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Dipole Radiation in a Conducting Half Space

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The problem of communication between antennas, submerged in a conducting medium such as sea water, is analyzed in terms of a dipole radiating in a conducting half space separated by a plane boundary from a dielectric half space. The theory is discussed for both horizontal and vertical, electric and magnetic dipoles.

Expressions for the Hertzian potentials of the dipole in the conducting half space can be reduced to integrals obtained by Sommerfeld (for a dipole at the boundary) multiplied by an exponential depth attenuation factor. The Hertzian potentials are used to determine the electric and magnetic field components.

This analysis shows that the main path of communication between submerged antennas is composed of three parts as follows: (a) energy flow from the transmitting dipole directly to the surface of the sea, (b) creation of a wave that travels along the surface refracting back into the sea, (c) energy flow normal to the surface to the receiving dipole.

1. Introduction

Radiation from electric and magnetic dipoles in the air above a plane earth has been considered by many writers beginning with Sommerfeld [1909]. Sommerfeld's original paper treated a vertical monopole placed at the surface of a flat earth having arbitrary dielectric and conductive properties. The solution of the boundary-value problem was based on the evaluation of Fourier-Bessel integrals which were solutions to the wave equation. Weyl [1919] attacked the problem by resolving the dipole radiation into a spectrum of plane waves and evaluating the resulting integral without resorting to Sommerfeld's cylindrical wave formulation. Sommerfeld [1926] expanded his original work to take into account vertical and horizontal, electric and magnetic dipoles above a plane earth.

Sommerfeld's original paper contained an error in sign which was a subject of much later discussion [Norton, 1935]. Although Sommerfeld's work is followed here, the effect of the error in sign does not appear because of the assumption in this paper that the transmitting antenna is located in a highly conducting half space.

In general, papers published prior to 1940 on the subject of radiation of dipoles were concerned with dipoles in air or on the surface of the earth. Moreover, several papers have been written on this subject since 1940. Study of electromagnetic radiation in the sea began about the turn of the century and limited experimental work was done around the end of the first world war. Tai [1947] analyzed an electric dipole in an infinite, conducting medium and found extremely high dissipation in the vicinity of the dipole because of high conductive losses.

The first extensive theoretical treatment of electromagnetic radiation by vertical and horizontal, electric and magnetic dipoles immersed in a conducting half space appeared in a thesis by Moore [1951]. This paper describes a portion of that work. Wait [1952] [1957] considered an insulated magnetic dipole in an infinitely conducting medium, showing that the fields are independent of the characteristics of the insulation for an antenna diameter much less than the radiation wavelength in the conducting medium. Then Wait and Campbell [1953] and Wait [1959] analyzed a magnetic dipole of this type in a semi-infinite medium including special cases of frequency, antenna depth, and separation between antennas. Their model was a horizontal magnetic dipole (axis parallel to the surface of the sea), submerged in the sea (a conducting half space).

Analysis of the electric dipole was done by Lien and Wait [1953] in which the evaluation of the complex integrals was reduced to forms suitable for the numerical computation. Baños and Wesley [1953, 1954] carried out a mathematical analysis of the general case of a horizontal electric dipole in a conducting half space. The results in this paper agree with those of Baños to the first order approximation. Kraichman [1960] experimentally verified Wait's and Baños' results in the intermediate distance region range. Also, the exponential increase of attenuation with depth was experimentally verified by Saran and Held [1960]. Anderson, in a thesis [1961], described the three-layered problem of earth-air-ionosphere for an electric dipole submerged in the sea. Further work by Wait [1960a, 1960b] has recently been carried out in an effort to unify the various approaches to the problem.

It is concluded from our analysis that the path of electromagnetic energy, between transmitting and receiving dipoles in the sea, is the following: (a) propagation from the transmitter directly to the surface, (b) propagation along the surface of the sea allowing refraction of the energy back into the sea, (c) leakage directly downward into the sea to the receiver.

A fundamental assumption in this analysis is that the displacement current (in the sea) is negligible compared to the conduction current. Since the conductivity of the sea is about 4 mhos/m and the frequencies of practical use in undersea communication are less than 50 kc/s, this is not at all a severe restriction for the case considered. Application to related problems depends upon the appropriateness of this assumption for each problem. The only other restriction is that the energy traveling directly through the sea between the transmitter and receiver is neglected. The ratio of the magnitude of the direct wave (through the sea) to the over-the-surface wave is of the order of $n\,e^{-(R-z)/\delta}$ where R, z, δ, and n, are respectively antenna separation, antenna depth, skin depth, and index of refraction of the sea with respect to the air. Thus for $R>>z$, the ratio is considerably less than unity.

Because of the "over-the-surface" mode of communication, the really important direction for transmitting dipole radiation is directly toward the surface of the sea. Since the radiation pattern of the vertical dipoles has a null in this direction, the vertical dipoles are not as effective in launching the required wave as are the horizontal dipoles. Electric and magnetic fields have been calculated for both the vertical and the horizontal dipoles, but the latter are of much greater importance in submarine communications.

2. Waves in Conducting Media

Discussions of the nature of electromagnetic radiation in a conducting medium and a conducting half space are presented by Stratton [1941] and Sunde [1949]. However, a brief review of the pertinent parts of the theory is included here for the purpose of defining the variables employed throughout this paper, as well as stating the fundamental conditions of the problem of dipole radiation in a conducting half space.

The Maxwellian curl equations for a conducting medium in which displacement current can be neglected are

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{1}$$

$$\nabla \times \mathbf{H} = \sigma\mathbf{E} \tag{2}$$

where the time variation $e^{j\omega t}$ has been suppressed

- $\mathbf{E}$ is the electric field vector (volts/meter)
- $\mathbf{H}$ is the magnetic field vector (ampere-turns/meter)
- ω is the angular frequency (radians/second)
- μ is the magnetic permeability (henrys/meter)
- σ is the conductivity (mhos/meter).

The diffusion equation for $\mathbf{E}$ is obtained by combining (1) and (2):

$$\nabla^2\mathbf{E} = j\omega\mu\sigma\mathbf{E}. \tag{3}$$

The solution of (3) for a plane wave traveling in the positive z-direction is:

$$\mathbf{E} = \mathbf{E}_0 e^{-(1+j)z/\delta} \quad \text{with} \quad \mathbf{H} = e^{-j\pi/4}\sqrt{2}\,\frac{\mathbf{n}\times\mathbf{E}}{\mu\omega\delta}$$

where $\delta=\sqrt{2/\omega\mu\sigma}$ is the skin depth (meters)

n is the unit vector in the direction of propagation.

It is convenient in determining the fields of a dipole to determine first either the vector potential **A** or the Hertzian potential **Π**. In this paper, the Hertzian potentials are used for both the electric and magnetic dipoles. The electric and magnetic field vectors are functions of the Hertzian potential as follows:

Electric dipole	*Magnetic dipole*	
$\mathbf{H}_1=\sigma\nabla\times\mathbf{\Pi}_1$	$\mathbf{E}_1=-j\omega\mu\nabla\times\mathbf{\Pi}_1^*$	(4)
$\mathbf{E}_1=\nabla\nabla\cdot\mathbf{\Pi}_1+k_1^2\mathbf{\Pi}_1$	$\mathbf{H}_1=\nabla\nabla\cdot\mathbf{\Pi}_1^*+k_1^2\mathbf{\Pi}_1^*$	(5)

where $\mathbf{\Pi}_1$, $\mathbf{\Pi}_1^*$ are the Hertzian potentials for the electric and magnetic dipoles respectively, and

$$k_1=\sqrt{-j\omega\mu\sigma} \qquad (\text{meters}^{-1}).$$

The subscript 1 denotes a conducting medium. Throughout this paper the expressions relating to the electric dipole are presented in the left column while those for the magnetic dipole are in the right column, side by side, for convenient comparison. The Hertzian potentials **Π** and **Π*** (the Green's function for a dipole source) in the infinite, conducting medium are

$$\mathbf{\Pi}_1=\mathbf{p}_1\frac{e^{-jk_1R}}{R} \qquad \mathbf{\Pi}_1^*=\mathbf{p}_1^*\frac{e^{-jk_1R}}{R} \tag{6}$$

and

$$\mathbf{p}_1=\frac{\widehat{I\mathbf{l}}}{4\pi\sigma} \qquad \mathbf{p}_1^*=\frac{\widehat{NI\mathbf{S}}}{4\pi} \tag{7}$$

where R is the distance from the dipole to the observation point

$\widehat{I\mathbf{l}}=\lim_{\substack{I\to\infty\\ \mathbf{l}\to\mathbf{0}}}(I\mathbf{l})$, is the electric moment (ampere-meters)

$\widehat{NI\mathbf{S}}=\lim_{\substack{I\to\infty\\ \mathbf{S}\to\mathbf{0}}}(NI\mathbf{S})$ is the magnetic moment (ampere-meters ^{2})

I is the current (amperes).
l is the electric dipole length vector (meters).
N is the number of turns in the magnetic loop.
S is the loop area vector (meters ^{2}).

