [1961], but there the fundamental simplicity and beauty of the matrix formulation is often disguised under a multitude of symbols and approximations. Moreover, certain important features of the matrices occurring in the coupled equations together with the possibility of nonsingular embedding do not as yet appear to have been recognized.

The coupled equations exhibit those ranges of height in which the propagation of the four characteristic waves takes place almost independently. Any point at which many of the coefficients in the equations become singular denotes that coupling takes place there between the characteristic waves concerned, the other characteristic waves not embraced by the coupling still being propagated independently.

The present investigation consists of an examination of the equations near the coupling points without the necessity of introducing approximations. It will be proved that the equations governing the coupling process at a coupling point are embedded in the equations governing the propagation of the waves as a whole, such that all singularities in the coefficients of all equations are removed at the coupling point. Moreover, although ionospheric propagation is represented in terms of four first order equations, yielding four characteristic waves in a homogeneous or slowly-varying medium, generalization to wave processes governed by n first order simultaneous differential equations yielding n characteristic waves is easily possible using matrix notation. The sphere of usefulness of the present investigation will thereby be extended if n equations are considered.

2. Transformation of an rth Order Linear Equation

The preliminary study of a given linear differential equation rather than a set of first order simultaneous linear differential equations is necessary in order to define a *principal embedded coupled matrix*.

The rth order linear equation in the dependent variable u

$$u^{(r)}=y_{r-1}(z)u^{(r-1)}+v_{r-2}(z)u^{(r-2)}+\ldots+v_1(z)u'+v_0(z)u, \tag{1}$$

where bracketed superscripts denote differentiation with respect to z, and where the r coefficients $v_0, v_1, \ldots, v_{r-1}$ are given functions of z, may be expressed as a linear system using matrix notation as follows:

$$\frac{d}{dz}\begin{bmatrix} u \\ u' \\ \cdots \\ u^{(r-2)} \\ u^{(r-1)} \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ v_0 & v_1 & v_2 & v_3 & \ldots & v_{r-2} & v_{r-1} \end{bmatrix}\begin{bmatrix} u \\ u' \\ \cdots \\ u^{(r-2)} \\ u^{(r-1)} \end{bmatrix}, \tag{2}$$

or

$$\mathbf{u}'=\mathbf{T}\,\mathbf{u} \tag{3}$$

say, where **u** denotes the column consisting of r elements and **T** denotes the square $r\times r$ matrix shown consisting of the coefficients occurring in (1).

When the matrix **T** is diagonalized by means of the transformation

$$\mathbf{u}=\mathbf{A}\,\mathbf{f},$$

the roots of the characteristic equation

$$\det(\mathbf{T}-q\mathbf{I})=0$$

are required. This equation has the explicit form

$$q^r - v_{r-1}q^{r-1} - v_{r-2}q^{r-2} - \ldots - v_1 q - v_0 = 0,$$

possessing the r characteristic roots $q_1, q_2, \ldots, q_r$.

The elements of the characteristic vector corresponding to the root q_j consist of the cofactors of any row of the matrix $\mathbf{T}-q_j\mathbf{I}$; in particular, the cofactors of the last row are 1, $q_j, q_j^2, \ldots, q_j^{r-1}$ respectively. Hence the matrix **A** of the transformation may be taken to be

$$\mathbf{A}=\begin{pmatrix} 1 & 1 & \ldots & 1 \\ q_1 & q_2 & \ldots & q_r \\ \ldots & \ldots & \ldots & \ldots \\ q_1^{r-1} & q_2^{r-1} & \ldots & q_r^{r-1} \end{pmatrix};$$

this is an *alternant* matrix, whose determinant is given by

$$\det \mathbf{A} = \prod_{\substack{i,j=1 \\ i>j}}^{r} (q_i - q_j). \tag{4}$$

It follows that **A** is nonsingular throughout any domain in the complex z-plane in which all the r characteristic roots are distinct.

Equation (3) becomes

$$\mathbf{A}'\mathbf{f} + \mathbf{A}\mathbf{f}' = \mathbf{T}\mathbf{A}\mathbf{f},$$

or

$$\mathbf{f}' = \mathbf{A}^{-1}\mathbf{T}\mathbf{A}\mathbf{f} - \mathbf{A}^{-1}\mathbf{A}'\mathbf{f}.$$

The matrix $\mathbf{A}^{-1}\ \mathbf{T}\ \mathbf{A}$ is now a diagonal matrix **Q**, whose diagonal elements consist of the r characteristic roots in order; hence

$$\mathbf{f}' = \mathbf{Q}\mathbf{f} - \mathbf{A}^{-1}\mathbf{A}'\mathbf{f}. \tag{5}$$

If solutions of (5) are known, then $\mathbf{u} = \mathbf{A}\ \mathbf{f}$ are the solutions of (3), and in particular the solution of (1) is

$$u = f_1 + f_2 + \ldots + f_r. \tag{6}$$

The r eqs (5) constitute a set of first order *coupled* equations equivalent to the original eq (1). The diagonal matrix **Q** will be called a *principal uncoupled matrix,* while the terms $\mathbf{A}^{-1}\ \mathbf{A}'\ \mathbf{f}$ will be called the *coupling terms,* though strictly speaking coupling arises only through the nondiagonal elements of the matrix $\mathbf{A}^{-1}\mathbf{A}'$. This product $\mathbf{A}^{-1}\mathbf{A}'$, whose explicit elements are calculated in section 5, may be called the *coupling matrix.* It is the presence of this coupling matrix that renders the r eqs (5) simultaneous and not independent, but the coupling matrix vanishes if the coefficients in the original eq (1) are constants, owing to the presence of derivatives in every element. The r transformed equations are then completely independent, and their r solutions provide r characteristic waves independently propagated when a physical picture is employed for their interpretation. Then

$$f_j = C_j \exp\ (q_j z),$$

where the C_j are arbitrary constants of integration.

When the coefficients in (1) are not constants, usually some parameter such as the frequency or some property of the medium attains values that cause the elements of the coupling matrix to be small in magnitude compared with the elements of the principal uncoupled matrix, provided a domain is considered in which **A** is nonsingular so that its reciprocal exists. Under these circumstances, the nondiagonal elements in the coupling matrix are neglected, yielding r independent equations, whose solutions represent r characteristic waves propagated independently through the slowly varying medium.

In this paper, equations of the form (1) are considered only when all the r characteristic roots become equal at a particular point z_0. To this end, r new functions $\alpha, \beta, \ldots, \rho$ are introduced, defined by the r linear equations

$$q_j=\alpha+k^{j-1}\beta+k^{2(j-1)}\gamma+\ldots+k^{(r-1)(j-1)}\rho, \tag{7}$$

where $k=\exp(2\pi i/r)$. The matrix of the coefficients in these equations is nonsingular, since it is an alternant whose elements consist of the rth roots of unity $1, k, k^2, \ldots, k^{r-1}$ raised to the appropriate powers. The r new functions are therefore uniquely defined.

Now

$$\begin{aligned} v_{r-1}&=\sum_{j=1}^{r} q_j \\ &=r\alpha+\beta\sum_{j=1}^{r} k^{j-1}+\gamma\sum_{j=1}^{r} k^{2(j-1)}+\ldots+\rho\sum_{j=1}^{r} k^{(r-1)(j-1)}. \end{aligned}$$

But if p is an integer,

$$\left[\sum_{j=1}^{r} k^{p(j-1)}\right](1-k^p)=1-(k^p)^r=1-(k^r)^p=0,$$

so if $p\neq 0$, it follows that

$$\sum_{j=1}^{r} k^{p(j-1)}=0. \tag{8}$$

Hence

$$\alpha=v_{r-1}/r.$$

For example, when $r=2$, α and β are merely the two terms that arise when the quadratic equation is solved using the standard formula, the familiar $\pm$ signs being associated with the β. When $r=3$, β and γ arise when the reduced cubic is solved by Cardan's method.

To lay the basis for the following sections, v_{r-1} is now chosen to vanish identically, indicating that $\alpha=0$. It will be seen in the general theory, however, that no loss of generality occurs on account of this assumption. Coupling is then studied by postulating that the r characteristic roots become equal at $z=z_0$. Under these circumstances, when α is omitted from (7) and when the r q's are given equal non-zero values, the rank of the matrix of the coefficients of the right-hand side is $r-1$, since the r linear expressions are dependent, while the rank of the augmented matrix is r. The equations are therefore inconsistent. If on the other hand the r equal values of the q's are zero, the first $r-1$ equations (7) yield $\beta=\gamma=\ldots=\rho=0$, since the matrix of the coefficients of these first $r-1$ equations is nonsingular.

