JOURNAL OF RESEARCH of the National Bureau of Standards—D. Radio Propagation
Vol. 65D, No. 6, November–December 1961

On the Spectrum of Terrestrial Radio Noise at Extremely Low Frequencies[1,2]

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(Received May 30, 1961; revised June 15, 1961)

A theory of the frequency spectrum of radio noise at extremely low frequency (ELF) is presented and the results compared with recent measurements of the first five "Schumann" resonant modes (between 8 and 34 c/s) made by Balser and Wagner [1960]. The source of this noise is assumed to be return strokes in vertical cloud-ground lightning flashes distributed randomly in time, uniformly in angular displacement along the earth relative to the observer and with statistics of stroke duration, interstroke intervals and strokes per flash taken from studies of thunderstorms reported by J. C. Williams. Thus, the mathematical model for the noise sources is an extremely simple one, being analogous to the shot effect in electron devices. The electromagnetic model employs the familiar waveguide mode theory, assumes a sharply bounded homogeneous ionosphere, and neglects the earth's magnetic field.

Agreement between the shape of the theoretical and observed spectrum is good for the first three modes and rather poor for the higher modes.

It is found by matching the theoretical resonant frequencies to the observed resonances that the product of effective ionosphere height h and the square root of effective conductivity $\sqrt{\sigma_i}$ is a decreasing function of frequency. The functional dependence of this quantity on frequency is determined and used in the calculation of the mode spectrum.

Discrepancies between the theory and experimental results are believed to be partially due to the artificiality of the sharply bounded homogeneous ionosphere model and to failure to give sufficient probability weighting to equatorial regions of abnormally high thunderstorm activity.

These last items are the subjects of continuing work on the extension of the theory.

1. Introduction

The theory of extremely low frequency (ELF) propagation, as originally developed by Schumann [1957] and extended by Wait [1960a] and others, predicts the existence of a series of modes whose resonant frequencies, relative amplitudes, and Q's can be found by considering the earth-air-ionosphere cavity as a linear filter and calculating its frequency response function directly from electromagnetic theory. Recently, Balser and Wagner [1960] have observed these modes and experimentally determined the detailed shape of the mode spectrum.

The basis of the cavity analog is the classical Sommerfeld theory of propagation around a spherical earth, which leads to the representation of the field between earth and ionosphere as a series of "waveguide modes." If the ionosphere height is extremely small compared to wavelength, as is true at ELF, the electric field is essentially radial and a single waveguide mode is sufficient to describe the fields at a distance from a localized source. This waveguide mode when considered in the time domain can, in turn, be shown (by an appropriate expansion of the hypergeometric series into a series of Legendre polynomials) to be a superposition of waveforms each closely approximating a damped sinusoid of frequency $\frac{1}{2\pi}\frac{c}{a}\sqrt{n(n+1)}$, where c and a are velocity of light and earth radius, respectively and n is a mode index number running from zero to infinity. This phenomenon is entirely analogous to the ringing of a resonant cavity in response to impulse excitation.

Experimentally observed VLF noise is thought to consist of a superposition of responses from lightning flashes all over the world [Williams, 1959]. Therefore, assuming that this hypothesis is correct down to ELF, a theoretical model based on it should yield accurate

[1] Work supported by the Office of Naval Research.
[2] A preliminary note on this work appeared in the J. Geophys. Research, May 1961.

calculations of the spectrum of observed ELF noise. It is the purpose of the study reported here to construct such a model and use it to calculate the expected ELF noise spectrum. The results are compared with the measurements of Balser and Wagner [1960].

2. ELF Cavity Mode Theory

Following Wait [1960a] one can express the significant (radial) component of the electric field at an observation point due to the harmonic excitation (at frequency ω) of a vertical electric dipole of moment Ids on the surface of the earth in the form

$$E_r(i\omega) = Ids \sum_{n=0}^{\infty} C_n(\theta) F_n(i\omega) \tag{1}$$

where the ionosphere is assumed to be electrically uniform and sharply bounded at height h, and the following definitions apply:

$$C_n(\theta) = \frac{i}{4\pi a^2 \epsilon_0 h} P_n(\cos\theta)(2n+1)$$

ϵ_0 = Dielectric constant of free space

$P_n(\cos\theta)$ = Legendre polynomial of order n

θ = Polar angle relative to the source

and

$$F_n(i\omega) = \frac{1}{i\omega} \frac{-\omega_v^2(i\omega)}{[-\omega_v^2(i\omega)+\omega_n^2]}$$

with

$$\omega_v^2(i\omega) = \frac{c^2}{a^2} v(v+1) = -(i\omega)^2 - \alpha(i\omega)^{3/2}$$

$$\omega_n^2 = \frac{c^2}{a^2} n(n+1)$$

and

$$\alpha = \frac{1}{h\sqrt{\mu\bar{\sigma}_i}} \left(1+\sqrt{\frac{\bar{\sigma}_i}{\bar{\sigma}_g}}\right)$$

where

$$\bar{\sigma}_g = \sigma_g + i\epsilon_g\omega_g \approx \sigma_g = \text{complex conductivity of the earth}$$

and

$$\bar{\sigma}_i = \frac{e^2}{m}\left\{\frac{N}{(g-i\omega)}\right\} = \sigma_i + i\omega\epsilon_i \approx \sigma_i = \text{complex conductivity}$$

of the ionosphere, where N, e, m, g, are the electron density, charge, mass, and collision frequency [Stratton, 1941], and it is assumed that $\omega << g$. In the calculation of the appropriate ionospheric electrical parameters the effects of the earth's magnetic field and the motions of the heavy ions are neglected; in addition, the electron collision frequency is taken to be independent of the nonrandom electron energy. In general $|\bar{\sigma}_g| >> |\bar{\sigma}_i|$, so in what follows α, which will be called the "ionospheric loss parameter," is approximately a real quantity, equal to $1/(h\sqrt{\mu\sigma_i})$.