The corresponding statements for the electric and magnetic fields in an infinite, nonconducting medium are the following:

$\mathbf{H}_2=j\omega\epsilon_0\nabla\times\mathbf{\Pi}_2$	$\mathbf{E}_2=-j\omega\mu\nabla\times\mathbf{\Pi}_2^*$	(8)
$\mathbf{E}_2=\nabla\nabla\cdot\mathbf{\Pi}_2+k_2^2\mathbf{\Pi}_2$	$\mathbf{H}_2=\nabla\nabla\cdot\mathbf{\Pi}_2^*+k_2^2\mathbf{\Pi}_2^*$	(9)

where $k_2=2\pi/\lambda_0$, is the wave number (meters $^{-1}$)
λ_0 is free-space-wavelength (meters)
ϵ_0 is free space permittivity (farads/meter).

The Hertzian potentials in this case are

$$\mathbf{\Pi}_2 = \mathbf{p}_2 \frac{e^{-jk_2R}}{R} \qquad \mathbf{\Pi}_2 = \mathbf{p}_2^* \frac{e^{-jk_2R}}{R} \tag{10}$$

$$\mathbf{p}_2 = -j \frac{\widehat{Il}}{4\pi\omega\epsilon_0} \qquad \mathbf{p}_2^* = \frac{\widehat{NI\mathbf{S}}}{4\pi}. \tag{11}$$

The above statement for the fields in an infinite medium must be modified for the case of a source in a conducting half space separated by a plane boundary from a nonconducting half space. For the modification, the source (transmitting dipole) is located in the conducting half space at coordinate position $(0, 0, z_t)$ from the boundary. Similarly, the point of observation (receiving dipole) is located at coordinate position (r, ϕ, z_r) (see fig. 1). The conducting half space (sea) is medium 1 (subscript 1), and the nonconducting half space (air) is medium 2 (subscript 2).

The Hertzian potentials in the following are normalized for convenience such that $|\mathbf{p}_1| = |\mathbf{p}_2| = |\mathbf{p}_1^*| = |\mathbf{p}_2^*| = 1$. The Hertzian potentials in both media must satisfy the wave equation (condition I below) and the radiation condition at infinity (condition II) below:

(I)
$$\nabla^2\mathbf{\Pi}_1 + k_1^2\mathbf{\Pi}_1 = \mathbf{0} \qquad \nabla^2\mathbf{\Pi}_1^* + k_1^2\mathbf{\Pi}_1^* = \mathbf{0}, \; z_r > 0 \text{ (sea)}$$
$$\nabla^2\mathbf{\Pi}_2 + k_2^2\mathbf{\Pi}_2 = \mathbf{0} \qquad \nabla^2\mathbf{\Pi}_2^* + k_2^2\mathbf{\Pi}_2^* = \mathbf{0}, \; z_r < 0 \text{ (air)}$$

(II)
$$\lim_{R\to\infty} \mathbf{\Pi}_1 = \mathbf{0} \qquad \lim_{R\to\infty} \mathbf{\Pi}_1^* = \mathbf{0} \qquad , \; z_r > 0$$
$$\lim_{R\to\infty} \mathbf{\Pi}_2 = \mathbf{0} \qquad \lim_{R\to\infty} \mathbf{\Pi}_2^* = \mathbf{0} \qquad , \; z_r < 0$$

where $R = \sqrt{r^2 + (z_t - z_r)^2}$.

Since the source is in the conducting half space, it is also necessary that Π_1 satisfy the radiation condition near the source:

(III)
$$\lim_{R\to 0} \Pi_1 = \frac{e^{-jk_1R}}{R} = \lim_{R\to 0} \Pi_1^*.$$

In addition to the above three conditions, the electric and magnetic fields must satisfy the boundary conditions at the surface of the sea (the plane, $z=0$). Thus the boundary conditions for the components of the Hertzian potentials and their derivatives are the following:[1]

(IV)
$$-jg\Pi_{z1} = \Pi_{z2} \qquad \Pi_{z1}^* = \Pi_{z2}^*$$
$$-jg\Pi_{x1} = \Pi_{x2} \qquad -jg\Pi_{x1}^* = \Pi_{x2}^*$$
$$-jg\frac{\partial\Pi_{x1}}{\partial z} = \frac{\partial\Pi_{x2}}{\partial z} \qquad \frac{\partial\Pi_{x1}^*}{\partial z} = \frac{\partial\Pi_{x2}^*}{\partial z}$$
$$\frac{\partial\Pi_{x1}}{\partial x} + \frac{\partial\Pi_{z1}}{\partial z} = \frac{\partial\Pi_{x2}}{\partial x} + \frac{\partial\Pi_{z2}}{\partial z} \qquad \frac{\partial\Pi_{x1}^*}{\partial x} + \frac{\partial\Pi_{z1}^*}{\partial z} = \frac{\partial\Pi_{x2}^*}{\partial x} + \frac{\partial\Pi_{z2}^*}{\partial z}$$

where $g = \sigma/\omega\epsilon_0 = |k_1^2/k_2^2|$. In condition (IV), it is assumed that the dipole is oriented either vertically in the z-direction, or horizontally in x-direction. (See fig. 1.) Sommerfeld [1949] has shown that only a Π_z component of the Hertzian potential is required to describe the fields of a vertical dipole. However, he has shown that both Π_x and Π_z components are required to describe the fields of a horizontal dipole (oriented in the x-direction).

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[1] In condition (IV) the quantity $-jg$ is an approximation for the complex index of reflection squared, n^2. If the frequency or conductivity is such that $n^2 \neq -jg$, then one can substitute the exact expression $n^2 = -jg + \kappa$, where κ is the relative permittivity of the medium, throughout this paper.

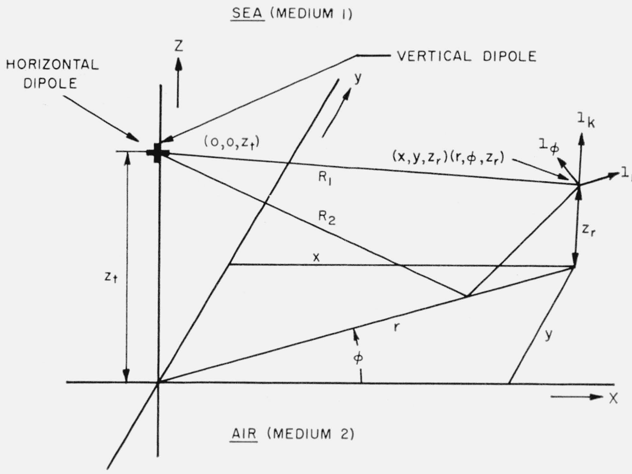

FIGURE 1. *Distances and coordinate system.*

3. Hertzian Potential Integrals

The Hertzian potential can be obtained in integral form for the four basic dipole configurations: vertical and horizontal, electric and magnetic dipoles. The method used here for obtaining the potential integrals was first used by Sommerfeld [1909, 1926]. The method is straightforward and available [Stratton, 1941; Sommerfeld 1949], so it will not be presented here. A general Hertzian potential for each of the four dipole configurations is obtained satisfying conditions (I) through (III). The general solution is then substituted into condition (IV) allowing the boundary conditions to determine the arbitrary coefficients of the general solution.