Hence, at $z=z_0$, it follows that

$$q_1=q_2=\ldots=q_r=0;$$

moreover, the coefficients of (1) in which $v_{r-1}\equiv 0$ consist of sums of products of these r roots, so

$$v_0(z_0)=v_1(z_0)=\ldots=v_{r-2}(z_0)=0.$$

This is too general for useful progress to be made, although the theory to be presented is equally valid without any further simplifying assumptions. The special choice is made that

$$v_1(z)\equiv v_2(z)\equiv \ldots \equiv v_{r-2}(z)\equiv 0,$$

leaving a differential equation

$$\frac{d^r u}{dz^r}=v_0(z)u. \tag{9}$$

When $r=2$, it should be noticed that no assumption is in fact made, and it is just this case that usually governs wave reflection and wave coupling. The r values of the characteristic roots are now given by

$$q_j=k^{j-1}(v_0)^{1/r}\equiv k^{j-1}\beta.$$

Another reason why these simplifying assumptions are advantageous concerns the possibility of writing down approximate solutions of (9), since (9) will form what may be termed a *comparison* equation embedded in an equation of higher order. Approximate solutions of this latter general equation are given in terms of approximate solutions of the comparison equation. When $r=2$, (9) can be solved approximately in terms of the Airy integral, while if $r>2$, solutions may be obtained in terms of functions investigated by Heading in a series of papers [1957 a and b; 1960], though such approximations have not as yet been published. But if (1) is considered with $v_{r-1}\equiv 0$ but with no further assumptions, useful approximate analytical solutions are almost impossible to obtain near a point $z=z_0$ at which r characteristic roots become equal.

Under these assumptions, matrix **T** becomes

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ v & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \tag{10}$$

where v_0 is now written simply as v.

The characteristic equation $q^r-v=0$ has the r roots

$$q_j=k^{j-1}v^{1/r}$$

and the alternant matrix **A** factorizes thus:

$$\mathbf{A}=\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & v^{1/r} & 0 & \ldots & 0 \\ 0 & 0 & v^{2/r} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & v^{(r-1)/r} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & k & k^2 & \ldots & k^{r-1} \\ 1 & k^2 & k^4 & \ldots & k^{2(r-1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & k^{r-1} & k^{2(r-1)} & \ldots & k^{(r-1)^2} \end{bmatrix} \tag{11}$$

$$=\mathbf{VK}$$

say, where **V** and **K** are defined by these two square matrices.

The matrix **K** being constant, it follows that

$$\mathbf{A}^{-1}\mathbf{A}'=\mathbf{K}^{-1}\mathbf{V}^{-1}\mathbf{V}'\mathbf{K}$$

$$=\mathbf{K}^{-1}\begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & v^{-1/r} & 0 & \cdots \\ 0 & 0 & v^{-2/r} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & v^{1/r}v'/rv & 0 & \cdots \\ 0 & 0 & 2v^{2/r}v'/rv & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}\mathbf{K}$$

$$=\frac{v'}{rv}\mathbf{K}^{-1}\begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}\mathbf{K}.$$

Clearly coupling only becomes large near a point at which v'/v is singular.

The explicit elements of this product have been calculated by Heading [1960]. They are

$$(\mathbf{A}^{-1}\mathbf{A}')_{jj}=\frac{v'}{v}\cdot\frac{r-1}{2r}, \tag{12}$$

$$(\mathbf{A}^{-1}\mathbf{A}')_{ij}=\frac{v'}{v}\cdot\frac{1}{r(k^{j-i}-1)}. \tag{13}$$

The coupled eqs (5) then take the form

$$=\begin{bmatrix} v^{1/r} & 0 & 0 & \cdots \\ 0 & kv^{1/r} & 0 & \cdots \\ 0 & 0 & k^2v^{1/r} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}\mathbf{f}-\frac{v'}{v}\cdot\frac{r-1}{2r}\mathbf{f}-\frac{v'}{rv}\begin{bmatrix} 0 & \frac{1}{k-1} & \frac{1}{k^2-1} & \cdots \\ \frac{1}{k^{-1}-1} & 0 & \frac{1}{k-1} & \cdots \\ \frac{1}{k^{-2}-1} & \frac{1}{k^{-2}-1} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}\mathbf{f}. \tag{14}$$

The first square matrix on the right hand side of (14) is the principal uncoupled matrix. The coupling terms become singular whenever v'/v is singular; such a point is known as a reflection or coupling point. Mathematically speaking, such a point is called a *transition point*. The coupling terms are singular at this point, but the original equation

$$\mathbf{u}'=\mathbf{T}\mathbf{u}$$

possesses a special matrix (10), associated in this particular case with no additional coupling terms. An $r\times r$ matrix of the form (10) will be called a *principal embedded coupled matrix*.

3. General System

The equations governing ionospheric radio propagation may be expressed as four linear first order differential equations with four dependent variables, given for example by Clemmow and Heading [1954]. More generally, consider n dependent variables $e_1, e_2, \ldots, e_n$ satisfying the n linear first order equations

$$e'_1 = \sum_{j=1}^{n} T_{ij} e_j.$$

In maxtrix notation,

$$\mathbf{e}' = \mathbf{Te}, \tag{15}$$

where the $n \times n$ matrix $\mathbf{T}$ contains n^2 elements each of which is a given function of z. In the ionospheric case, $n=4$, and some of the elements of $\mathbf{T}$ are zero, while others are complicated functions of height, the explicit forms of which would obscure the essential simplicity of the present argument. The matrix $\mathbf{T}$ is therefore more complicated than that occurring in (3) or the principal embedded coupled matrix (10).

If the dependent column $\mathbf{e}$ is now transformed to the column $\mathbf{f}$ by the transformation

$$\mathbf{e} = \mathbf{Sf}, \tag{16}$$

it follows that

$$\mathbf{f}' = \mathbf{S}^{-1}\mathbf{TSf} - \mathbf{S}^{-1}\mathbf{S}'\mathbf{f}.$$

If the n characteristic roots of $\mathbf{T}$ are denoted by $q_1, q_2, \ldots, q_n$, let $\mathbf{s}_j$ be the column matrix formed from the cofactors of $\mathbf{T} - q_j\mathbf{I}$ taken along any suitable row. Then the transformation matrix

$$\mathbf{S} = (\mathbf{s}_1 \mathbf{s}_2 \ldots \mathbf{s}_n)$$

diagonalizes the matrix $\mathbf{T}$; that is, $\mathbf{S}^{-1}\mathbf{T}\ \mathbf{S} = \mathbf{Q}$ consists of the n roots arranged in order down the leading diagonal. The coupled equations become

$$\begin{aligned} \mathbf{f}' &= \mathbf{Qf} - (\mathbf{s}_1\mathbf{s}_2 \ldots \mathbf{s}_n)^{-1}(\mathbf{s}_1\mathbf{s}_2 \ldots \mathbf{s}_n)'\mathbf{f} \\ &= \mathbf{Qf} - \mathbf{S}^{-1}\mathbf{S}'\mathbf{f}, \end{aligned} \tag{17}$$

valid at all points at which $\mathbf{S}$ is nonsingular.

If desired, a change of independent variable may easily be effected. If $\zeta = \zeta(z)$, then

$$\frac{d\mathbf{f}}{dz} = \frac{d\mathbf{f}}{d\zeta}\zeta',$$

yielding

$$\frac{d\mathbf{f}}{d\zeta} = \frac{1}{\zeta'}\mathbf{Qf} - \mathbf{S}^{-1}\frac{d\mathbf{S}}{d\zeta}\mathbf{f}.$$

Similarly, a change of dependent variables may be made by placing

$$f_j = \phi_j(z) g_j,$$

where the n functions $\phi_j(z)$ denotes n given functions. In matrix form, let

$$\mathbf{f} = \mathbf{\Phi g},$$

where $\mathbf{\Phi}$ is a diagonal matrix consisting of the n functions $\phi_j(z)$ arranged in order down the leading diagonal. Then

$$\mathbf{\Phi g}' + \mathbf{\Phi}'\mathbf{g} = \mathbf{Q\Phi g} - \mathbf{S}^{-1}\mathbf{S}'\mathbf{\Phi g},$$

or

$$\begin{aligned} \mathbf{g}' &= \mathbf{\Phi}^{-1}\mathbf{Q\Phi g} - (\mathbf{\Phi}^{-1}\mathbf{\Phi}' + \mathbf{\Phi}^{-1}\mathbf{S}^{-1}\mathbf{S}'\mathbf{\Phi})\mathbf{g} \\ &= \mathbf{Qg} - (\mathbf{S\Phi})^{-1}(\mathbf{S\Phi})'\mathbf{g}, \end{aligned}$$

implying that these equations would have been produced had $\mathbf{S\Phi}$ been used for the transformation matrix (16) rather than $\mathbf{S}$. The characteristic vector $\mathbf{s}_j$ is merely multiplied throughout by ϕ_j.