The function $E_r(i\omega)$ is actually the Fourier transform of the field in response to a (vertical) current impulse of magnitude I at $\theta=0$ and at $t=0$, i.e., it can be regarded as the characteristic frequency response function of the earth-air-ionosphere cavity. $F_n(i\omega)$ is the Fourier transform of the nth mode excited by an impulse of dipole current of magnitude I. From knowledge of all of the $F_n(i\omega)$'s one can predict the frequency response to any time varying source current using standard linear system theory.

3. Superposition of Fields From ELF Sources

The radial field component is expressed here in a spherical coordinate system wherein the source is at the pole. Because of azimuthal symmetry about the source, we can without changing the form of (1) consider $E_r(\omega)$ as expressed in a system with pole at the observer and source at polar angle θ. Suppose that lightning flashes, assumed equivalent to vertical dipole sources, are distributed around the earth's surface. We define $E_r(\omega;\theta,\phi)$ as the Fourier transform of the radial field component at the pole due to a vertical dipole source at the point θ,ϕ. The source current moment waveform at θ,ϕ and its Fourier transform are denoted by $\underset{\vee}{i}(t;\theta,\phi)$ and $\underset{\vee}{I}(\omega;\theta,\phi)$, respectively.

The voltage induced in a receiver at the observation point at time t due to the presence of lightning sources all over the world is proportional to

$$v(t)=\sum_{k=1}^{N}\int_{-\infty}^{t} dt' f(t-t';\,\theta_k,\phi_k)\,\underset{\vee}{i}(t';\,\theta_k,\phi_k) \tag{2}$$

where the index k is used to number the source positions, from 1 through N, and

$$f(t;\,\theta,\phi)=\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)f_n(t)$$

where $f_n(t)$ is the inverse Fourier transform of $F_n(i\omega)$, i.e., the impulse response of the *nth* earth-air-ionosphere cavity node.

4. Calculation of the Power Spectrum

The evaluation of the power spectrum of $v(t)$ falls within a familiar class of problems in random noise theory, specifically, the spectral analysis of the voltage arising from the "shot effect" in electron devices. This problem has been treated by Rice [Wax, 1954] and others [Davenport and Root, 1958]. The analysis to follow is mathematically equivalent to that of the shot effect.

Consider a voltage, $v(t)$, appearing at the output of a linear filter with impulse response $f(t)$ due to a train of input pulses, each of which is denoted by $\xi(t-t_k;\beta)$. The shape of the *kth* pulse is determined by the components of a vector β, its magnitude (time integral of amplitude over duration of pulse) is a_k, and its time of occurence is t_k. The output voltage is thus

$$v(t)=\sum_{k} a_k\int_{-\infty}^{\infty} f(t-t')\xi(t'-t_k;\,\underline{\beta}_k)dt' \tag{3}$$

We assume a time interval [3] $-T\leq t\leq T$ over which all the action takes place and that there are exactly N pulses within that interval. Because of finite filter "memory" and the filter realizability requirement,[4] we can regard as zero the response to pulses occurring at values t_k outside the interval $(t-T_0)$ to t. Here T_0 is the time of occurence of the earliest imput influencing the filter output at time t. The interval $(t-T_0)$ is less than T but somewhat greater than the filter time constant. Numbering from 1 to N the pulses within the interval $-T$ to T, and using (3) we can write

$$v(t)=\sum_{k=1}^{N} a_k g(t-t_k;\,\underline{\beta}_k) \tag{4}$$

where

$$g(t-t_k;\,\beta_k)=\int_{t-T_0}^{t} f(t-t')\xi(t'-t_k;\,\underline{\beta}_k)dt';\ -T\leq t_k\leq T;\qquad k=1,\,.\,.\,.\,,N,$$

the response of the filter at time t to a pulse occurring at time t_k.

[3] The time T washes out in the final results; the finite time assumption is strictly for mathematical convenience in the derivation.
[4] i.e., the requirement that the filter can respond only to past (and not to future) inputs.

We now assume that: (1) The a_k's are independent and identically distributed random variables with common probability density function $p(a)$; (2) for a fixed number of pulses within the interval $t-T_0$ to t, the t_k's are statistically independent random variables, each uniformly distributed throughout the interval $-T$ to T; (3) the probability of finding exactly N pulses within the interval $t-T_0$ to t, denoted by $p(N)$, is independent of the statistics of the a_k's and β_k's; (4) the β_{ki}s are independent and identically distributed random variables with common probability density function $p(\beta)$; and (5) the random variables a_k, t_k, and β_k are all mutually statistically independent.

By calculating the autocorrelation function of $v(t)$ under the assumptions (1) through (5) and taking its Fourier transform, we obtain for the spectrum [5]

$$S(\omega)=\frac{1}{4T^2}\{\langle N(N-1)\rangle\langle a\rangle^2\overline{\langle g(t)\rangle}\overline{\langle g(t+\tau)\rangle}\delta(\omega)\}+\hat{S}(\omega) \tag{5}$$

where

$$\hat{S}(\omega)=\frac{\langle N\rangle\langle a^2\rangle}{2T}\int_{\text{All }\underline{\beta}\text{-space}} d\underline{\beta}p(\underline{\beta})|G(\omega;\underline{\beta})|^2$$

and

$$G(\omega;\underline{\beta})=\int_{-\infty}^{\infty} dte^{-i\omega t}g(t;\underline{\beta})$$

By the well-known convolution theorem for Fourier transforms

$$G(\omega;\underline{\beta})=F(i\omega)\tilde{\xi}(\omega;\underline{\beta}) \tag{6}$$

where $F(i\omega)$ is the frequency response function of the filter and $\tilde{\xi}(\omega;\underline{\beta})$ is the Fourier transform of the pulse.