The Hertzian potentials for the observation point in the sea are presented in integral form for the four basic dipole configurations. The superscripts v and h denote vertical and horizontal dipole respectively. For $z_t>0$, $z_r>0$,

$$\Pi^v_{z1}=\mathscr{R}+I_{a1} \qquad \Pi^{*v}_{z1}=\mathscr{R}+I_{b1} \tag{12}$$

and

$$\Pi^h_{x1}=\mathscr{R}+I_{b1} \qquad \Pi^{*h}_{x1}=\mathscr{R}+I_{a1} \tag{13}$$

$$\Pi^h_{z1}=\cos\varphi\frac{\partial I_{c1}}{\partial r}=\Pi^{*h}_{z1} \tag{14}$$

where

$$\mathscr{R}=\frac{e^{-jk_1R_1}}{R_1}-\frac{e^{-jk_1R_2}}{R_2}$$

$$R_1=\sqrt{r^2+(z_r-z_t)^2} \qquad \text{(meters)}$$

$$R_2=\sqrt{r^2+(z_r+z_t)^2} \qquad \text{(meters)}$$

and where

$$I_{a1}=2\int_0^\infty \frac{e^{-F(z_r+z_t)}J_0(\xi r)\xi d\xi}{F-jgG} \tag{15a}$$

$$I_{b1}=2\int_0^\infty \frac{e^{-F(z_r+z_t)}J_0(\xi r)\xi d\xi}{F+G} \tag{15b}$$

$$I_{c1}=2(-jg-1)\int_0^\infty \frac{e^{-F(z_r+z_t)}J_0(\xi r)\xi d\xi}{(F+G)(F-jgG)} \tag{15c}$$

$$F=\sqrt{\xi^2-k_1^2} \qquad \text{(meters}^{-1}\text{)}$$

$$G=\sqrt{\xi^2-k_2^2} \qquad \text{(meters}^{-1}\text{)}.$$

The function $\mathscr{R}$ has two parts: $\exp(-jk_1R_1)/R_1$ representing the primary source at the position $(0, 0, z_t)$, and $\exp(-jk_1R_2)/R_2$ representing a secondary source at the image position $(0, 0, -z_t)$. In the case of the horizontal electric and the vertical magnetic antennas, the secondary source represents the image of the primary source. However, in the case of the vertical electric and the horizontal magnetic antennas, the secondary source represents an inverse image—that is, an image dipole of the opposite polarity. The primary source radiates over the direct path R_1, and the secondary source radiates over the reflected or image path R_2 (see fig. 1). Both paths R_1 and R_2 are entirely within the sea. Consequently, each of the Hertzian potentials in (12) and (13) is composed of three components: (a) a primary source function, (b) a secondary source function, and (c) an integral. It is shown later that the integral in every case represents the major contribution to the Hertzian potentials if $r>>(z_r+z_t)$. Therefore it is tacitly assumed that $\mathscr{R}$, composed of the primary and secondary sources, can be neglected. Consequently, (12) and (13) can be approximated for $z_r>0$, $z_t>0$, $r>>(z_r+z_t)$, as follows:

$$\Pi_{z1}^{v}\approx I_{a1} \qquad \Pi_{z1}^{*h}\approx I_{b1} \tag{16}$$

$$\Pi_{x1}^{h}\approx I_{b1} \qquad \Pi_{x1}^{*h}\approx I_{a1}. \tag{17}$$

I_{a1}, and I_{b1}, and I_{c1} are interdependent. It can be shown that [2] for $z_t>0$, $z_r>0$,

$$\frac{\partial I_{c1}}{\partial z_r}=I_{b1}+jgI_{a1} \tag{18a}$$

$$I_{c1}=\frac{\partial}{\partial z_r}\left[-\frac{2}{k_1^2}\frac{e^{-jk_1R_2}}{R_2}+\left(\frac{1}{k_2^2}+\frac{1}{k_1^2}\right)I_{a1}\right]\approx\frac{1}{k_2^2}\frac{\partial I_{a1}}{\partial z_r}. \tag{18b}$$

The Hertzian potentials for an observation point located in medium 2 are obtained in the same manner as were the Hertzian potentials for a point in medium 1. In the case of the potentials in medium 2, the function $\mathscr{R}$ is not involved, so the potentials can be written directly for $z_t>0$, $z_r<0$,

$$\Pi_{z2}^{v}=-jgI_{a2} \qquad \Pi_{z2}^{*v}=I_{b2} \tag{19}$$

and

$$\Pi_{x2}^{h}=-jgI_{b2} \qquad \Pi_{x2}^{*h}=-jgI_{a2} \tag{20}$$

$$\Pi_{z2}^{h}=-jg\cos\varphi\frac{\partial I_{c2}}{\partial r} \qquad \Pi_{z2}^{*h}=\cos\varphi\frac{\partial I_{c2}}{\partial r} \tag{21}$$

where

$$I_{a2}=2\int_0^\infty\frac{e^{-Fz_t+Gz_r}J_0(\xi r)\xi d\xi}{F-jgG} \tag{22a}$$

$$I_{b2}=2\int_0^\infty\frac{e^{-Fz_t+Gz_r}J_0(\xi r)\xi d\xi}{F+G} \tag{22b}$$

$$I_{c2}=2(-jg-1)\int_0^\infty\frac{e^{-Fz_t+Gz_r}J_0(\xi r)\xi d\xi}{(F+G)(G-jgG)} \tag{22c}$$

as in (18), the integrals I_{a2}, I_{b2}, and I_{c2} are also interdependent. For $z_t>0$, $z_r<0$,

$$\frac{\partial I_{c2}}{\partial z_r}=I_{b2}-I_{a2} \tag{23a}$$

$$I_{c2}=\frac{1}{k_2^2}\left(\frac{\partial I_{a2}}{\partial z_t}+\frac{\partial I_{a2}}{\partial z_r}\right). \tag{23b}$$

[2] (18) requires that the term $(-jg-1)$ in (15c) be kept in this form. However, for any practical calculation $(-jg-1)\approx-jg$ since $g>>1$ for the conductivity and frequencies of interest in this paper.

Now it is possible to write the electric and magnetic fields in medium 1 (sea) as a function of integrals I_{a1} and I_{b1}; and the fields in medium 2 (air) as functions of I_{a2} and I_{b2}. Since it is the purpose of this paper to determine the fields in the sea and at the surface of the sea, attention is focused on the integrals I_{a1} and I_{b1}. If the fields in the sea are determined, the boundary conditions can be used conveniently to determine the fields in air at the boundary ($z_r = 0-$). Consequently, *all* fields to be discussed in this paper can be expressed in terms of integrals I_{a1} and I_{b1}. It is convenient, however, to replace I_{a1} and I_{b1} with two new integrals I_a and I_b with a corresponding change of variables as follows:

$$\frac{c}{\omega} I_{a1} = I_a = 2\int_0^\infty \frac{e^{-L\zeta} J_0(\rho\psi)\psi d\psi}{L - jgM} \tag{24}$$

$$\frac{c}{\omega} I_{b1} = I_b = 2\int_0^\infty \frac{e^{-L\zeta} J_0(\rho\psi)\psi d\psi}{L + M} \tag{25}$$

where

$$L = \frac{c}{\omega} F = \sqrt{\psi^2 + jg}$$

$$M = \frac{c}{\omega} G = \sqrt{\psi^2 - 1}$$

$$\psi = \frac{c}{\omega} \xi$$

$$\rho = \frac{\omega}{c} r = \frac{r}{\lambda_0/2\pi}$$

$$\zeta = \frac{\omega}{c} z = \frac{z}{\lambda_0/2\pi}$$

$z = z_r + z_t$ (meters)

c=speed of light in free space (meters/second)

λ_0=free-space wave length (meters).