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Now each cofactor of $\mathbf{T}-q_j\mathbf{I}$ is a polynomial of degree $n-1$ at most in q_j, the coefficients being independent of the suffix j. Generally,

$$\mathbf{s}_j=\begin{pmatrix}\text{polynomial 1 of degree } n-1 \text{ at most in } q_j\\ \text{polynomial 2 of degree } n-1 \text{ at most in } q_j\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \text{polynomial } n \text{ of degree } n-1 \text{ at most in } q_j\end{pmatrix}$$

$$=\begin{pmatrix}\text{the } n \text{ coefficients of polynomial 1}\\ \text{the } n \text{ coefficients of polynomial 2}\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \text{the } n \text{ coefficients of polynomial } n\end{pmatrix}\begin{pmatrix}1\\ q_j\\ \cdots\\ q_j^{n-1}\end{pmatrix}$$

$$=\mathbf{P}\begin{pmatrix}1\\ q_j\\ \cdots\\ q_j^{n-1}\end{pmatrix}$$

say, where $\mathbf{P}$ denotes the $n\times n$ matrix formed from the coefficients of the n polynomials. The last column of $\mathbf{P}$ consists of $n-1$ zero elements, while the remaining elements is unity (when n is even), since only one polynomial can be of degree $n-1$, the others being of degree $n-2$.

Then

$$\mathbf{S}=(\mathbf{s}_1\mathbf{s}_2\ldots\mathbf{s}_n)=\mathbf{P}\begin{pmatrix}1 & 1 & 1 & \cdots\\ q_1 & q_2 & q_3 & \cdots\\ q_1^2 & q_2^2 & q_3^2 & \cdots\\ \cdots & \cdots & \cdots & \cdots\end{pmatrix}=\mathbf{PA},$$

where $\mathbf{A}$ denotes the alternant matrix consisting of the roots and their powers arranged in order. It follows that

$$\mathbf{S}^{-1}\mathbf{S}'=\mathbf{A}^{-1}\mathbf{P}^{-1}(\mathbf{PA}'+\mathbf{P}'\mathbf{A})$$

$$=\mathbf{A}^{-1}\mathbf{A}'+\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A}.$$

The coupled equations (17) take the form

$$\mathbf{f}'=\mathbf{Qf}-\mathbf{A}^{-1}\mathbf{A}'\mathbf{f}-\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{Af},$$

similar to the set (5) apart from the additional coupling terms $\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A}\mathbf{f}$. As before, $\mathbf{Q}$ denotes the principal uncoupled matrix, $\mathbf{A}^{-1}\mathbf{A}'$ is the *primary coupling matrix* and $\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A}$ is the *secondary coupling matrix*.

4. Integral Equation

Let the matrix $\mathbf{A}^{-1}\mathbf{A}'$ be separated into the sum of two parts, $\mathbf{D}$ and $(\mathbf{A}^{-1}\mathbf{A}'-\mathbf{D})$ say, where $\mathbf{D}$ is an arbitrary diagonal matrix, though this may preferably consist of certain diagonal terms chosen from the explicit representation of $\mathbf{A}^{-1}\mathbf{A}'$ given in the following section. Then

$$\mathbf{f}'=(\mathbf{Q}-\mathbf{D})\mathbf{f}-(\mathbf{A}^{-1}\mathbf{A}'-\mathbf{D}+\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A})\mathbf{f},$$

where the matrix $\mathbf{Q}-\mathbf{D}$ is diagonal.

Let $\mathbf{f}_0$ be a solution of the equation

$$\mathbf{f}'=(\mathbf{Q}-\mathbf{D})\mathbf{f},$$

this equation representing in effect n independent linear equations. Let $\mathbf{F}_0$ denote the $n\times n$ diagonal matrix containing the elements of $\mathbf{f}_0$ arranged down the leading diagonal. Consider the integral equation

$$\mathbf{f}=\mathbf{N}\mathbf{f}_0-\mathbf{F}_0\int_a^z \mathbf{F}_0^{-1}(\mathbf{A}^{-1}\mathbf{A}'-\mathbf{D}+\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A})\mathbf{f}dz, \tag{19}$$

where $\mathbf{N}$ denotes an arbitrary constant diagonal matrix and a an arbitrary constant.

This is the required integral equation, for differentiation yields

$$\begin{aligned}\mathbf{f}'&=\mathbf{N}\mathbf{f}_0'-\mathbf{F}_0'\int_a^z \mathbf{F}_0^{-1}(\mathbf{A}^{-1}\mathbf{A}'-\mathbf{D}+\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A})\mathbf{f}dz-\mathbf{F}_0\mathbf{F}_0^{-1}(\mathbf{A}^{-1}\mathbf{A}'-\mathbf{D}+\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A})\mathbf{f}\\ &=\mathbf{N}\mathbf{f}_0'-\mathbf{F}_0'\mathbf{F}_0^{-1}(\mathbf{N}\mathbf{f}_0-\mathbf{f})-(\mathbf{A}^{-1}\mathbf{A}'-\mathbf{D}+\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A})\mathbf{f}\end{aligned}$$

from (19).

But

$$\mathbf{F}_0'=(\mathbf{Q}-\mathbf{D})\mathbf{F}_0$$

since all matrices are diagonal, so

$$\begin{aligned}\mathbf{f}'&=\mathbf{N}(\mathbf{Q}-\mathbf{D})\mathbf{f}_0-(\mathbf{Q}-\mathbf{D})(\mathbf{N}\mathbf{f}_0-\mathbf{f})-(\mathbf{A}^{-1}\mathbf{A}'-\mathbf{D}+\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A})\mathbf{f}\\ &=(\mathbf{Q}-\mathbf{D})\mathbf{f}-(\mathbf{A}^{-1}\mathbf{A}'-\mathbf{D}+\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A})\mathbf{f},\end{aligned}$$

since $\mathbf{N}$ commutes with $(\mathbf{Q}-\mathbf{D})$. This demonstrates the equivalence of the differential and integral equations.

5. Explicit Form of the Primary Coupling Matrix

In order to appreciate more clearly the character of the coupling terms $\mathbf{A}^{-1}\mathbf{A}'\mathbf{f}$, it is necessary to evaluate the explicit form of the product $\mathbf{A}^{-1}\mathbf{A}'$.

The reciprocal of the nth order alternant matrix

$$\mathbf{A}=\begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ q_1 & q_2 & q_3 & \cdots & q_n\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ q_1^{n-1} & q_2^{n-1} & q_3^{n-1} & \cdots & q_n^{n-1}\end{bmatrix}$$

may be calculated to have the form

$$\begin{bmatrix}\dfrac{(-1)^{n-1}S_{n-1}(q_1=0)}{(q_1)} & \dfrac{(-1)^{n-2}S_{n-2}(q_1=0)}{(q_1)} & \cdots & \dfrac{-S_1(q_1=0)}{(q_1)} & \dfrac{S_0}{(q_1)}\\[2ex] \dfrac{(-1)^{n-1}S_{n-1}(q_2=0)}{(q_2)} & \dfrac{(-1)^{n-2}S_{n-2}(q_2=0)}{(q_2)} & \cdots & \dfrac{-S_1(q_2=0)}{(q_2)} & \dfrac{S_0}{(q_2)}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \dfrac{(-1)^{n-1}S_{n-1}(q_n=0)}{(q_n)} & \dfrac{(-1)^{n-2}S_{n-2}(q_n=0)}{(q_n)} & \cdots & \dfrac{-S_1(q_n=0)}{(q_n)} & \dfrac{S_0}{(q_n)}\end{bmatrix}.$$

Here, $S_0(\equiv 1)$, S_1, S_2, . . ., S_{n-1} are the elementary symmetric functions of the n quantities $q_1, q_2, \ldots, q_n$. The symbol $S_p(q_s=0)$ is used to denote the algebraic expression S_p when q_s is replaced by zero. The symbol (q_p) is used to denote the product

$$(q_p)\equiv(q_p-q_1)q_p-q_2)\ldots(q_p-q_{p-1})(q_p-q_{p+1})\ldots(q_p-q_n)=(\partial F/\partial q)_{q=q_p}, \tag{20}$$

where

$$F=\det(q\mathbf{I}-\mathbf{T}).$$

It can then be seen that

$$(\mathbf{A}^{-1})_{ij}=(-1)^{n-j}S_{n-j}(q_i=0)/(q_i),$$

and

$$(\mathbf{A}')_{ij}=(i-1)q_j^{i-2}q_j',$$

so the general element in the product $\mathbf{A}^{-1}\mathbf{A}'$ is given by

$$(\mathbf{A}^{-1}\mathbf{A}')_{ij}=\sum_{k=1}^{n}(\mathbf{A}^{-1})_{ik}(\mathbf{A}')_{kj}$$

$$=\sum_{k=1}^{n}(-1)^{n-k}S_{n-k}(q_i=0)(k-1)q_j^{k-2}q_j'/(q_i). \tag{21}$$

In order to simplify this expression, the polynomial of degree $n-1$ is formed, whose zeros are $q_1, q_2, \ldots, q_n$ but with q_i omitted.