The first term of (5) is a "DC spike," and is not required to determine the shape of the mode spectrum in the frequency region of interest. We will, therefore, concern ourselves only with $\hat{S}(\omega)$.

5. Statistics of Lightning Source Currents

The result (5) can be adapted to the calculation of the ELF noise spectrum by allowing the pulses $\xi(t-t_k;\underline{\beta}_k)$ to represent lightning flashes whose shape parameters, time of occurrence t_k, and polar angles θ_k are assumed to be random variables, statistically independent when $j\neq k$. The angles θ_k are included among the components of the vector $\underline{\beta}_k$, as are the number of strokes in a flash [Williams, chapter VI], and the shaping parameters associated with individual strokes.

The statistics of the lightning flashes are assumed to obey all of the assumptions (1) through (5). It does not necessarily follow that we can with physical justification define a spatial probability density function $p(\theta)$ apart from consideration of other random variables associated with flashes. However, this assumption is basic to what follows and we note that it is tantamount to the assumption that statistics of lightning flash shape (i.e., those types of flash important as sources of VLF noise) are independent of those of terrestrial location of the flashes. The essential correctness of the latter assertion is supported by experimental evidence [Williams, tables 6–1, 6–2, 6–4, figures 6–1, 6–2, 6–7, 6–11, 6–14].

Using (5) and (6), the definition of $f(t)$ given in (2), and the assumptions about lightning flashes discussed above, we obtain for the ELF noise spectrum (exclusive of the DC component)

$$\hat{S}(\omega)=K\langle|\tilde{\xi}(\omega,\underline{\beta}')|\rangle\sum_{m,n=0}^{\infty} S_{mn}(\omega) \tag{7}$$

where K is a constant, $\underline{\beta}'$ represents all lightning flash parameters exclusive of θ, and

$$S_{mn}(\omega)=(2m+1)(2n+1)F_m(i\omega)F_n^*(i\omega)L_{mn},$$

[5] Ensemble averaging over β_k is denoted by <..>, averaging over the time variables t_k by a line over the symbol.

where

$$L_{mn}=2\pi\int_0^{\pi} d\theta \sin\theta P_m(\cos\theta)P_n(\cos\theta)p(\theta),$$

$$\langle|\tilde{\xi}(\omega;\underline{\beta}')|^2\rangle=\int_{\text{All }\underline{\beta}'\text{ space}} d\underline{\beta}' p(\underline{\beta}')|\tilde{\xi}(\omega;\underline{\beta}')|^2$$

$p(\underline{\beta}')$ is the probability density of the set of parameters $\underline{\beta}'$, and $p(\theta)\sin\theta d\theta$ is the probability that the lightning flash is located between θ and $\theta+d\theta$.

We are interested only in relative and not absolute amplitudes; therefore, in numerical computations we will fix K by assuming a normalization that equates the average power levels in the theoretical and experimental results.

6. Flash Shape Parameters

J. C. Williams [ch. V, sec. 5f] has concluded from his studies of thunderstorms that the overwhelming cause of VLF noise is the presence of vertical currents in the return strokes associated with cloud-to-ground lightning flashes. These flashes consist of from 2 to 12 strokes of similar shape. He has compiled statistics on flash parameters, including the shape of the median return stroke [ch. VI, sec. 1, 4, 5, and ch. VII, sec. 1], interstroke interval, and number of strokes per flash. He has also made spectral calculations for the VLF noise resulting from the flashes, but these are in a higher frequency range and inapplicable to our problem (ch. VII, sec. 3).

The current moment of Williams' median return stroke is proportional to

$$\underset{\vee}{}(t-t_k)=\left(\int_0^t \hat{v}(t')dt'\right) i(t-t_k)=\{-16.8(10^3(e^{-(t-t_k)/(1.7\times10^{-6})}+15.35(10^{-3})e^{-(t-t_k)/(33\times10^{-6})}$$
$$+10^3e^{-(t-t_k)/(5\times10^{-4})}+0.45(10^3)e^{-(t-t_k)/(6.8\times10^{-3})}\}u(t-t_k)\{1-e^{-5.55(10^{-3})(t-t_k)}\} \quad (8)$$

where $\hat{v}(t)$ is the velocity of the lightning charge along the cloud-ground channel, the unit step is inserted to assure that the stroke has zero value before time t_k and the lightning current $i(t-t_k)$, $v(t)$ and t_k are measured in amperes, kilometers/second and seconds, respectively.

The normalized spectrum of (8) is proportional to

$$|\underset{\vee}{I}(\omega)|^2=\left|e^{i\omega t_k}\sum_{l=1}^{4} I_l\tau_l\left\{\frac{1}{1-i\omega\tau_l}-\frac{\tau_v}{\tau_l+\tau_v- \quad (:}\right\}\right| \quad (9)$$

where I_l and τ_l are respectively the currents and time constants given in (8) and $\tau_k=1.80(10^{-4})$ sec.