The dimensionless distance factors ρ and ζ are used to express distances r and z in terms of (free-space-wavelength)/2π. Therefore the electric and magnetic field components in the sea, as functions of the integrals I_a and I_b are tabulated as follows:

Electric dipole — *Magnetic dipole*

Vertical dipole

$$E_r = \frac{\omega^3}{c^3} \frac{\partial^2 I_a}{\partial\rho\partial\zeta} \qquad H_r = \frac{\omega^3}{c^3} \frac{\partial^2 I_b}{\partial\rho\partial\zeta} \tag{26}$$

$$E_z = \frac{\omega^3}{c^3}\left(\frac{\partial^2 I_a}{\partial\zeta^2} - jgI_a\right) \qquad H_z = \frac{\omega^3}{c^3}\left(\frac{\partial^2 I_b}{\partial\zeta^2} - jgI_b\right) \tag{27}$$

$$H_\varphi = -\frac{\omega^2}{c^2}\sigma \frac{\partial I_a}{\partial\rho} \qquad E_\varphi = j\frac{\omega^3}{c^2}\mu \frac{\partial I_b}{\partial\rho} \tag{28}$$

$$E_\varphi = H_r = H_z = 0 \qquad H_\varphi = E_r = E_z = 0 \tag{29}$$

Horizontal dipole

$$H_r = \frac{\omega^2}{c^2}\sigma \sin\varphi\left(\frac{1}{\rho}\frac{\partial^2 I_a}{\partial\rho\partial\zeta} + \frac{\partial I_b}{\partial\zeta}\right) \qquad E_r = -j\frac{\omega^3}{c^2}\mu \sin\varphi\left(\frac{1}{\rho}\frac{\partial^2 I_a}{\partial\rho\partial\zeta} + \frac{\partial I_a}{\partial\zeta}\right) \tag{30}$$

$$H_\varphi = \frac{\omega^2}{c^2}\sigma \cos\varphi\left(\frac{\partial^3 I_a}{\partial\rho^2\partial\zeta} + \frac{\partial I_b}{\partial\zeta}\right) \qquad E_\varphi = -j\frac{\omega^3}{c^2}\mu \cos\varphi\left(\frac{\partial^3 I_a}{\partial\rho^2\partial\zeta} + \frac{\partial I_a}{\partial\zeta}\right) \tag{31}$$

$$H_z = -\frac{\omega^2}{c^2}\sigma \sin\varphi \frac{\partial I_b}{\partial \rho} \qquad E_z = j\frac{\omega^3}{c^2}\mu \sin\varphi \frac{\partial I_a}{\partial \rho} \tag{32}$$

$$E_r = -\frac{\omega^3}{c^3}\cos\varphi\, jg\left(\frac{\partial^2 I_a}{\partial \rho^2} + I_b\right) \qquad H_r = -\frac{\omega^3}{c^3}\cos\varphi\, jg\left(\frac{\partial^2 I_a}{\partial \rho^2} + I_a\right) \tag{33}$$

$$E_\varphi = \frac{\omega^3}{c^3}\sin\varphi\, jg\left(\frac{1}{\rho}\frac{\partial I_a}{\partial \rho} + I_b\right) \qquad H_\varphi = \frac{\omega^3}{c^3}\sin\varphi\left(\frac{1}{\rho}\frac{\partial I_a}{\partial \rho} + I_a\right) \tag{34}$$

$$E_z = -\frac{\omega^3}{c^3}\cos\varphi \frac{\partial^2 I_a}{\partial \rho \partial \zeta} \qquad H_z = -\frac{\omega^3}{c^3}\cos\varphi \frac{\partial^2 I_a}{\partial \rho \partial \zeta}. \tag{35}$$

As already stated, the electric and magnetic fields in the air at the boundary can be determined from the fields in the sea and the boundary conditions. The fields are related through the boundary conditions as follows, for $z_r = 0$:

$$H_{i2} = H_{i1} \qquad i = r, \varphi, z \tag{36a}$$

$$E_{i2} = E_{i1} \qquad i = r, \varphi \tag{36b}$$

$$E_{z2} = -jgE_{z1}. \tag{36c}$$

The boundary conditions (36) state that the magnetic fields and the tangential electric fields in air are equal, immediately adjacent to the boundary, to those in the sea, and that the vertical electric fields in the air are related to those in the sea by the proportionality constant $-jg$.

4. Evaluation of the Integrals

In the previous section the electric and magnetic fields in the sea and at the boundary were written as functions of the integrals I_a (24) and I_b (25). Apparently these integrals cannot be evaluated in closed form. Asymptotic expansions for I_a and I_b (as well as I_{a1}, (15a), I_{b1}, (15b), I_{a2} (22a), I_{b2} (22b)) and their appropriate derivatives can be determined by the method of critical points [Fredericks, 1953; Baños, 1953, 1954; Erdelyi, 1956].

However, it is possible to reduce the integrals I_a and I_b to those evaluated by Sommerfeld [1909, 1926, 1949] for the case in which the source and point of observation both lie on the boundary. Sommerfeld evaluated these integrals by transforming them into integrals along a contour in the complex plane. It is necessary to investigate the contours used to show the approximation possible for the case of the source and point of observation in the sea.

An integrand map is shown in figure 2. There are four branch points and two poles. The branch points are associated with both integrals, but I_b does not have a pole. The branch points arise because of the presence of the square roots in L and M. They occur where

$$L = \sqrt{\psi^2 + jg} = 0 \tag{37a}$$

$$M = \sqrt{\psi^2 - 1} = 0. \tag{37b}$$

B.P. 1 and B.P. 3 are associated with L, and B.P. 2 and B.P. 4 are associated with M.

Coordinates of B.P. 1 and B.P. 3 are given by

$$\text{B.P. 1} \qquad \psi = \sqrt{g/2}(-1 + j) \tag{38a}$$

$$\text{B.P. 3} \qquad \psi = \sqrt{g/2}(1 - j). \tag{38b}$$

It can be seen that their distance from the origin is $\sqrt{g}$. For low frequencies and high conductivity, as with sea water, this is a rather large value (generally $> 10^3$ for communication in the sea).

In the form shown in (37b) it would appear that the branch points 2 and 4 would be on the real axis. It is obvious from the limits of the integrals that the path of integration must

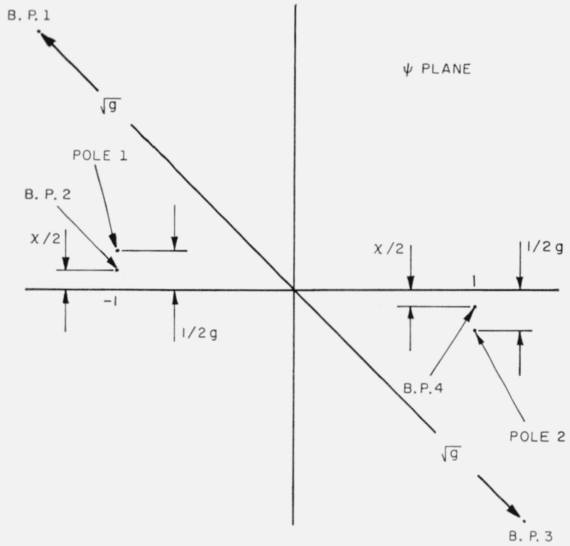

FIGURE 2. *Integrand map.*

lie along the real axis and this could cause certain complications. The reason this branch point appears to lie on the real axis is that medium 2 has been assumed to have zero conductivity. As a matter of fact, any practical medium, even air, has some conductivity and therefore the effective (complex) permittivity must contain a small imaginary component.