This is

$$\frac{F}{q-q_j}=q^{n-1}-S_1(q_i=0)q^{n-2}+\ldots+(-1)^pS_p(q_i=0)q^{n-p-1}+\ldots$$

$$=\sum_{p=0}^{n-1}(-1)^pS_p(q_i=0)q^{n-p-1}$$

$$=\sum_{k=1}^{n}(-1)^{n-k}S_{n-k}(q_i=0)q^{k-1},\qquad(\text{with } p=n-k)$$

yielding upon differentiation with respect to q

$$\frac{\partial}{\partial_q}\left(\frac{F}{q-q_i}\right)=\sum_{k=1}^{n}(-1)^{n-k}S_{n-k}(q_i=0)(k-1)q^{k-2}. \tag{22}$$

If $i\neq j$, this result transforms (21) thus:

$$(\mathbf{A}^{-1}\mathbf{A}')_{ij}=\frac{q_j'}{(q_i)}\left[\frac{\partial}{\partial q}\left(\frac{F}{q-q_i}\right)\right]_{q=q_j}$$

$$=\frac{q_j'}{(q_i)}\left[\frac{\partial F/\partial q}{q-q_i}-\frac{F}{(q-q_i)^2}\right]_{q=q_j}$$

$$=\frac{q_j'}{q_j-q_i}\cdot\frac{(q_j)}{(q_i)}, \tag{23}$$

since F vanishes when $q=q_j$. Out of the $n-1$ factors in (q_j), the factor (q_j-q_i) obviously cancels with the factor (q_i-q_j) occurring amongst the $n-1$ factors of (q_i).

On the other hand, when $i=j$, the left-hand side of (22) may be written as

$$\frac{\partial}{\partial q}\prod_{\substack{p=1\\ \neq i}}^{n}(q-q_p),$$

yielding

$$(\mathbf{A}^{-1}\mathbf{A}')_{ii}=\frac{q_1'}{(q_i)}\left[\frac{\partial}{\partial q}\prod_{\substack{p=1\\ \neq i}}^{n}(q-q_p)\right]_{q=q_i}$$

$$=q_i'\frac{\partial}{\partial q}\log\prod_{\substack{p=1\\ \neq i}}^{n}(q-q_p)\Bigg]_{q=q_i}$$

$$=\sum_{\substack{p=1\\ \neq i}}^{n}\frac{q_1'}{q_i-q_p},$$

being the sum of $n-1$ partial fractions.

6. Classification of Transition Points

A *specular transition point* (a reflection or coupling point in physical terms) exists at a point $z=z_0$ in the complex plane at which two or more characteristic roots attain equality.

A transition point of order two exists at a point where $q_1=q_2$; such a point may be denoted by $z=z_{12}$. The first and second diagonal elements each contain one partial fraction with a denominator (q_1-q_2), so this particular partial fraction (but no others down the leading digonal) is singular at the transition point.

Equation (23) shows that every nondiagnonal element is singular at z_{12} along rows one and two only. But only columns one and two have q_1' and q_2' respectively in their numerators; these derivatives may be singular at z_{12}. There is, however, a difference in the order of magnitude of the various coupling terms near the singularity z_{12}. For the coupling coefficients $(\mathbf{A}^{-1}\mathbf{A}')_{ij}$ $(i=1, 2)$ are larger in order of magnitude when $j=1, 2$ than when $j=3, 4, \ldots, n$. This implies that coupling exists mainly only between f_1 and f_2, and that $f_3, f_4, \ldots, f_n$ are relatively free from coupling with f_1 and f_2. In fact, the 2×2 matrix containing the first and second rows and the first and second columns is the distinct factor in determining the coupled solution near z_{12}.

If q_1 and q_2 are expressed in the form

$$q_1=\alpha+\beta,\ q_2=\alpha-\beta,$$

clearly $\beta=0$ at the transition point, while the first part $\alpha=\frac{1}{2}(q_1+q_2)$ is a more irrelevant term in determining the character of the coupling at the transition point.

Similarly, transition points of order two exist whenever $q_a=q_b$; such points may be represented by $z=z_{ab}$. Altogether, $\frac{1}{2}n(n-1)$ sets of second order transition points exist in the z-plane. The solution for f_a and f_b near z_{ab} is locally associated with a 2×2 matrix taken from the a- and bth rows and from the a- and bth columns of $\mathbf{A}^{-1}\mathbf{A}'$.

The determination just considered of the important terms governing the coupling process is clearly unsatisfactory, since infinite terms occur throughout. The embedding transformation discussed in the next sections removes this uncertainty and yields a clearer picture of the coupling process than the matrix $\mathbf{A}^{-1}\mathbf{A}'$, which merely shows where the coupling occurs.

Every nth order equation must of necessity possess these specular transition points of order two, since they are merely given by the solutions of the well-defined equations $q_a=q_b$. But some equations may possess transition points of higher order, not as a matter of necessity but exceptionally. If a point exists at which $q_a=q_b=q_c$, the point z_{abc} is called a third order transition point. All elements in rows a, b, c of the matrix $\mathbf{A}^{-1}\mathbf{A}'$ are now singular at the point z_{abc}, but only the columns a, b, c of these rows prove to be of importance, so the 3×3 matrix taken from these three rows and columns is the determining factor for the coupling beteen f_a, f_b, f_c.

Similar arguments may be used to define an rth $(r\leq n)$ order transition point. Such a point exists if $q_a=q_b=q_c= \ldots$ at $z=z_{abc\ldots}$, where r of the characteristic roots attain equality at this point. The essential coupling between $f_a, f_b, f_c, \ldots$ is determined by the $r\times r$ matrix selected from the appropriate rows and columns of $\mathbf{A}^{-1}\mathbf{A}'$.

These r values of the roots may be expressed according to the scheme (7), but for the purpose of this paper it is assumed that $q_j=\alpha+k^{j-1}\beta$. For the usual case when $r=2$, no assumption is thereby made.

It should be noted that of the remaining $n-r$ roots, s of these may also attain equality at $z=z_{abc}$ . . .. If their common value γ is not equal to α at z_{abc} . . ., then this set of s variables is not coupled to the previous set of r variables.

If various characteristic roots attain equality only as $z\rightarrow\infty$ for a certain range of arg z, the coupling is described as *continuous coupling* rather than specular.

Finally, irregular coupling may occur at points where the matrix **P** becomes singular, or at points where one or more of the characteristic roots become singular. It is not intended to discuss this form of coupling here, since it does not appear to be amenable to systematic treatment.

7. Embedding Transformation

Specular embedding is defined with respect to a particular transition point z_0 of order $r\leq n$. The first order coupled eqs (18) in the n variables are to be transformed in such a way that the r variables associated with the transition point appear in the new equations with a principal embedded coupled matrix (similar to (10)), while the remaining $n-r$ variables not associated with the transition point appear with a principal uncoupled matrix (a diagonal matrix), in such a way that all coupling terms in all equations are nonsingular at z_0.

The independent propagation of waves not affected by the coupling is thereby exhibited, together with the explicit form of the rth order equation satisfied by the r coupled waves. This coupling is seen to be independent of the remaining $n-r$ waves.

The general coupled equations

$$\mathbf{f}'=\mathbf{Qf}-\mathbf{A}^{-1}\mathbf{A}'\mathbf{f}-\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{Af}$$

are assumed to possess a transition point z_0 at which $q_1=q_2=\ .\ .\ .\ =q_r$. If $q_j=\alpha+k^{j-1}\beta$, where $k=e^{2\pi i/r}$, α and β being functions of z such that $\beta=0$ at z_0, a new column $\mathbf{g}$ is defined such that

$$\mathbf{f}=\begin{pmatrix}\mathbf{I}_r \exp\left(\int\alpha dz\right) & 0\\ 0 & \mathbf{I}_{n-r}\end{pmatrix}\mathbf{g}=\mathbf{Eg}$$

say, where $\mathbf{I}_p$ denotes the $p\times p$ unit matrix. Then

$$\mathbf{E}'\mathbf{g}+\mathbf{E}\mathbf{g}'=\mathbf{QEg}-\mathbf{A}^{-1}\mathbf{A}'\mathbf{Eg}-\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{AEg},$$

or

$$\mathbf{g}'=\mathbf{E}^{-1}\mathbf{QEg}-\mathbf{E}^{-1}\mathbf{E}'\mathbf{g}-\mathbf{E}^{-1}\mathbf{A}^{-1}\mathbf{A}'\mathbf{Eg}-\mathbf{E}^{-1}\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{AEg}. \tag{24}$$

Now

$$\mathbf{E}^{-1}\mathbf{QE}-\mathbf{E}^{-1}\mathbf{E}'=\mathbf{Q}-\begin{pmatrix}\alpha\mathbf{I}_r & 0\\ 0 & \mathbf{I}_{n-r}\end{pmatrix}$$

$$=\begin{bmatrix}\beta & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & k\beta & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 0 & k^2\beta & \cdots & 0 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & k^{r-1}\beta & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0 & q_{r+1} & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & q_n\end{bmatrix}$$

$$=\begin{pmatrix}\beta\mathbf{K}_r & 0\\ 0 & \mathbf{Q}_{n-r}\end{pmatrix}$$

say, where $\mathbf{K}_r$ is the diagonal matrix with elements $1, k, k^2, \ldots, k^{r-1}$ and $\mathbf{Q}_{n-r}$ is the diagonal matrix with elements $q_{r+1}, \ldots, q_n$.