It would be desirable in the interest of accuracy to consider the shaping parameters of the stroke model as random variables and average the spectrum over these variables. This is a formidable mathematical problem in general; and moreover, there is insufficient experimental data available to assign probability density functions to these parameters, a measure which would be required to evaluate the averages in the general case. The assumption that all shaping parameters have small standard deviations relative to their mean values would reduce the problem to one requiring only a knowledge of first and second moments of the distribution of each parameter. However, experimental knowledge of even these quantities is not available in enough detail to allow any meaningful assignment of numbers to the calculation of the spectrum of the median return stroke itself. This is tantamount to assuming that all return strokes are nearly identical in shape. This is not quite accurate, but the analysis will still provide useful insights into the effects of stroke shape on the ELF spectrum.

The waveform of a flash consisting of M identical strokes [6] is

$$\xi(t)=\sum_{l=1}^{M} \underset{\vee}{i}(t-t_l), \tag{10}$$

where $\underset{\vee}{i}(t-t_l)$ is given by (8).

The spectrum of a lightning flash is roughly proportional to the averaged absolute square of the Fourier transform of $\xi(t)$, i.e.,

$$\langle|\tilde{\xi}(\omega)^2\rangle=|\underset{\vee}{I}(\omega)|^2 \sum_{l,k=1}^{M} \langle e^{i\omega(t_l-t_k)}\rangle \tag{11}$$

where the indicated average is over the ensemble of the interstroke time intervals t_l—t_k.

An obvious classification of terms in the summation of (11) results in

$$\rho_M(\omega)\equiv\frac{\langle|\tilde{\xi}(\omega)|^2\rangle}{|I(\omega)|^2}=M+\sum_{\substack{l,k=1\\ l\neq k}}^{M} e^{i\omega(l-k)T}\langle e^{i\omega\Delta T_{lk}}\rangle \tag{12}$$

where T is the average interstroke interval and ΔT_{lk} is the deviation of the interval (t_l-t_k) from $(l-k)T$.

To calculate $\langle e^{i\omega\Delta T_{lk}}\rangle$ accurately, we would require a knowledge of the probability density function for ΔT_{lk}. An extremely rough estimate of this function is provided by Williams [ch. VI, figs. 6–7], but it is not specified in sufficient detail to allow a precise representation. A rough processing of Williams' curve shows a Gaussian distribution to be a reasonable approximation for this variable, with standard deviation σ_{lk}, the latter being assumed somewhat smaller than T. The result of a simple integration is

$$\langle e^{i\omega\Delta T_{lk}}\rangle=e^{-\omega^2\sigma^2_{lk}/2} \tag{13}$$

In the absence of experimental data on second order statistics of ΔT_{lk}, it will be assumed that the random variable $(t_l-\langle t_l\rangle)$ is not strongly correlated with $(t_k-\langle t_k\rangle)$, where $k\neq l$. It follows that

$$\sigma^2_{lk}\simeq\sigma^2_t \qquad \text{for all } l\neq k \tag{14}$$

Invoking (13) and (14) in (12), using the power series for $1/(1-x)$ to bring the summation into closed form and averaging over all possible values of M, we arrive at the results.

$$\rho(\omega)=\sum_{M=0}^{\infty} \rho_M(\omega)p(M) \tag{15}$$

where $p(M)$ is the probability of M strokes in a flash and

$$\rho_M(\omega)=M(1-e^{-\omega^2\sigma_t^2/2})+e^{-\omega^2\sigma_t^2/2}\left(\frac{\sin^2\frac{M\omega T}{2}}{\sin^2\frac{\omega T}{2}}\right) \tag{16}$$

Averaging (16) over values of M, we obtain

$$\rho(\omega)\simeq\langle M\rangle(1-e^{-\omega^2\sigma_t^2/2})+e^{-\omega^2\sigma_t^2/2}\sum_{n=0}^{\infty} A_n \cos n\,\omega T \tag{17}$$

where

$$A_k=\sum_{M=K}^{\infty}(M-K)p(M).$$

[6] Actually the strokes are not quite identical, the first being about twice as intense as succeeding ones. However, the approximation will not introduce serious errors into an analysis of this type.

The parameters T and σ_t have been roughly estimated from Williams (fig. 6–7) and found to be about 33 and 7 msec, respectively. The precision of these estimates is questionable, but they are probably correct to within 20 percent. The same remarks apply to similar rough estimates made of $p(M)$ from Williams' cumulative probability distribution curves for the number of strokes per flash (fig. 6–1). With these estimates we conclude that the probabilities $p(1)$ through $p(10)$ can be roughly approximated by the numbers 0.15, 0.22, 0.23, 0.15, 0.09, 0.06, 0.04, 0.03, 0.02, 0.01, and 0, that $\langle M\rangle$ is about 2.5 msec, that the numbers A_0 through A_9 are roughly equal to 3.5, 2.5, 1.6, 1.0, 0.61, 0.36, 0.20, 0.10, 0.04, and 0.01, and that A_n effectively vanishes for $n>9$.

7. Geographical Distribution of Lightning Activity

The assumption is now introduced that the probability of occurrence of a lightning flash within a range of angles θ, $\theta+d\theta$ during the time interval between $t-T_0$ and t is

$$\sin\theta p(\theta)d\theta=\int_0^{2\pi} d\phi(p(\theta,\phi)\sin\theta d\theta)=(p_0+\bar{p}(\theta))d\theta\sin\theta; \qquad 0\leq\theta\leq\pi \tag{18}$$

where $p(\theta,\phi)d\theta\sin\theta d\phi$ is the flash occurrence probability for a region of solid angle $d\theta\sin\theta d\phi$ at θ,ϕ, p_0 is the part of the θ distribution that is uniform in the angle θ, and $\bar{p}(\theta)$ accounts for the existence of regions of abnormally high or low average thunderstorm activity.