Thus, the complex permittivity, $\bar{\epsilon}$, is

$$\epsilon=\epsilon_0(1-j\chi) \tag{39}$$

where χ is a very small value associated with the conductivity of air. Utilizing the complex permittivity in (37b) we find the branch points are located by

$$\begin{aligned} &\text{B.P. 2} \qquad \psi\approx-1+j\chi/2 \\ &\text{B.P. 4} \qquad \psi\approx 1-j\chi/2. \end{aligned} \tag{40}$$

The poles occur where $(L-jgM)$ is zero. This occurs when

$$\psi^2=\frac{g^2}{1+g^2}\left(1-\frac{j}{g}\right). \tag{41}$$

For the case of large g (generally $> 10^6$) the location of the poles is given approximately by

$$\begin{aligned} &\text{Pole 1} \qquad \psi\approx-1+j/2g \\ &\text{Pole 2} \qquad \psi\approx 1-j/2g. \end{aligned} \tag{42}$$

The integrals as originally stated have a lower limit of zero and an infinite upper limit. To aid in evaluating them by contour methods it is convenient to write these integrals in forms such that the path of integration lies along the entire real axis. The principal consequence is the conversion of the Bessel functions of the first kind into Hankel functions of the first kind. Thus the integrals become

$$I_a=\int_{-\infty}^{\infty}\frac{e^{-L\zeta}}{L-jgM}H_0^{(1)}(\rho\psi)\psi d\psi \tag{43}$$

$$I_b=\int_{-\infty}^{\infty}\frac{e^{-L\zeta}}{L+M}H_0^{(1)}(\rho\psi)\psi d\psi. \tag{44}$$

It is therefore possible to close a contour by swinging a semicircle, whose radius is unbounded, from the positive real axis to the negative real axis through the upper half plane. Since there are two branch points in the upper half plane, one must form the contour such that the two associated branch cuts are not crossed. It is, of course, possible to choose convenient branch cuts, within limits.

Branch line 2 lies along the contour from $\psi=-1+j\chi/2$ to $\psi\rightarrow j\infty$ for which M is pure imaginary. It extends, in essence along the horizontal axis from -1 to 0 and then up along the imaginary axis. The contour of integration is shown in figure 3, where it can be seen that branch line 2 is the first branch cut encountered in swinging the infinite semicircle from positive real axis to negative real axis. The choice of values for the square root in M on the two Riemann surfaces separated by branch line 2 is such that for $z_r>0$, the exponent in the factor $e^{M\zeta}$ has a positive real part in the first quadrant of the ψ-plane and a negative real part in the second quadrant. (This aids in obtaining convergence of the integrals in the nonconducting medium (22) along branch line 1.) One observes as Sommerfeld did that in this case (high conductivity) Pole 1 and B.P. 2 are very close together so that the contours for I_a must pass around the branch point and the pole with the same loop. This problem has been treated in great detail by numerous authors and will not be elaborated here.

Branch line 1 extends from $\psi=\sqrt{g/2}\ (-1+j)$ to $\psi\rightarrow j\infty$. Along this line L has a large imaginary component and a small real component. The choice of Riemann surfaces is such that the real part of the exponent in the factor $e^{-L\zeta}$ is negative on the left side of the branch line, because this aids in exponential reduction of the integrand along branch line 2.

It can be shown readily that the integral along the infinite semicircle is zero. Following the terminology used by other writers in the past, the integral along branch line 2 is called $P+Q_2$ (except for I_b where it is just Q_2). The integral along branch line 1 is called Q_1. Thus for each of the integrals we may state that the value is given by

$$I=P+Q_1+Q_2 \tag{45}$$

where P is, of course, zero for I_b.

For a highly conducting medium, except very close to the source dipole, the contribution to the integral along branch line 1 is negligible. One can validate this statement by examining the asymptotic expansion of the zero order Hankel function of the first kind [Stratton, 1940] which is

$$H_0^{(1)}(\rho\psi)\sim\sqrt{\frac{2}{\pi\rho\psi}}\,e^{j(\rho\psi-\pi/4)}. \tag{46}$$

It is seen that the Hankel function vanishes exponentially as ψ increases along branch line 1. Since the smallest imaginary part of ψ associated with branch line 1 is $\sqrt{g/2}$, the contribution Q_1 is justifiably neglected. Consequently, it can be assumed that the significant contribution to the integral comes from the integration along branch line 2; that is

$$I\approx P+Q_2. \tag{47}$$

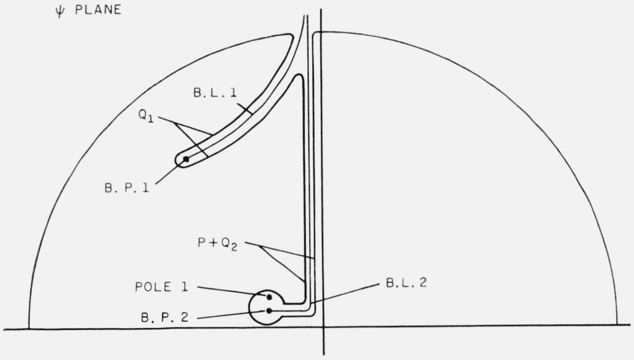

FIGURE 3. *Contour of integration.*

The *most important* approximation made in this treatment is the one allowing the removal of the exponential $e^{-L\zeta}$ from the integrand. To see that this is possible, it is necessary to express I, in the form of either (43) or (44), as the sum of two integrals for integration along branch line 2. The limits of the first integral are B.P. 2 and a point ψ_0 on branch line 2; and the limits of the second integral are ψ_0 and $j\infty$. The point ψ_0 is chosen such that $1<<|\psi_0|^2<<g$. For example, using $g=10^7$, one might choose $\psi_0=j10^3$. Consequently, one can approximate $L=\sqrt{\psi^2+jg}\approx\sqrt{jg}$ over the first integral (B.P. 2, ψ_0) with negligible error introduced in the evaluation of the first integral. Over the second integral $(\psi_0, j\infty)$, the inequality

$$|e^{-\sqrt{jg}\zeta}|\leq|e^{-L\zeta}| \tag{48}$$

permits the following inequality to be written:

$$\left|\int\limits_{\substack{\text{B.L.2}\\(\psi_0<\psi<j\infty)}}\frac{e^{-\sqrt{jg}\zeta}}{L+aM}H_0^{(1)}(\rho\psi)\psi d\psi\right|\leq\left|\int\limits_{\substack{\text{B.L.2}\\(\psi_0<\psi<j\infty)}}\frac{e^{-L\zeta}}{L+aM}H_0^{(1)}(\rho\psi)\psi d\psi\right|=B \tag{49}$$

where $a=\begin{cases}-jg \text{ for } I_a\\ \;\;1 \text{ for } I_b\end{cases}$

Using the approximation $M=\sqrt{\psi^2-1}\approx\psi$, over $(\psi_0, j\infty)$, and (46) with $10^{-2}<\rho<10^2, |\psi|\geq|\psi_0|\geq10^3$, the following upper bound is written for the inequality (49):

$$B\leq\left|\sqrt{\frac{2}{\pi\rho}}\int_{\psi_0}^{j\infty}\frac{e^{-L\zeta}e^{j\rho\psi-j\pi/4}2aj\psi^{3/2}d\psi}{jg+(1-a^2)\psi^2}\right|\leq\left|\sqrt{\frac{8}{\pi}}\frac{e^{-\sqrt{jg}\zeta+j\rho\psi_0}}{5jg\rho^{7/2}}(-j\rho^2\psi_0^2+2\rho\psi_0+2)\right| \tag{50}$$

It is shown later that a contribution whose magnitude is less than the upper bound (50) is negligible. Consequently, for the following approximation in I:

$$L\approx\sqrt{jg} \tag{51}$$

one observes that the error is negligible in the first integral and that the contribution of the second integral, where the approximation is relatively poor, is also negligible. Hence the approximation is justified. Using (51), the integrals I_a and I_b for the conductivity and frequency range considered in this treatment are written as follows:

$$I_a\approx P+Q_2\approx 2e^{(-1-j)\sqrt{g/2}\zeta}\int\limits_{\text{B. L. 2}}\frac{H_0^{(1)}(\rho\psi)\psi d\psi}{L-jgM} \tag{52}$$

and

$$I_b\approx Q_2\approx 2e^{(-1-j)\sqrt{g/2}\zeta}\int\limits_{\text{B. L. 2}}\frac{H_0^{(1)}(\rho\psi)\psi d\psi}{L+M}. \tag{53}$$

This approximation is important because it places outside the integral the factor associated with the distances of the dipole and of the point of observation from the boundary. Thus the remaining integral is simply that for observation at the surface of fields due to a source also located at the surface. This was the problem Sommerfeld [1909] treated for the vertical dipole, and the horizontal dipole problem has also been treated since then by many writers. *Thus the problem for fields due to a dipole in a conducting half space has been reduced to the known problem for fields due to a dipole at an interface.*