Equation (24) now becomes

$$\mathbf{g}' = \begin{pmatrix} \beta\mathbf{K}_r & 0 \\ 0 & \mathbf{Q}_{n-r} \end{pmatrix} \mathbf{g} - (\mathbf{E}^{-1}\mathbf{A}^{-1}\mathbf{A}'\mathbf{E} + \mathbf{E}^{-1}\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A}\mathbf{E})\mathbf{g}. \tag{25}$$

Finally, a new column $\mathbf{h}$ is defined, satisfying

$$\mathbf{g} = \mathbf{R}^{-1}\mathbf{h},$$

where

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 0 \\ \beta & k\beta & k^2\beta & \ldots & k^{r-1}\beta & 0 \\ \beta^2 & k^2\beta^2 & k^4\beta^2 & \ldots & k^{2(r-1)}\beta^2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta^{r-1} & k^{r-1}\beta^{r-1} & k^{2(r-1)}\beta^{r-1} & \ldots & k^{(r-1)^2}\beta^{r-1} & 0 \\ 0 & 0 & 0 & \ldots & 0 & \mathbf{I}_{n-r} \end{bmatrix}$$

$$= \begin{pmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{I}_{n-r} \end{pmatrix} \tag{26}$$

say, where $\mathbf{B}$ is an alternant matrix similar to (11).

Equation (25) now becomes

$$\mathbf{R}^{-1\prime}\mathbf{h} + \mathbf{R}^{-1}\mathbf{h}' = \begin{pmatrix} \beta\mathbf{K}_r & 0 \\ 0 & \mathbf{Q}_{n-r} \end{pmatrix} \mathbf{R}^{-1}\mathbf{h} - (\mathbf{E}^{-1}\mathbf{A}^{-1}\mathbf{A}'\mathbf{E} + \mathbf{E}^{-1}\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A}\mathbf{E})\mathbf{R}^{-1}\mathbf{h},$$

or

$$\mathbf{h}' = \mathbf{R} \begin{pmatrix} \beta\mathbf{K}_r & 0 \\ 0 & \mathbf{Q}_{n-r} \end{pmatrix} \mathbf{R}^{-1}\mathbf{h} - \mathbf{R}\mathbf{R}^{-1\prime} - \mathbf{R}(\mathbf{E}^{-1}\mathbf{A}^{-1}\mathbf{A}'\mathbf{E} + \mathbf{E}^{-1}\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A}\mathbf{E})\mathbf{R}^{-1}\mathbf{h}. \tag{27}$$

The original eq (15) is transformed into this form by the transformation

$$\begin{aligned} \mathbf{e} &= \mathbf{S}\mathbf{f} \\ &= \mathbf{P}\mathbf{A}\mathbf{f} \\ &= \mathbf{P}\mathbf{A}\mathbf{E}\mathbf{g} \\ &= \mathbf{P}\mathbf{A}\mathbf{E}\mathbf{R}^{-1}\mathbf{h} \end{aligned} \tag{28}$$

In the next sections, it will be shown that (27) possesses the required embedding properties.

8. Embedded Equation

The coupling matrix occurring in (27) will be considered in two stages, firstly

$$\mathbf{R}\mathbf{R}^{-1\prime} + \mathbf{R}\mathbf{E}^{-1}\mathbf{A}^{-1}\mathbf{A}'\mathbf{E}\mathbf{R}^{-1} \tag{29}$$

and secondly

$$\mathbf{R}\mathbf{E}^{-1}\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A}\mathbf{E}\mathbf{R}^{-1}. \tag{30}$$

Since $\mathbf{R}\,\mathbf{R}^{-1} = \mathbf{I}$, it follows that

$$\mathbf{R}\mathbf{R}^{-1\prime} = -\mathbf{R}'\mathbf{R}^{-1}.$$

Moreover, $\mathbf{E}$ and $\mathbf{R}$ commute owing to the special forms of $\mathbf{E}$ and $\mathbf{R}$; hence (29) becomes

$$
\begin{aligned}
&-\mathbf{R}'\mathbf{R}^{-1}+\mathbf{E}^{-1}\mathbf{R}\mathbf{A}^{-1}\mathbf{A}'\mathbf{R}^{-1}\mathbf{E}\\
&=-\mathbf{E}^{-1}\mathbf{R}'\mathbf{A}^{-1}\mathbf{A}\mathbf{R}^{-1}\mathbf{E}-\mathbf{E}^{-1}\mathbf{R}\mathbf{A}^{-1\prime}\mathbf{A}\mathbf{R}^{-1}\mathbf{E}\\
&=-\mathbf{E}^{-1}(\mathbf{R}'\mathbf{A}^{-1}+\mathbf{R}\mathbf{A}^{-1\prime})\mathbf{A}\mathbf{R}^{-1}\mathbf{E}\\
&=-\mathbf{E}^{-1}(\mathbf{R}\mathbf{A}^{-1})'(\mathbf{A}\mathbf{R}^{-1})\mathbf{E}\\
&=\mathbf{E}^{-1}(\mathbf{A}\mathbf{R}^{-1})^{-1}(\mathbf{A}\mathbf{R}^{-1})'\mathbf{E}. \qquad (31)
\end{aligned}
$$

Also matrix (30) takes the form

$$
(\mathbf{A}\mathbf{E}\mathbf{R}^{-1})^{-1}\mathbf{P}^{-1}\mathbf{P}'(\mathbf{A}\mathbf{E}\mathbf{R}^{-1}). \qquad (32)
$$

At all points where coupling may be neglected (and this includes z_0 since the next section will demonstrate that all coupling matrices are nonsingular at z_0), equation (27) becomes

$$
\begin{aligned}
\mathbf{h}'&=\mathbf{R}\begin{pmatrix}\beta\mathbf{K}_r & 0\\ 0 & \mathbf{Q}_{n-r}\end{pmatrix}\mathbf{R}^{-1}\,\mathbf{h}\\
&=\begin{pmatrix}\mathbf{B} & 0\\ 0 & \mathbf{I}_{n-r}\end{pmatrix}\begin{pmatrix}\beta\mathbf{K}_r & 0\\ 0 & \mathbf{Q}_{n-r}\end{pmatrix}\begin{pmatrix}\mathbf{B}^{-1} & 0\\ 0 & \mathbf{I}_{n-r}\end{pmatrix}\mathbf{h}\\
&=\begin{pmatrix}\beta\mathbf{B}\,\mathbf{K}_r\mathbf{B}^{-1} & 0\\ 0 & \mathbf{Q}_{n-r}\end{pmatrix}\mathbf{h}.
\end{aligned}
$$

The elements of $\mathbf{h}$ are thus separated into two distinct parts. If the first r elements are denoted by $\mathbf{h}_r$ and the remaining $n-r$ elements by $\mathbf{h}_{n-r}$, it follows that

$$
\mathbf{h}'_r\ =\beta\mathbf{B}\,\mathbf{K}_r\mathbf{B}^{-1}\mathbf{h}_r, \qquad (33)
$$

$$
\mathbf{h}'_{n-r}=\mathbf{Q}_{n-r}\,\mathbf{h}_{n-r}. \qquad (34)
$$

It may immediately be verified that

$$
\beta\mathbf{B}\,\mathbf{K}_r\mathbf{B}^{-1}=\frac{1}{r}\begin{bmatrix}1 & 1 & 1 & \cdots\\ \beta & k\beta & k^2\beta & \cdots\\ \beta^2 & k^2\beta^2 & k^4\beta^2 & \cdots\\ \cdots & \cdots & \cdots & \cdots\end{bmatrix}\begin{bmatrix}\beta & 0 & 0 & \cdots\\ 0 & k\beta & 0 & \cdots\\ 0 & 0 & k^2\beta & \cdots\\ \cdots & \cdots & \cdots & \cdots\end{bmatrix}\begin{bmatrix}1 & \beta^{-1} & \beta^{-2} & \cdots\\ 1 & k^{-1}\beta^{-1} & k^{-2}\beta^{-2} & \cdots\\ 1 & k^{-2}\beta^{-1} & k^{-4}\beta^{-2} & \cdots\\ \cdots & \cdots & \cdots & \cdots\end{bmatrix}
$$

$$
=\begin{bmatrix}0 & 1 & 0 & \cdots\\ 0 & 0 & 1 & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ \beta^r & 0 & 0 & \cdots\end{bmatrix}. \qquad (35)
$$

Hence the elements of $\mathbf{h}_r$ are associated with an equation containing a principal embedded coupled matrix, while the elements of $\mathbf{h}_{n-r}$ are associated with a principal uncoupled matrix. In fact,

$$
h_1^{(r)}=\beta^r h_1=vh_1, \qquad (36)
$$

where v vanishes at z_0. The separation of the principal terms has therefore been achieved.