Initially, we will neglect $\bar{p}(\theta)$ and assume a completely uniform distribution p_0. The result of applying this assumption to (7) (see appendix) with the aid of the orthonormality of the Legendre polynomials is

$$\frac{\hat{S}(\omega)}{p_0\langle|\tilde{\xi}(\omega,\underline{\beta}')|^2\rangle}=\sum_{n=0}^{\infty}(2n+1)|F_n(i\omega)|^2 \tag{19}$$

Considering now the term $\bar{p}(\theta)$, it has been established that there are three equatorial regions of abnormally high average thunderstorm frequency [Handbook of Geophysics, 1960]. These regions are located in (1) South America, (2) Africa, and (3) the Southwest Pacific area. We will denote them respectively by R_1, R_2, and R_3. The regions center approximately at (50° W, −10° S), (15° E, +10° N), and (110° E, +10° N) respectively, and during all seasons of the year they show an average thunderstorm frequency somewhat higher than is observed elsewhere in the world. Speaking more quantitatively, the spatial probability density in those regions is from 100 to 1,000 times higher than that throughout most of the earth's surface.

There is another near-equatorial region in Southeast Asia (~90° E, +20° N), showing substantial year-round thunderstorm activity, but not as high as in R_1, R_2, and R_3. This region has not been accounted for in our calculations. Certain nonequatorial land regions show intense activity during their summers, e.g., the Southeastern United States (centered about (75° W, +30° N)), but this has also been excluded from our analysis. It is desired here to concentrate on those features of the ELF noise spectrum that are not season-dependent, and this objective is best fulfilled by studying the contributions from the three most prominent equatorial regions R_1, R_2, and R_3.

In view of the approximate nature of meteorological data, it would not be feasible to attempt to construct a precise thunderstorm probability function for the regions under investigation. We will, therefore, construct an idealized theoretical model which is convenient for computation and which embodies the essential features of the observed distributions.

We will approximate the probability density functions within the regions R_j by Gaussian functions

$$p_j(\theta)=K_j\frac{e^{-(\theta-\theta_j)^2/2\sigma_j^2}}{\sqrt{2\pi\sigma_j^2}} \tag{20}$$

where K_j is a constant, θ_j is the center of the distribution in R_j and σ_j is the standard deviation.

Using (7), (18), and (20), we have

$$L_{mn}=\frac{4\pi p_0}{2n+1}\,\delta_{mn}+2\pi\sum_{j=1}^{N}K_j l_{mnj} \tag{21}$$

where

$$l_{mnj}=\int_0^{\pi} d\theta\,\sin\theta P_m(\cos\theta)P_n(\cos\theta)\,\frac{e^{-(\theta-\theta_j)^2/2\sigma_j^2}}{\sqrt{2\pi\sigma_j^2}}$$

The integrals l_{mnj} have no simple orthogonality property to aid in their evaluation. The method to be used here will involve the expansion of the Legendre polynomials $P_n(\cos\theta)$ in a Fourier Series of the form [Jahnke-Emde, 1945]

$$P_n(\cos\theta)=\sum_{\mu=0}^{n}\tilde{a}_{n\mu}\cos\mu\theta \tag{22}$$

where the coefficients $\tilde{a}_{n\mu}$ are given in reference 9. From (21) and (22) we have

$$l_{mnj}=\sum_{\mu=0}^{m}\sum_{u=0}^{n}\tilde{a}_{m\mu}\tilde{a}_{n\nu}\mathsf{U}_{m\mu n\nu} \tag{23}$$

where

$$\mathsf{U}_{m\mu n\nu}\equiv\int_0^{\pi}d\theta\,\sin\theta\,\frac{e^{-(\theta-\theta_j)^2/2\sigma_j^2}}{\sqrt{2\pi\sigma_j^2}}\cos\mu\theta\cos\nu\theta$$

Using trigonometric identities, assuming (justifiably, with the parameter values of regions R_1, R_2, R_3) that (1) σ_j is small compared to π and (2) θ_j is not near 0 or π (so that the limits of the integral are effectively $-\infty$ and $+\infty$), and integrating by completion of the square in the exponential, we obtain

$$\frac{\hat{S}(\omega)}{p_0\langle|\tilde{\xi}(\omega,\underline{\beta}')|^2\rangle}-\sum_{n=0}^{\infty}(2n+1)|F_n(i\omega)|^2=\frac{1}{P_0}\sum_{j=1}^{N}2\pi K_j\sum_{l=1}^{\infty}\sin l\theta_j e^{-l^2\sigma_j^2/2}B_l(\omega) \tag{24}$$

where

$$B_l(\omega)=\frac{\hat{A}_{l-1}(\omega)-\hat{A}_{l+1}(\omega)}{2}\qquad \text{if } l\neq 1$$

$$B_1(\omega)=\hat{A}_0(\omega)-\frac{\hat{A}_2(\omega)}{2}$$

$$\hat{A}_0(\omega)=\frac{1}{2}\left\{\sum_{\mu=0}^{\infty}\{[\hat{a}_\mu(\omega)]_R^2+[\hat{a}_\mu(\omega)]_I^2\}\right.$$

$$\left.+[\hat{a}_0(\omega)]_R^2+[\hat{a}_0(\omega)]_I^2\right\}$$

and

$$\hat{A}_k(\omega)=\sum_{v=0}^{\infty}\{[\hat{a}_{v+k}(\omega)]_R[\hat{a}_v(\omega)]_R$$