Both convergent and asymptotic series were developed by Sommerfeld and others for this integral. The expression for I_a was found [Moore, 1951] for $0<\rho<1$, to be

$$I_a = j2\,\frac{e^{-j\rho-\sqrt{jg}\zeta}}{\rho g}\left\{1-\frac{1}{2}\frac{\rho}{g}+\frac{1}{2}\frac{j\rho}{1\cdot 3g^2}(1+j\rho)+\frac{1}{2}\frac{\rho}{1\cdot 5g^3}\left[1+j\rho+\frac{(j\rho)^{3/2}}{3}\right]+\dots\right.$$

$$\left.+\frac{1}{2}\frac{j\pi}{\sqrt{jg}}H_0^{(1)}(-\rho)+\frac{1}{2}\frac{j\pi}{(jg)^{3/2}}H_1^{(1)}(-\rho)+\dots\right\}. \qquad (54)$$

This may be expressed as

$$I_a \approx j2\,\frac{e^{-j\rho-\sqrt{jg}\zeta}}{\rho g}\,T. \qquad (55)$$

For distances such that $10^{-2}<\rho<1$, conductivity $\sigma\approx 4$ mhos/m, and frequency $f<50$ kc/s, then $T\approx 1$. A more convenient form for I_a given by Sommerfeld [1949] is the following for $\rho>>1$:

$$I_a = j2\,\frac{e^{-j\rho-\sqrt{jg}\zeta}}{g\rho}\left[1+\frac{j}{2}\sqrt{2\pi\eta}\,e^{-\eta/2}-\eta e^{-\eta/2}\int_0^{\sqrt{\eta}} e^{-u^2/2}du\right]. \qquad (56)$$

An expression for I_b similar to that for I_a (54) was also obtained by Moore. However, I_b can be expressed in closed form [Wait, 1952] as follows:

$$I_b = 2\,\frac{e^{-j\rho-\sqrt{jg}\zeta}}{(jg-1)\rho^3}\left[1+j\rho-(1+\sqrt{jg}\rho)e^{-(\sqrt{jg}-j)\rho}\right]. \qquad (57)$$

For the practical purpose of submarine communications; that is, $\sigma\approx 4$ mhos/m, $f<50$ kc/s, it is sufficient to approximate I_a (54) or (56) and I_b (57) as follows:

$$I_a = j2\,\frac{e^{-j\rho-\sqrt{jg}\zeta}}{g\rho}, \quad 10^{-2}<\rho<10^2, \quad \rho>>\zeta \qquad (58)$$

and

$$I_b = -j2\,\frac{e^{-j\rho-\sqrt{jg}\zeta}}{g\rho^3}(1+j\rho), \quad 10^{-2}<\rho\leq 1, \quad \rho>>\zeta \qquad (59a)$$

or

$$I_b = 2\,\frac{e^{-j\rho-\sqrt{jg}\zeta}}{g\rho^2}\left(1-\frac{j}{\rho}\right), \quad 1\leq\rho<10^2, \quad \rho>>\zeta. \qquad (59b)$$

Now that the integrals I_a and I_b have been evaluated, the derivatives of I_a and I_b involved in the field components ((26) through (36)) must be determined. It can be shown [Moore 1951] that the power series (54) can be differentiated, allowing the necessary derivatives to the same order as of (58) to exist. I_b (57) can be differentiated; consequently, the necessary derivatives of (59) also exist. Returning to the statement that $\mathscr{R}$ can be neglected in (16) and (17), one can use (16), (17), (24), (25), (58), and (59) to verify this approximation. Similarly, the approximation (51) can be justified by comparing the magnitude of the upper bound (50) to (58) or (59). For the case $g=10^7$, $\psi_0=j10^3$ and $10^{-2}<\rho<10^2$, the error is less than 1 percent.

5. Fields in a Conducting Half Space

After performing the indicated differentiations, substituting numerical values for permittivity, permeability, and velocity of propagation in free space, and inserting dipole moments, the resulting far field expressions (26) through (35) for $\rho>1$, are

$$E_r^{ve} \approx -\sqrt{j}1.76\times 10^{-32}\,\frac{\omega^{7/2}Ile^{-z/\delta}}{\sigma^{3/2}\rho}\,T\left(1-\frac{j}{\rho}\right) \qquad (60)$$

$$H_\varphi^{ve} \approx -1.57\times 10^{-29}\,\frac{\omega^3 Ile^{-z/\delta}}{\sigma\rho}\,T\left(1-\frac{j}{\rho}\right) \qquad (61)$$

$$E_\varphi^{vm} \approx 1.96\times 10^{-35}\,\frac{\omega^4 ISNe^{-z/\delta}}{\sigma\rho^2}\left(1-\frac{j2}{\rho}\right) \qquad (62)$$

$$H_r^{vm} \approx \sqrt{-j}\, 1.76 \times 10^{-32} \frac{\omega^{7/2} ISNe^{-z/\delta}}{\sigma^{1/2}\rho^2} \left(1 - \frac{j2}{\rho}\right) \tag{63}$$

$$H_r^{he} \approx -\sqrt{j} 1.05 \times 10^{-23} \frac{\omega^{5/2} Ile^{-z/\delta}}{\sigma^{1/2}\rho^2} \left[\frac{1+T}{2} - \frac{jT}{2\rho}\right] \sin\varphi \tag{64}$$

$$H_\varphi^{he} \approx \sqrt{-j}\, 5.26 \times 10^{-24} \frac{\omega^{5/2} Ile^{-z/\delta}}{\sigma^{1/2}\rho} T \left[1 - \frac{j}{T\rho}(2T-1) - \frac{2}{\rho^2}\right] \cos\varphi \tag{65}$$

$$E_r^{he} \approx -5.92 \times 10^{-27} \frac{\omega^3 Ile^{-z/\delta}}{\sigma\rho} T \left[1 - \frac{j}{T\rho}(2T-1) - \frac{2}{\rho^2}\right] \cos\varphi \tag{66}$$

$$E_\varphi^{he} \approx j 1.18 \times 10^{-26} \frac{\omega^3 Ile^{-z/\delta}}{\sigma\rho^2} \left[\frac{1+T}{2} - \frac{jT}{2\rho}\right] \sin\varphi \tag{67}$$

$$E_r^{hm} \approx -\sqrt{j}\, 6.64 \times 10^{-30} \frac{\omega^{7/2} ISNe^{-z/\delta}}{\sigma^{1/2}\rho} T \left(1 - \frac{j}{\rho} - \frac{1}{\rho^2}\right) \sin\varphi \tag{68}$$

$$E_\varphi^{hm} \approx -\sqrt{-j}\, 1.33 \times 10^{-29} \frac{\omega^{7/2} ISNe^{-z/\delta}}{\sigma^{1/2}\rho^2} T \left(1 - \frac{j}{\rho}\right) \cos\varphi \tag{69}$$

$$H_r^{hm} \approx j\, 1.17 \times 10^{-26} \frac{\omega^3 ISNe^{-z/\delta}}{\rho^2} T \left(1 - \frac{j}{\rho}\right) \cos\varphi \tag{70}$$

$$H_\varphi^{hm} \approx -5.83 \times 10^{-27} \frac{\omega^3 ISNe^{-z/\delta}}{\rho} T \left(1 - \frac{j}{\rho} - \frac{1}{\rho^2}\right) \sin\varphi. \tag{71}$$