9. Nonsingular Coupling Terms

Every element in the coupling matrices (31) and (32) must now be shown to be nonsingular at z_0.

The elements of $\mathbf{A}\ \mathbf{R}^{-1}$ occurring in (31) are considered. Since

$$\mathbf{AR}^{-1}=\frac{1}{r}\begin{pmatrix} 1 & 1 & 1 & \dots \\ q_1 & q_2 & q_3 & \dots \\ q_1^2 & q_2^2 & q_3^2 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}\begin{pmatrix} 1 & \beta^{-1} & \dots & \beta^{-r+1} & 0 & \dots & 0 \\ 1 & k^{-1}\beta^{-1} & \dots & k^{-r+1}\beta^{-r+1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & k^{-r+1}\beta^{-1} & \dots & k^{-(r-1)^2}\beta^{-r+1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & r & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & r \end{pmatrix}$$

it follows that the element $(\mathbf{AR}^{-1})_{ij}$ when $1\leq i\leq n$, $1\leq j\leq r$, is given by

$$r(\mathbf{AR}^{-1})_{ij}=(q_1^{i-1}q_2^{i-1}\dots q_n^{i-1})\begin{pmatrix} 1 \\ (k^{-1})^{j-1} \\ (k^{-2})^{j-1} \\ \dots \\ (k^{-r+1})^{j-1} \end{pmatrix}\beta^{-j+1}$$

$$=\sum_{p=1}^{r} q_p^{i-1}(k^{-p+1})^{j-1}\beta^{-j+1}$$

$$=\sum_{p=1}^{r} (\alpha+k^{p-1}\beta)^{i-1}k^{-(p-1)(j-1)}\beta^{-j+1}$$

$$=\sum_{p=1}^{r}\left\{\sum_{l=0}^{i-1} {}_{i-1}C_l\alpha^{i-1-l}(k^{p-1}\beta)^l k^{-(p-1)(j-1)}\beta^{-j+1}\right.$$

$$=\sum_{l=0}^{i-1}\left[\sum_{p=1}^{r} k^{(p-1)(l-j+1)}\right]{}_{i-1}C_l\alpha^{i-1-l}\beta^{l-j+1}.$$

Now if $(l-j+1)$ is a multiple of r,

$$\sum_{p=1}^{r} k^{(p-1)(l-j+1)}=r,$$

while if $(l-j+1)$ is not a multiple of r, this sum vanishes by (8). Hence the only terms in β arising in $r(\mathbf{AR}^{-1})_{ij}$ are those of the form β^{Nr} where N is an integer. Moreover, the lowest value of $(l-j+1)$ is $-r+1$, so N can only equal 0, 1, 2, ... Hence

$$(\mathbf{AR}^{-1})_{ij}=\sum_{l} {}_{i-1}C_l\alpha^{i-1-l}\beta^{Nr}$$

where $l=Nr+j-1$, where N is a non-negative integer chosen so that

$$j-1\leq l\leq i-1.$$

In other words, this element is a polynomial in β^r.

It follows that $(\mathbf{AR}^{-1})'_{ij}$ contains β only through terms such as $Nr\ \beta^{Nr-1}\beta'$.

Finally, since $(\mathbf{AR}^{-1})^{-1}=(\text{adj }\mathbf{AR}^{-1})/(\det\mathbf{AR}^{-1})$ the elements of adj $\mathbf{AR}^{-1}$ are further polynomials in β^r, while

$$\det \mathbf{AR}^{-1}=(\det \mathbf{A})/(\det \mathbf{R})$$

$$=\prod_{\substack{i,j=1\\ i>j}}^{n} (q_i-q_j)/\beta^{1+2\cdots+(r-1)} \det \mathbf{K}$$

from (4) and (26). Now when i and j are both less than or equal to r, the relevant factors in the numerator are

$$(q_r-q_{r-1}) \ldots (q_r-q_1)(q_{r-1}-q_{r-2}) \ldots (q_{r-1}-q_1) \ldots (q_2-q_1)$$

which is proportional to

$$\beta^{r-1}\beta^{r-2} \ldots \beta^2\beta.$$

Hence $\det \mathbf{AR}^{-1}$ does not contain β as a factor, so the matrix $\mathbf{AR}^{-1}$ is nonsingular at z_0.

In the product $(\mathbf{AR}^{-1})^{-1}$ $(\mathbf{AR}^{-1})'$, each element contains terms of the form $\beta^{Mr}(\beta^{Nr})'$, where $M\geq 0$, $N\geq 0$ are integers. This is proportional to $\beta^{(M+N)r-1}\beta'$, with $M\geq 0$, $N\geq 1$, the lowest power of β yielding $\beta^{r-1}\beta'$. If β, when expanded in terms of $z-z_0$ commences with a term $(z-z_0)^b$, then $\beta^{r-1}\beta'$ contains the factor $(z-z_0)^{br-1}$. This is nonsingular at z_0 if $br-1\geq 0$; that is, if $b\geq 1/r$. This implies that $v(z)\equiv\beta^r$, when expanded as a power series in terms of $z-z_0$, should commence with a term that is linear at least.

Hence the elements of the primary coupling matrix (31) are nonsingular at z_0 under these circumstances. The secondary coupling matrix (32) is also nonsingular at z_0, since it has been shown that $\mathbf{AR}^{-1}$ is nonsingular there. It is assumed of course that $\mathbf{P}$ is nonsingular at z_0 in the definition of specular embedding.

10. An Explicit Case

To deal with an explicit case, consider a second order transition point associated with the third order differential equation

$$u'''=v_2u''+v_1u'+v_0u.$$

Equation (3) is

$$\frac{d}{dz}\begin{pmatrix} u \\ u' \\ u'' \end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ v_0 & v_1 & v_2 \end{pmatrix}\begin{pmatrix} u \\ u' \\ u'' \end{pmatrix}.$$

The transformation

$$\mathbf{u}=\begin{pmatrix} 1 & 1 & 1 \\ q_1 & q_2 & q_3 \\ q_1^2 & q_2^2 & q_3^2 \end{pmatrix}\mathbf{f}$$

yields equation (5), which, when expanded by the results of section 5, becomes

$$\mathbf{f}'=\begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{pmatrix}\mathbf{f}-\begin{pmatrix} \frac{q_1'}{q_1-q_3}+\frac{q_1'}{q_1-q_2} & \frac{q_2'}{q_1-q_2}+\frac{q_2'}{q_3-q_1} & \frac{q_3'}{q_1-q_3}+\frac{q_3'}{q_2-q_1} \\ \frac{q_1'}{q_2-q_1}+\frac{q_1'}{q_3-q_2} & \frac{q_2'}{q_2-q_1}+\frac{q_2'}{q_2-q_3} & \frac{q_3'}{q_2-q_3}+\frac{q_3'}{q_1-q_2} \\ \frac{q_1'}{q_3-q_1}+\frac{q_1'}{q_2-q_3} & \frac{q_2'}{q_3-q_2}+\frac{q_2'}{q_1-q_3} & \frac{q_3'}{q_3-q_1}+\frac{q_3'}{q_3-q_2} \end{pmatrix}\mathbf{f}.$$

If z_0 is a point at which $q_1=q_2$, and if $q_1=\alpha+\beta$, $q_2=\alpha-\beta$, the coupling matrix has the form

$$\begin{bmatrix} \dfrac{\alpha'+\beta'}{q_1-q_3}+\dfrac{\alpha'+\beta'}{2\beta} & \dfrac{\alpha'-\beta'}{2\beta}+\dfrac{\alpha'-\beta'}{q_3-q_1} & \dfrac{q_3'}{q_1-q_3}-\dfrac{q_3'}{2\beta} \\ -\dfrac{\alpha'+\beta'}{2\beta}+\dfrac{\alpha'+\beta'}{q_3-q_2} & -\dfrac{\alpha'-\beta'}{2\beta}+\dfrac{\alpha'-\beta'}{q_2-q_3} & \dfrac{q_3'}{q_2-q_3}+\dfrac{q_3'}{2\beta} \\ \dfrac{\alpha'+\beta'}{q_3-q_1}+\dfrac{\alpha'+\beta'}{q_2-q_3} & \dfrac{\alpha'-\beta'}{q_3-q_2}+\dfrac{\alpha'-\beta'}{q_1-q_3} & \dfrac{q_3'}{q_3-q_1}+\dfrac{q_3'}{q_3-q_2} \end{bmatrix}.$$

Singularities arise on account of the terms β', $1/\beta$, and β'/β. If β^2 is approximately linear near z_0, of the form $z-z_0$, $\beta\doteqdot(z-z)^{\frac{1}{2}}$, so the three singular terms are of the forms $(z-z_0)^{-\frac{1}{2}}$, $(z-z_0)^{-\frac{1}{2}}$ and $(z-z_0)^{-1}$ respectively. The β'/β terms are obviously of a higher order of magnitude than the others; these give rise to the particular matrix

$$\frac{\beta'}{2\beta}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

but other singular elements arise in rows 1 and 2 and columns 1 and 2.