$$+[\hat{a}_{v+k}(\omega)]_I[\hat{a}v(\omega)]_I\}$$

$$+\frac{1}{2}\left\{\frac{[a_k(\omega)]_R^2}{2}+\frac{[a_k(\omega)]_I^2}{2}\right\}\left\{\frac{1+(-1)^k}{2}\right\}$$

$$+\sum_{v=0}^{\left[\frac{k}{2}-1,\frac{k-1}{2}\right]}\{[\hat{a}_{k-v}(\omega)]_R[\hat{a}_v(\omega)]_R$$

$$+[\hat{a}_{k-v}(\omega)]_I[\hat{a}_v(\omega)]_I\}$$

for $k>0$ where $\left[\frac{k}{2}-1, \frac{k-1}{2}\right]$ means $\frac{k}{2}-1$ if k is even, $\frac{k-1}{2}$ if k is odd, where the subscripts R and I refer to "Real part" and "Imaginary part" respectively, and where

$$[a_\mu(\omega)]_R=\sum_{n=0}^{\infty}(2n+1)[F_n(i\omega)]_R\tilde{a}_{n\mu}$$

$$[\hat{a}_\mu(\omega)]_I=\sum_{n=0}^{\infty}(2n+1)[F_n(i\omega)]_I\tilde{a}_{n\mu}$$

Normalization of $p(\theta)$ can be accomplished by integrating (18) subject to the above assumptions (1) and (2). The result is

$$2p_0+\sum_{j=1}^{N}K_j\sin\theta_j e^{-\sigma_j^2/2}=1. \tag{25}$$

The condition (25) will serve as a constraint on the relative values of p_0 and the Kj coefficients.

8. Results and Conclusions

Note that the theoretical resonant frequencies for a lossless ionosphere are given by the numbers $\frac{1}{(2\pi)}\sqrt{\left(\frac{c}{a}\right)^2 n(n+1)}$, which for $c/a=46.7$ are 10.5, 18.2, 25.6, 33.1, and 40.5. These are sufficiently different from the experimentally observed resonances to warrant the conclusion that ionospheric losses play a major role in the mode spectrum. The real positive quantity α, equal to $1/(h\sqrt{\mu\sigma_i})$ is a measure of these losses. To study its effect, the functions $\omega_n F_n(i\omega)$ were calculated for various values of α. In the earliest work, the mode series was computed for a few values of α (assumed independent of frequency) corresponding to $h\approx 80$ km (a compromise between day and night ionospheres) and values of σ_i between (10^{-8}) and (10^{-6})mhos/m. It was found that deviations from observed mode resonant frequencies were very substantial (e.g., 20 to 30 percent) and that no constant value of α would produce near-agreement on all five observed modes. The next approach was to allow α to vary with frequency and use the Balser and Wagner results to infer its frequency variation. This was initially done by graphical methods based on plots of real and imaginary parts of $\omega_n F_n(i\omega)$ [Raemer, 1961]. The method has been revised to increase accuracy. The function $|\omega_n F_n(i\omega)|^2$ was computed in terms of normalized variables $x_n\equiv\omega/\omega_n$ and $y_n\equiv\frac{\alpha}{\sqrt{2\omega_n}}$ (plots are shown in fig. 1) and its derivative with respect to x_n equated to zero to find the maximum. Corresponding values of y_n were then found for $n=1$ through 5, resulting in a function $\alpha(f)$ required to match theoretical and observed resonant frequencies. This function was then fitted to an analytical curve, by the method of least squares and (see fig. 2) a satisfactory fit was found to be

$$\alpha(f)=7.649-0.1863f+0.01362f^2-0.00006602f^3-0.000003388f^4 \tag{26}$$

Note that this differs from the function originally reported by the author [1961], i.e.,

$$\alpha(f)=6.44\log_{10}f-0.0131(f-8.0) \tag{27}$$

This is due to the difference in computational method, and the function (26) should be regarded as the more accurate of the two.

The fundamental assumption behind the mode-matching technique described above is that the observed modes are those arising from the individual nth order frequency response functions of the earth-air-ionosphere cavity and are not significantly influenced either by cross-terms between modes, overlap between modes, or the details of the lightning flash spectrum.