Similarly, the resulting near field expressions ($\rho < 1$) are

$$E_r^{ve} \approx \sqrt{-j}\, 1.76 \times 10^{-32} \frac{\omega^{7/2} Ile^{-z/\delta}}{\sigma^{3/2}\rho^2} (1 + j\rho) \tag{72}$$

$$H_\varphi^{ve} \approx j 1.57 \times 10^{-29} \frac{\omega^3 Ile^{-z/\delta}}{\sigma\rho^2} (1 + j\rho) \tag{73}$$

$$E_\varphi^{vm} \approx -5.90 \times 10^{-35} \frac{\omega^4 ISNe^{-z/\delta}}{\sigma\rho^4} \left(1 + j\rho - \frac{\rho^2}{3}\right) \tag{74}$$

$$H_r^{vm} \approx -\sqrt{-j}\, 5.26 \times 10^{-32} \frac{\omega^{7/2} ISNe^{-z/\delta}}{\sigma^{1/2}\rho^4} \left(1 + j\rho - \frac{\rho^2}{3}\right) \tag{75}$$

$$H_r^{he} \approx \sqrt{-j}\, 1.05 \times 10^{-23} \frac{\omega^{5/2} Ile^{-z/\delta}}{\sigma^{1/2}\rho^3} (1 + j\rho) \sin\varphi \tag{76}$$

$$H_\varphi^{he} \approx -\sqrt{-j}\, 5.26 \times 10^{-24} \frac{\omega^{5/2} Ile^{-z/\delta}}{\sigma^{1/2}\rho^3} (1 + j\rho - \rho^2) \cos\varphi \tag{77}$$

$$E_r^{he} \approx 5.92 \times 10^{-27} \frac{\omega^3 Ile^{-z/\delta}}{\sigma\rho^3} (1 + j\rho - \rho^2) \cos\varphi \tag{78}$$

$$E_\varphi^{he} \approx 1.18 \times 10^{-26} \frac{\omega^3 Ile^{-z/\delta}}{\sigma\rho^3} (1 + j\rho) \sin\varphi \tag{79}$$

$$E_r^{hm} \approx \sqrt{j}\, 6.64 \times 10^{-30} \frac{\omega^{7/2} ISNe^{-z/\delta}}{\sigma^{1/2}\rho^3} (1 + j\rho - \rho^2) \sin\varphi \tag{80}$$

$$E_\varphi^{hm} \approx -\sqrt{j}\, 1.33 \times 10^{-29} \frac{\omega^{7/2} ISNe^{-z/\delta}}{\sigma^{1/2}\rho^3} (1 + j\rho) \cos\varphi \tag{81}$$

$$H_r^{hm} \approx 1.17 \times 10^{-26} \frac{\omega^3 ISNe^{-z/\delta}}{\rho^3} (1+j\rho) \cos \varphi \tag{82}$$

$$H_\varphi^{hm} \approx 5.84 \times 10^{-27} \frac{\omega^3 ISNe^{-z/\delta}}{\rho^3} (1+j\rho-\rho^2) \sin \varphi. \tag{83}$$

Here the superscripts *ve, vm, he,* and *hm* refer respectively to vertical-electric, vertical-magnetic, horizontal-electric, and horizontal-magnetic dipoles. T is determined from (54) and (55).

The fields are all expressed in terms of ρ, the radial distance in units of (free-space-wavelength)/2π. Since ρ is directly proportional to both ω and r, the frequency dependence of the result is not what it would be if ρ were independent of frequency. For example, at values of $10^{-2}<\rho<1$, several of the fields are independent of frequency except for the depth attenuation factor: E_r^{he}, E_φ^{he}, H_r^{hm}, and H_φ^{hm}.

The z-components of the fields in the conducting halfspace (sea) are all small (zero to a first-order approximation for the vertical electric dipole) compared with the horizontal components, and therefore they have not been included. On the other hand, the z-component of the electric field in the air, for the horizontal dipoles, is the predominant component. The expressions for the electric field vertical component in the nonconducting region at the surface can be obtained by applying the boundary condition (36c) for $\rho>1$; they are

$$E_{z2}^{he} \approx -\sqrt{-j}\, 1.98 \times 10^{-21} \frac{\omega^{5/2} Ile^{-z_t/\delta}}{\sigma^{1/2}\rho} T \left(1-\frac{j}{\rho}\right) \cos \varphi \tag{84}$$

$$E_{z2}^{hm} \approx 2.22 \times 10^{-24} \frac{\omega^3 ISNe^{-z_t/\delta}}{\rho} T \left(1-\frac{j}{\rho}\right) \sin \varphi. \tag{85}$$

Similarly, the results for $\rho<1$ are

$$E_{z2}^{he} \approx -\sqrt{j}\, 1.98 \times 10^{-21} \frac{\omega^{5/2} Ile^{-z_t/\delta}}{\sigma^{1/2}\rho^2} (1+j\rho) \cos \varphi \tag{86}$$

$$E_{z2}^{hm} \approx -j2.22 \times 10^{-24} \frac{\omega^3 ISNe^{-z_t\ \delta}}{\rho^2} (1+j\rho) \sin \varphi. \tag{87}$$

These fields are given only at the surface as no effort has been made to evaluate the necessary field integrals at any distance away from the surface in the nonconducting medium. The horizontal electric antenna is the most practical for use in submarine communications. As an illustration of the manner in which the electric and magnetic fields vary as a function of distance from the source; the fields for the horizontal electric antenna have been plotted as a function of ρ in figures 4, 5, and 6, for typical values of frequency, conductivity, and practical antenna current, length, and depth.

6. Discussion and Conclusions

The form taken by the field equations indicates the mode of propagation that prevails as long as the distance from the boundary to both point of observation and source dipole is small compared with the horizontal separation. Each of the field expressions may be considered to consist of 3 parts: a multiplying factor which includes the dipole strength and parameters such as frequency and conductivity of the medium; an exponential attenuation factor whose exponent is the sum of the distance from the dipole to the surface and from the surface to the point of observation, expressed in units of skin depth; and a factor associated with variation in the radial (horizontal) direction. The latter factor is the same as that for surface fields of surface dipoles.

Thus it appears that propagation occurs as indicated in figure 7. The wave starts at the dipole, proceeds by the shortest path to the surface (the path of minimum attenuation), is refracted at the surface, travels along the surface as a wave in air, and comes to the point of observation by the path of least attenuation (straight down).

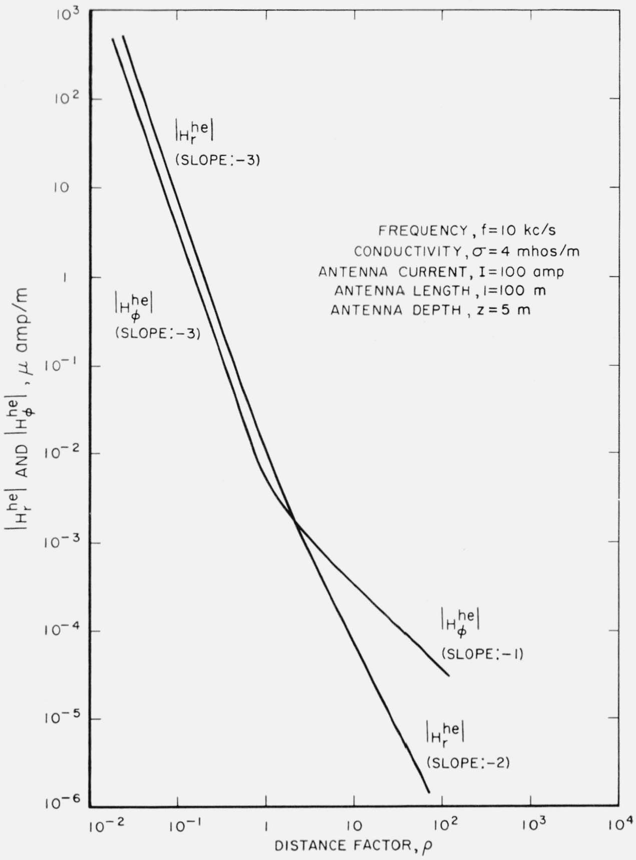

FIGURE 4. *Variation of the maximum absolute value of the magnetic field components in the sea for a submerged horizontal electric dipole with radial distance in units of free-space-wavelength/2π.*