The embedding transformation (28), in which $\mathbf{P}\equiv\mathbf{I}$, becomes

$$\mathbf{u}=\mathbf{AER}^{-1}\mathbf{h}$$

$$=\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ q_1 & q_2 & q_3 \\ q_1^2 & q_2^2 & q_3^2 \end{bmatrix}\begin{bmatrix} \exp\left(\int\alpha dz\right) & 0 & 0 \\ 0 & \exp\left(\int\alpha dz\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \beta^{-1} & 0 \\ 1 & -\beta^{-1} & 0 \\ 0 & 0 & 2 \end{bmatrix}\mathbf{h}$$

$$=\begin{bmatrix} \exp\left(\int\alpha dz\right) & 0 & 1 \\ \alpha\exp\left(\int\alpha dz\right) & \exp\left(\int\alpha dz\right) & q_3 \\ (\alpha^2+\beta^2)\exp\left(\int\alpha dz\right) & 2\alpha\exp\left(\int\alpha dz\right) & q_3^2 \end{bmatrix}\mathbf{h},$$

while the coupled eqs (27) become

$$\mathbf{h}'=\begin{bmatrix} 0 & 1 & 0 \\ \beta^2 & 0 & 0 \\ 0 & 0 & q_3 \end{bmatrix}\mathbf{h}-\mathbf{E}^{-1}(\mathbf{AR}^{-1})^{-1}(\mathbf{AR}^{-1})'\mathbf{Eh}$$

using results (31) and (35). The principal matrices yield approximately

$$\frac{d}{dz}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ \beta^2 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix},$$

$$h_3'=q_3h_3.$$

exhibiting the embedding property. It should be pointed out that when approximate solutions are written down, it may be necessary to retain some of the coupling terms. For example, in the last equation, an additional term in h_3 must be retained on the right hand side if the W.K.B. approximation for h_3 is to be obtained.

11. A Special Property of the Coupling Matrices

Under the original transformation (16), the coupling matrix is

$$\mathbf{A}^{-1}\mathbf{A}' + \mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A}, \tag{37}$$

while under the embedding transformation (28), the coupling matrix is

$$\mathbf{E}^{-1}(\mathbf{AR}^{-1})^{-1}(\mathbf{AR}^{-1})\mathbf{E} + (\mathbf{AER}^{-1})^{-1}\mathbf{P}^{-1}\mathbf{P}'(\mathbf{AER}^{-1}). \tag{38}$$

It is supposed that the last coupling matrix occurs when the first r elements of **f** or **h** are coupled together at z_0.

However, the $n-r\times n-r$ square matrix formed by crossing out the first r rows and the first r columns of both (37) and (38) are identical. This may be simply demonstrated by partitioning all the matrices between rows r and $r+1$ and between columns r and $r+1$, and expanding the products.

Hence the $n-r$ equations $(r+1\leq j\leq n)$ in the elements of **f**

$$f_j' = q_j f_j + \text{coupling terms in } f_1, \ldots, f_r + \text{coupling terms in } f_{r+1}, \ldots f_n$$

and the $n-r$ equations $(r+1\leq j\leq n)$ in the elements of **h**

$$h_j' = q_j h_j + \text{coupling terms in } h_1, \ldots, h_r + \text{coupling terms in } h_{r+1}, \ldots h_n$$

are identical as far as the coupling coefficients multiplying $f_{r+1}, \ldots, f_n$ and $h_{r+1}, \ldots, h_n$ respectively, are concerned.

Hence any coupling diagonal terms not neglected in the equation for f_j' $(r+1\leq j\leq n)$ are identical with the coupling diagonal terms not neglected in the equation for h_j' $(r+1\leq j\leq n)$.

12. Försterling-Type Coupled Equations

As far as ionospheric radio propagation is concerned, two differential equations each of the second order may be coupled together. The original form given by Försterling [1942] and that considered by Budden and Clemmow [1957] are specially simple, in that the characteristic equation for q reduces to a biquadratic. This special feature disguises the more general formulation that is possible.

The general formulation of this new type of coupled equation is as follows. Let z_0 be a transition point of order r, namely a point at which $q_1=q_2= \ldots =q_r$. Then the n coupled eqs (27) (with no singularities in the coupling coefficients at z_0) may be re-expressed in the form of $n-r+1$ equations in $n-r+1$ variables, taking the form

$$\begin{aligned} h_r^{(r)} &= \beta^r h_r + \text{coupling terms} \\ \mathbf{h}_{n-r}' &= \mathbf{Q}_{n-r}\mathbf{h}_{n-r} + \text{coupling terms}. \end{aligned} \tag{39}$$

The coupling terms involve only the $n-r+1$ variables h_r and $\mathbf{h}_{n-r}$. This result may be shown merely by eliminating $h_1, h_2, \ldots, h_{r-1}$ from the last $n-r+1$ eqs (27), the equation involving h_r' being differentiated $r-1$ times in the process. The coupling terms now involve derivatives of h_r, but the coefficients themselves are also derivatives, which would be small in a slowly-varying medium.

Similarly, if another transition point of order s $(2\leq s\leq n-r)$ occurs such that s of the q's in $\mathbf{Q}_{n-r}$ become equal, the number of equations may further be reduced.

This process is explicitly exhibited for $n=4$, such that a transition point of order 2 occurs at z_0 at which $q_1=q_2$, and another transition point of order 2 at z_1 at which $q_3=q_4$.

The given equation

$$\mathbf{e}' = \mathbf{T}\mathbf{e}$$

is transformed into

$$\mathbf{h}'=\begin{pmatrix}0 & 1 & 0 & 0\\ \beta^2 & 0 & 0 & 0\\ 0 & 0 & q_3 & 0\\ 0 & 0 & 0 & q_4\end{pmatrix}\mathbf{h}+\text{coupling terms}$$

by the transformation

$$\mathbf{e}=\mathbf{PAER}^{-1}\mathbf{h},$$

where

$$\mathbf{E}=\begin{pmatrix}\exp(\int\alpha dz) & 0 & 0 & 0\\ 0 & \exp(\int\alpha dz) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\quad \mathbf{R}=\begin{pmatrix}1 & 1 & 0 & 0\\ \beta & -\beta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Similarly, if $q_3=\gamma+\delta$, $q_4=\gamma-\delta$, and if

$$\mathbf{F}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \exp(\int\gamma dz) & 0\\ 0 & 0 & 0 & \exp(\int\gamma dz)\end{pmatrix},\quad \mathbf{S}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & \delta & -\delta\end{pmatrix}.$$

then the transformation

$$\mathbf{e}=\mathbf{PAER}^{-1}\mathbf{FS}^{-1}\mathbf{j}$$

(in which $\mathbf{E}$, $\mathbf{R}^{-1}$, $\mathbf{F}$, $\mathbf{S}^{-1}$ are mutually commutative) transforms the equation into

$$\mathbf{j}'=\begin{pmatrix}0 & 1 & 0 & 0\\ \beta^2 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & \delta^2 & 0\end{pmatrix}\mathbf{j}+\text{coupling terms} \tag{40}$$

where the coupling terms are nonsingular at z_0 and z_1.

Now let

$$\mathbf{m}=\begin{pmatrix}j_1\\ j_3\end{pmatrix},\qquad \mathbf{n}=\begin{pmatrix}j_2\\ j_4\end{pmatrix},$$

and two simultaneous matrix equations are formed involving $\mathbf{m}$ and $\mathbf{n}$; that is, eq (40) is multiplied first by $\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}$ and then by $\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$. This yields

$$\mathbf{m}'=\mathbf{n}+\mathbf{Wm}+\mathbf{Xn}, \tag{41}$$

$$\mathbf{n}'=\begin{pmatrix}\beta^2 & 0\\ 0 & \delta^2\end{pmatrix}\mathbf{m}+\mathbf{Ym}+\mathbf{Zn}$$

$$=\mathbf{\Delta m}+\mathbf{Ym}+\mathbf{Zn}, \tag{42}$$

say, where $\mathbf{W}$, $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ are 2×2 matrices originating by partitioning the 4×4 coupling matrix in (40).