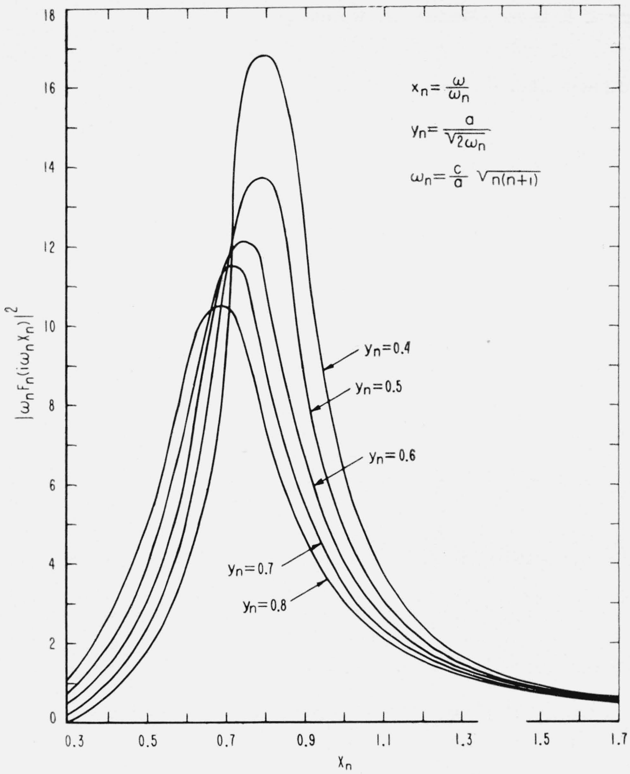

FIGURE 1. *$|\omega_n F_n(i\omega_n x_n)|^2$ versus x_n with y_n as parameter.*

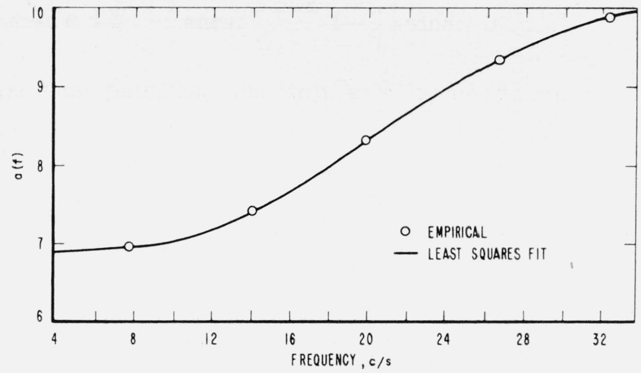

FIGURE 2. *Ionospheric loss parameter versus frequency.*

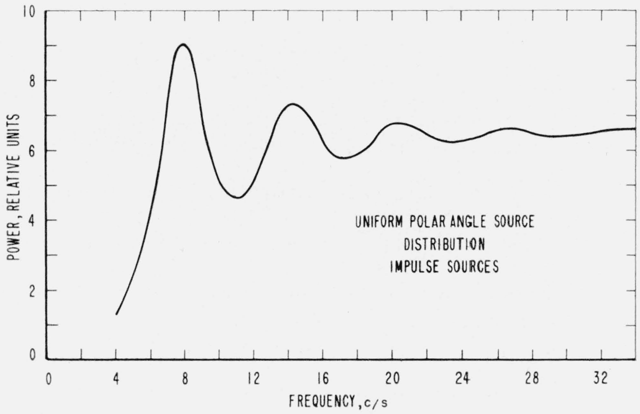

FIGURE 3. *Idealized ELF noise spectrum.*

For the moment we confine attention to the uniform angular distribution of sources. Using (26) to represent α, the quantity given by (19) has been calculated and is shown plotted in figure 3.

The plot of figure 3 is not appropriate for comparison with the experimental curve, since it assumes a white lightning flash spectrum and neglects the fact (see (1)) that the field amplitude contains a factor proportional to the reciprocal of the effective ionosphere height h. If it is assumed that the frequency dependence of α is due entirely to the effective ionosphere height and not to frequency dependence in the effective conductivity σ_i, then for consistency the power should contain a factor $[\alpha(f)]^2$. Multiplication of the quantity plotted in figure 3 by this factor leads to degradation rather than enhancement of agreement with experimental results, i.e., the envelope increases rapidly with frequency. Assuming our basic model to be essentially correct, one of two alternative possibilities follows: (1) The frequency dependence in α is entirely due to h and the spectrum of lightning flashes within the frequency range of interest is substantially different from a "white noise" spectrum or (2) the frequency dependence in α is largely due to σ_i, in which case the lightning spectrum still cannot be "white" noise but need not decay very rapidly with frequency within the range of interest.

Assuming that the ELF noise is largely caused by return lightning strokes, the suspicion that there is a significant decay in the spectrum with increasing frequency is borne out by figure 4. This figure, showing the estimated spectrum of a lightning flash compounded from (9) and (17) shows an average spectral decay of between 3 and 4 db per octave of frequency in the region of interest. Inserting this factor into (19) results in the theoretical curve (a) of figure 5. Incorporating the frequency dependence of h into the calculation results in the theoretical curve of figure 6. On both figures 5 and 6, the experimental curve is also shown for purposes of comparison. In both figures the experimental curve has been normalized in such a way that its average power is equal to the average power of the theoretical curve.