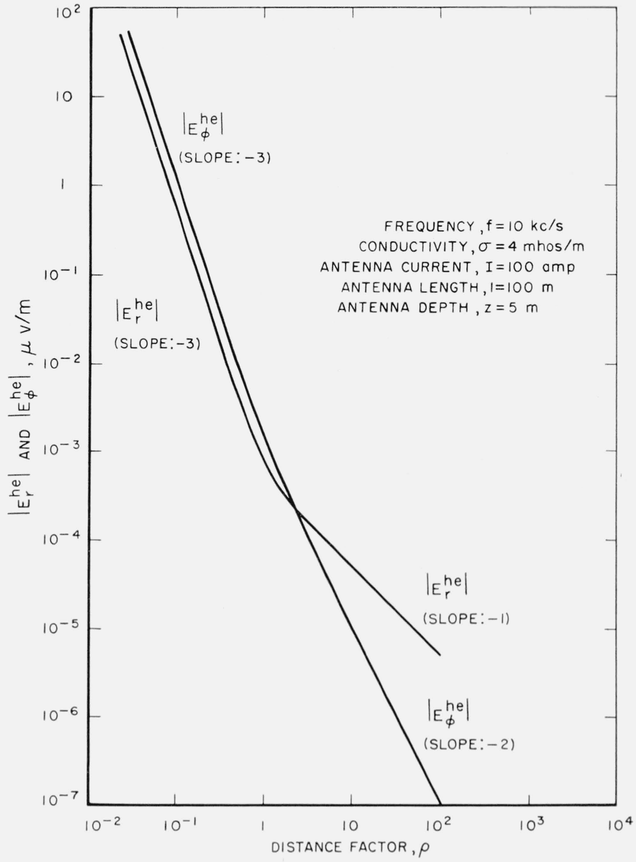

FIGURE 5. *Variation of the maximum absolute value of the electric field components in the sea for a submerged horizontal dipole with radial distance in units of free-space-wavelength/2π.*

It is possible to consider that an equivalent source on the surface sets up the wave in the nonconducting medium. This wave then travels as would be expected for such a source. Since the sea is not a perfect conductor, there is a slight tilt to the wave front, as indicated by the presence of both radial and axial components of the electric field; the radial (horizontal) component is much smaller than the axial (vertical) component. The Poynting vector for such a wave is nearly horizontal, but tilted slightly toward the conducting medium. The component of the Poynting vector associated with that part of the wave traveling into the conducting medium is small but finite. This wave which travels straight down is that which is seen by the observer in the conducting medium.

A similar situation prevails for waves in an imperfectly conducting waveguide. It is customary to calculate the losses in the wall of the waveguide by calculating the power flowing into the wall per unit area. In the waveguide this represents a leak of power from the desired wave. In this situation, where the observer is actually located in the conductor, the desired signal is the same as the "leak" for the waveguide.

Because of this mode of propagation, it is possible for radiation starting with the dipole in a conducting half space to be observed at much greater distances than would be possible if the wave were generated in an infinite homogeneous conducting medium. Thus, the exponential attenuation of 55 db per wavelength applies in the case of the conducting half space only to that part of the path from the dipole to the surface and from the surface to the point of observation. The rest of the path undergoes the same sort of attenuation that a wave generated by a dipole in air would encounter.

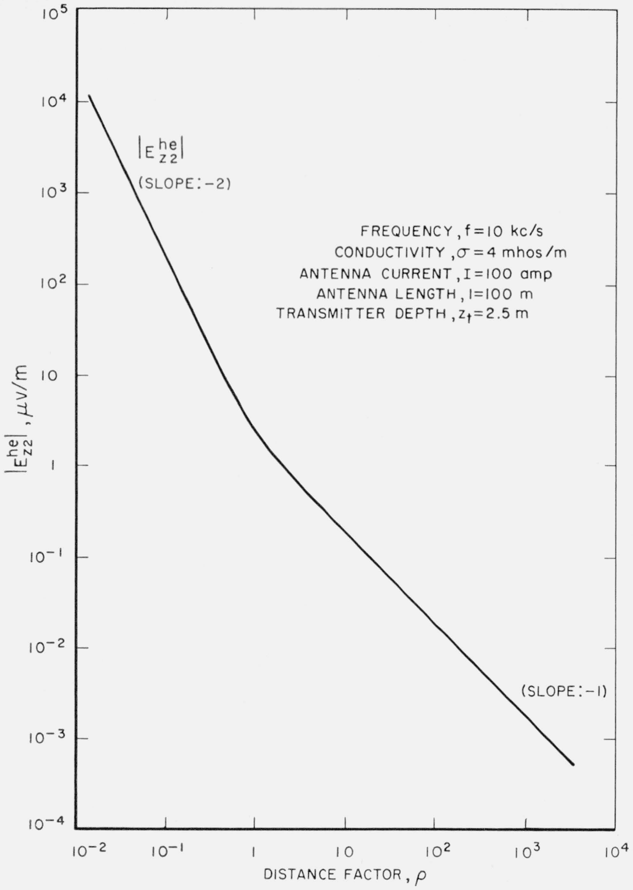

FIGURE 6. *Variation of the maximum absolute value of the normal electric field component in the air for a submerged horizontal electric dipole with radial distance in units of free-space-wavelength/2π.*

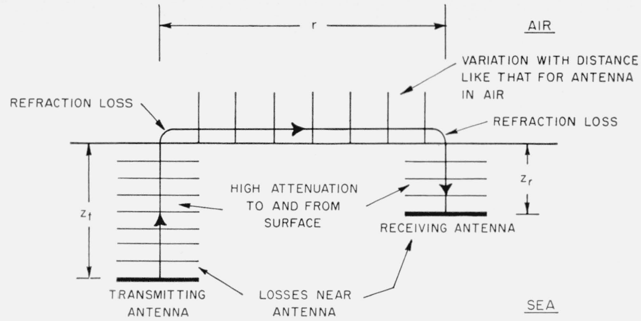

FIGURE 7. *Path of propagation.*

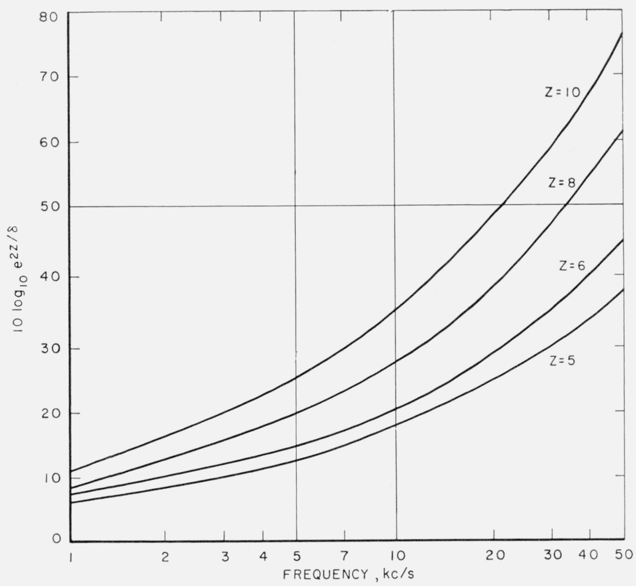

FIGURE 8. *Variation of depth attenuation factor with frequency.*

Although this development has applied to the case of a conducting half space and the expressions given are those associated with the approximation of the earth's surface by a plane, the fact that it is possible to separate out the horizontal attenuation of the wave in air means that it is also possible to apply the transmission loss that would be developed using spherical, rather than plane, earth geometry. In fact, it is also possible to use the attenuation calculated when the ionosphere is taken into account. Thus the results obtained here are considerably more general, within the limits of the approximations used, than would be indicated by the assumption of a plane earth.

It is interesting to note the magnitude of the attenuation of the wave in going through the conducting medium for the special case of sea water. This is shown for several depths as a function of frequency in figure 8. The depth indication is in meters. It is obvious that if the attenuation is to be kept low, the frequency must be kept low also.

It has been shown here that the fields created by a dipole in a conducting medium, which is near a plane boundary with a nonconducting medium, may be expressed in a form that may be considered as propagation from the dipole to the interface, travel through the nonconducting medium along the boundary, and propagation back into the conducting medium to the point of observation. Expressions have been presented for the fields of both vertical and horizontal, electric and magnetic dipoles. The fields of the horizontal dipoles are considerably stronger than those of the vertical dipoles, as would be expected because of the vertical nulls in the radiation from the vertical dipoles.

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