Column $\mathbf{m}$ is now eliminated by differentiating (42):

$$\mathbf{n}''=\mathbf{\Delta m}'+\mathbf{\Delta}'\mathbf{m}+\mathbf{Y}'\mathbf{m}+\mathbf{Y}\mathbf{m}'+\mathbf{Z}\mathbf{n}'+\mathbf{Z}'\mathbf{n}$$

$$=(\mathbf{\Delta}+\mathbf{Y})\mathbf{m}'+(\mathbf{\Delta}'+\mathbf{Y}')\mathbf{m}+\mathbf{Z}\mathbf{n}'+\mathbf{Z}'\mathbf{n}$$

$$=(\mathbf{\Delta}+\mathbf{Y})(\mathbf{n}+\mathbf{W}\mathbf{m}+\mathbf{X}\mathbf{n})+(\mathbf{\Delta}'+\mathbf{Y}')\mathbf{m}+\mathbf{Z}\mathbf{n}'+\mathbf{Z}'\mathbf{n} \qquad \text{(from 41)}$$

$$=\mathbf{\Delta n}+(\mathbf{\Delta W}+\mathbf{Y}\mathbf{W}+\mathbf{\Delta}'+\mathbf{Y}')\mathbf{m}+\mathbf{Z}\mathbf{n}'+(\mathbf{\Delta X}+\mathbf{Y}+\mathbf{Y}\mathbf{X}+\mathbf{Z}')\mathbf{n}$$

$$=\mathbf{\Delta n}+(\mathbf{\Delta W}+\mathbf{Y}\mathbf{W}+\mathbf{\Delta}'+\mathbf{Y}')(\mathbf{\Delta}+\mathbf{Y})^{-1}(\mathbf{n}'-\mathbf{Z}\mathbf{n})+\mathbf{Z}\mathbf{n}'+(\mathbf{\Delta X}+\mathbf{Y}+\mathbf{Y}\mathbf{X}+\mathbf{Z}')\mathbf{n}$$

from (42). $\mathbf{\Delta n}$ is the only term on the right hand side whose coefficient does not involve a derivative. Hence $\mathbf{\Delta}$ is the principal uncoupled matrix, while all the other terms are coupling terms.

The coefficient of $\mathbf{n}'$ is $(\mathbf{\Delta W}+\mathbf{Y}\mathbf{W}+\mathbf{\Delta}'+\mathbf{Y}')(\mathbf{\Delta}+\mathbf{Y})^{-1}$. A subsidiary change of variable will eliminate n_1' from the first equation and n_2' from the second equation if required.

The above process may obviously be carried out for any number of variables.

13. Continuous Coupling

Let the elements of the original matrix $\mathbf{T}$ (see eq (15)) be functions of z that tend to constant limits as, say, $z\rightarrow-\infty$ along the negative real axis. Then the n characteristic roots also tend to limiting values as $z\rightarrow-\infty$.

If a range of real z existed such that all the elements of $\mathbf{T}$ were constants throughout that range, then all the coupling coefficients would be zero, in which case the n coupled eqs (18) reduce to

$$\mathbf{f}'=\mathbf{Q}\mathbf{f},$$

or

$$f_j'=q_jf_j,$$

implying n-independently propagated waves.

More generally, if a domain of z is chosen throughout which the coupling coefficients are small, the coupled equations reduce to

$$f_j'=q_jf_j-(\mathbf{A}^{-1}\mathbf{A}'+\mathbf{A}^{-1}\mathbf{P}^{-1}\mathbf{P}'\mathbf{A})_{jj}f_j,$$

where all nondiagonal coupling terms have been neglected; such approximate equations yield the W.K.B.-type solutions with arbitrary constants. These approximate solutions represent the n characteristic waves in the medium, and they are independently propagated whenever coupling may be neglected. In particular, only one characteristic wave may be considered, namely f_1, with $f_2=f_3=\ldots=f_n=0$. For this particular solution

$$\mathbf{e}=\mathbf{S}\mathbf{f},$$

so

$$e_1:e_2:\ldots:e_n=S_{11}:S_{21}:\ldots:S_{n1},$$

thereby providing in this domain the varying ratios of the values of the n original dependent variables associated with this characteristic wave.

The problem demanding attention concerns what happens to these ratios as $z\rightarrow-\infty$.

The coupling coefficients may not be neglected when:

i. coupling is specular; the previous embedding theory yields equations dealing with this case;

ii. coupling is irregular; namely, when $\mathbf{P}^{-1}\mathbf{P}'$ or when one of the q's is singular. This paper is not concerned with this case;

iii. coupling is continuous, namely when some of the q's tend to equality as $z\to-\infty$.

To be definite, let $q_1, q_2, \ldots q_r$ have a common limit as $z\to-\infty$. Then the denominators of many of the coupling coefficients considered in section 5 tend to zero. Thus continuous coupling between $q_1, q_2, \ldots, q_r$ is embedded in the background provided by the complete n equations. Previously, specular coupling has been explicitly exhibited without approximations. Here, however, it proves advantageous to use approximations in the equations.

Moreover, it is assumed that the coupling coefficients do not become large in the range of continuous coupling. Every denominator of the form (q_1-q_j), $1\leq i,j\leq r$, contains a derivative in its numerator, and as $z\to-\infty$ it is assumed that the derivative is such that the whole ratio remains finite at least.

Above the region of continuous coupling, a definite approximate solution is considered, namely f_1, with $f_2=f_3=\ldots f_n=0$. Within the region, $\mathbf{e}=\mathbf{S}\ \mathbf{f}$, with $f_{r+1}=f_{r+2}=\ldots=f_n=0$ throughout. Then

$$e_1:e_2:\ldots:e_n=\sum_{j=1}^{r} S_{1j}f_j+\sum_{j=1}^{r} S_{2j}f_j:\ldots:\sum_{j=1}^{r} S_{nj}f_j$$

$$=\sum_{j=1}^{r} S_{1j}(f_j/f_1):\sum_{j=1}^{r} S_{2j}(f_j/f_1):\ldots:\sum_{j=1}^{r} S_{nj}(f_j/f_1) \qquad (43)$$

ratios depending upon the $r-1$ values $f_2/f_1, f_3/f_1, \ldots, f_r/f_1$.

These ratios are denoted by $r_2, r_3, \ldots, r_r$ respectively, and their initial values above the region of continuous coupling are all zero.

The coupled equations

$$\mathbf{f}'=\mathbf{Q}\ \mathbf{f}-\mathbf{S}^{-1}\mathbf{S}'\mathbf{f}$$

simplify to

$$f_i'=q_if_i-\sum_{j=1}^{r}(\mathbf{S}^{-1}\mathbf{S}')_{ij}f_j, \qquad (1\leq i\leq r)$$

in which the diagonal elements in $\mathbf{S}^{-1}\mathbf{S}'$ may oftentimes be neglected since these would be small compared with the q_i.

A new set of $r-1$ equations may now be derived, with $r_2, r_3, \ldots, r_r$ as the dependent variables. The differentiation of r_i $(2\leq i\leq r)$ yields

$$r_i'=\frac{d}{dz}\left(\frac{f_i}{f_1}\right)=\frac{f_i'}{f_1}-\frac{f_i}{f_1}\cdot\frac{f_1'}{f_1}$$

$$=\frac{1}{f_1}\left[q_if_i-\sum_{j=1}^{r}(\mathbf{S}^{-1}\mathbf{S}')_{ij}f_j\right]-r_i\frac{1}{f_1}\left[q_1f_1-\sum_{j=1}^{r}(\mathbf{S}^{-1}\mathbf{S}')_{1j}r_j\right]$$

$$=q_ir_i-\sum_{j=1}^{r}(\mathbf{S}^{-1}\mathbf{S}')_{ij}r_j-r_i\left[q_1-\sum_{j=1}^{r}(\mathbf{S}^{-1}\mathbf{S}')_{1j}r_j\right],$$

being $r-1$ simultaneous nonlinear differential equations for $r_2, r_3, \ldots, r_r$. These are the appropriate differential equations governing continuous coupling. Integration starts with zero values, yielding $r_2, r_3, \ldots, r_r$ as $z\to-\infty$, which yield in turn the required ratios (43) of the e's associated with this characteristic wave. In the region of continuous coupling, the original ratios $S_{11}:S_{21}:\ldots:S_{n1}$ are converted into the final values.

14. References

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