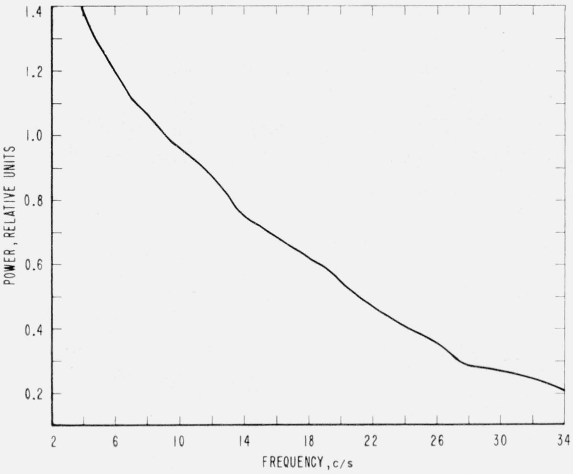

FIGURE 4. *Estimated lightning flash spectrum.*

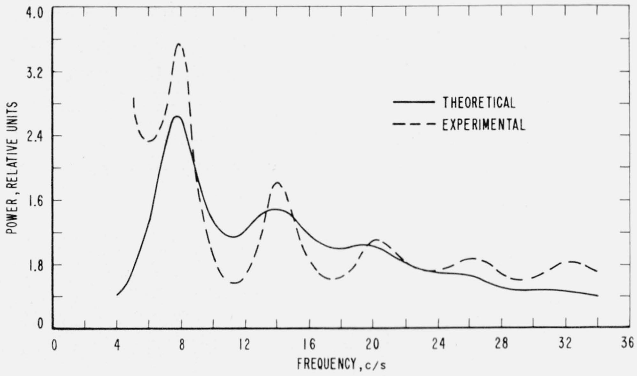

FIGURE 5. *ELF noise spectrum—effective ionospheric height assumed constant.*

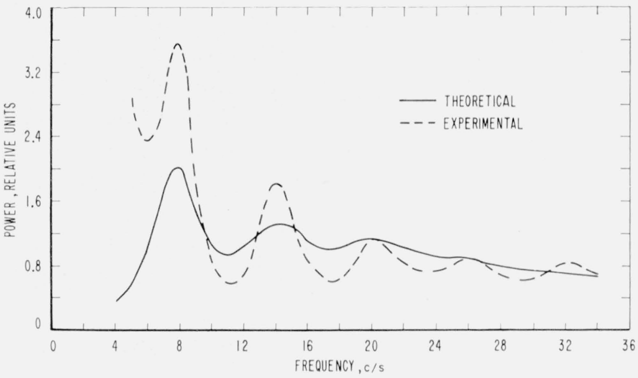

FIGURE 6. *ELF noise spectrum—effective ionospheric height assumed frequency dependent.*

The minima in the experimental curves are somewhat deeper than those of the theoretical spectra and the envelope of the minima, which is roughly constant on the experimental curves, decays substantially with frequency in the theoretical results. The rate of decay of the envelope of the maxima is comparable, but too rapid at the high frequency end of the region of interest. Widths of the modes are roughly comparable. Generally the agreement of spectral shape is seen to be reasonably good near the lower order modes and rather poor near the higher order modes. On the theoretical curves of both figures 5 and 6, the fourth mode becomes barely visible and the fifth is completely suppressed. Agreement is somewhat better in general in the case where the frequency dependence of $1/h^2$ is not accounted for, providing some evidence that the frequency dependence of α is predominantly due to the conductivity σ_i.

The computations on the equatorial region analysis are currently in progress. Preliminary results seem to indicate that accounting for these regions has no effect on the depths of the modes, but will have some influence on the envelope of the spectrum.

Other extensions which might reduce the discrepancies between the theory presented here and the Balser and Wagner results are as follows:

(1) The lightning statistics used here were obtained from data taken at many times of year and at many locations. Long term (e.g., year-round) averages of lightning parameters are not necessarily the same parameters averaged over a single day. The ELF noise results obtained by Balser and Wagner [1960] were obtained on a single day and at a single location. Only observed spectra averaged over many locations and times of year would be strictly comparable with the theory discussed here, unless the ELF noise spectrum is found to be essentially independent of time and location, a supposition that would require many more experiments to ascertain.

(2) The same remarks apply to the assumption that the lightning which gives rise to ELF noise consists entirely of cloud-ground return strokes and that the lightning current is entirely vertical. Wait [1960a] has pointed out that there may be significant contributions from horizontal current components at the frequencies of interest here.

(3) Assuming the return lightning stroke moment to be a random function and averaging over all random variables would probably make the lightning stroke more "noisy," i.e., give it a flatter spectrum. This could result in a theoretical spectrum whose envelope bears a closer resemblance to that of the observed spectrum.

(4) The ionosphere model could be improved by accounting for the earth's magnetic field and the effects of heavy ions.

(5) The probable reason for the fact that our theoretical curves are not as oscillatory as the experimental results is that the assumed ionospheric loss parameter is too large, tending to "flatten" the modes. This implies that the homogeneous ionsophere model is not sufficiently realistic for precise calculation of ELF noise effects. In a model accounting for the variation of conductivity with height near the lower edge of the ionosphere, e.g., the exponential model of J. R. Wait [1960b, 1960c], the loss parameter would be a complex number.[7] This would result in a more oscillatory spectral curve, as indicated in recent work by J. Galejs [private communication], as yet unpublished, and in some preliminary calculations by the author. Work by the author to infer information about the ionosphere from the Balser and Wagner curves with the assumption of a complex loss parameter is currently in progress.

Appreciation is expressed to Dr. R. Row and Dr. J. Galejs for helpful discussions pertaining to the work reported here, and to Mrs. J. Van Horn and Mr. C. Roche for computational assistance.

9. Appendix. Remarks on Analytical Points

Note that with a uniform distribution, a source could be at the observer's position. The discontinuity of the field at this point is accounted for in the fundamental theory by the fact that the hypergeometric series $P_v(-\cos\theta)$ used in the representation of the field has a discontinuity at $\theta=0$. The expansion of this function into a series of Legendre polynomials [Magnus and Oberhetinger, 1949], which is basic to our formulation, is valid everywhere but at $\theta=0$. Sommerfeld [1949] has shown that $P_v(-\cos\theta)$ becomes proportional to $\log\theta^2$ as θ approaches zero. In carrying out the integration indicated in (7) in the vicinity of this point, the product $(\sin\theta\log\theta^2)$ appearing in the integrand is an indeterminate form which can be shown to vanish as $\theta\to 0$. The contribution to the integral at $\theta=0$ therefore vanishes, and no difficulties are encountered in extending the expansion to this point.

Another analytical point worthy of note is that the expansion mentioned above is not valid at $v=0, \pm 1, \pm 2, \pm 3, \ldots$ etc. The point $v=0$ occurs at zero frequency, which we exclude from the analysis. Also, it can be shown that, except at $\omega=0$, v always has an imaginary part and therefore cannot take on values $\pm 1, \pm 2, \ldots$ etc.

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[7] This would account for both the ionosphere attenuation loss undergone by a propagating wave and a phase shift due to the conductivity gradient, whereas the present treatment accounts only for the former. Neglect of the phase shift would be valid with a sharply bounded homogeneous ionosphere.

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(Paper 65D6–162)