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A Note Concerning the Excitation of ELF Electromagnetic Waves

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Previous solutions for the ELF mode series are discussed briefly. Particular attention is paid to the height-gain functions. The excitation of the modes for vertical and horizontal dipole excitation is also considered.

1. Introduction

In a number of earlier papers [Wait, 1960a and b] the author discussed the theory of electromagnetic wave propagation in the extremely low frequency range (i.e., 1.0 to 3000 c/s). The model adopted is a smooth homogeneous earth surrounded by a concentric reflecting layer which is to represent the ionosphere. The field was represented by a sum of modes for both vertical and horizontal electric dipole excitation. Essentially the same model has been used by Schumann [1954] but some of his results differ from those of Wait [1960a]. It is the purpose of this note to discuss certain extensions and limitations of the theory.

Initially the earth is taken to be flat. This idealization simplifies the discussion. Furthermore, for ELF, influence of sphericity can be introduced later in the analysis.

2. Formulation

Choosing the usual cylindrical coordinate system (ρ,ϕ,z) the earth's surface is at $z=0$ and the ionospheric reflecting layer is at $z=h$. Free space of electrical constant ϵ_0,μ_0 occupies the space $0<z<h$. At $z=0$ and $z=h$, it is further assumed that the fields satisfy the Leontovich boundary conditions. Again this is not an essential restriction but it does simplify the discussion somewhat. Thus

$$\left.\begin{aligned} E_\rho &= -Z_g H_\phi \\ E_\phi &= Z_g H_\rho \end{aligned}\right\} \text{ at } z=0 \tag{1}$$

and

$$\left.\begin{aligned} E_\rho &= Z_i H_\phi \\ E_\phi &= -Z_i H_\rho \end{aligned}\right\} \text{ at } z=h \tag{2}$$

where

$$Z_g=(i\mu_0\omega/\sigma_g)^{\frac{1}{2}}=\Delta_g/\eta,$$

and

$$Z_i=(i\mu_0\omega/\sigma_i)^{\frac{1}{2}}=\Delta_i/\eta,$$

in terms of the (complex) conductivities σ_g and σ_i of the earth and the ionosphere, respectively. The preceding forms for Z_g and Z_i are valid at ELF when the earth and the ionosphere are both equivalent to isotropic conductors [Wait, 1960b]. In the above, $\eta \cong 120\pi$ ohms and Δ_g and Δ_i are dimensionless complex quantities.

3. General Form of the Mode Series

First the source is taken to be a vertical electric dipole (of current I and length ds) located at $z=z_0$, where $0<z_0<h$. From a previous analysis the fields, in the space $0<z<h$, are obtained from

$$E_z=WE_0,$$
$$E_\rho=-SE_0,$$
$$H_\phi=-TE_0/\eta, \tag{3a}$$

where

$$E_0=i(\eta/\lambda)\,Ids(e^{-ik\rho})/\rho, \tag{3b}$$

is a reference field and W, S, and T are dimensionless factors which are expressible as a sum of modes. In the usual notation, the latter are given explicitly by

$$W=-i\pi\frac{\rho}{h}e^{ik\rho}\sum_{n=0}^{\infty}\delta_n S_n^2 H_0^{(2)}(kS_n\rho)f_n(z_0)f_n(z) \tag{4}$$

$$S=\frac{\pi\rho}{h}e^{ik\rho}\sum_{n=0}^{\infty}\delta_n S_n H_1^{(2)}(kS_n\rho)f_n(z_0)g_n(z) \tag{5}$$

$$T=-\frac{\pi\rho}{h}e^{ik\rho}\sum_{n=0}^{\infty}\delta_n S_n H_1^{(2)}(kS_n\rho)f_n(z_0)f_n(z) \tag{6}$$

where

$$\delta_n=\left[1+i\,\frac{\partial[R_i(C)R_g(C)]/\partial C}{2khR_i(C)R_g(C)}\right]^{-1}_{C=C_n}, \tag{7}$$

$$R_g(C)=\frac{C-\Delta_g}{C+\Delta_g}, \tag{8}$$

$$R_i(C)=\frac{C-\Delta_i}{C+\Delta_i}, \tag{9}$$

and

$$2f_n(z)=[R_g(C_n)]^{-\frac{1}{2}}e^{ikC_nz}+[R_g(C_n)]^{\frac{1}{2}}e^{-ikC_nz}, \tag{10}$$

$$2g_n(z)=C_n[R_g(C_n)]^{-\frac{1}{2}}e^{ikC_nz}-C_n[R_g(C_n)]^{\frac{1}{2}}e^{-ikC_nz}. \tag{11}$$

$H_0^{(2)}$ and $H_1^{(2)}$ are the Hankel functions of the second kind of order zero and one. C_n are the solutions of the mode equation

$$R_g(C)R_i(C)e^{-i2khC}=e^{-i2\pi n} \tag{12}$$

and

$$S_n=(1-C_n^2)^{\frac{1}{2}}.$$

4. Height-Gain Functions

While the above result was given in previous papers [Wait, 1960a and b] its significance was not fully discussed there. Furthermore, the heights of source and observer above the ground were taken to be zero. It is of interest to look into the nature of the solution under more general conditions.

The function W is considered first. It is the ratio of the actual vertical electric field E_z to the reference field E_0 of the same dipole source if it, and the observer, were on the surface of a flat perfectly conducting plane. If $k\rho \gg 1$, corresponding to the "far-zone", it follows that

$$W \cong \frac{(\rho/\lambda)^{\frac{1}{2}}}{h/\lambda} e^{i\left(\frac{2\pi\rho}{\lambda}-\frac{\pi}{4}\right)} \sum_{n=0}^{\infty} \delta_n S_n^{3/2} e^{-i2\pi S_n\rho/\lambda} f_n(z_0) f_n(z), \tag{13}$$

where λ is the free-space wavelength. The vertical field E_z is thus proportional to a sum of modes whose amplitudes are proportional to

$$\frac{1}{\rho^{\frac{1}{2}}} \exp\left[-\mathrm{Im}\ 2\pi S_n\rho/\lambda\right].$$

At short distances, such that

$$\rho \ll h,\ W \rightarrow \left(1-\frac{i}{k\rho}-\frac{1}{k^2\rho^2}\right), \tag{14}$$

provided $z=z_0=0$. This is the expected behavior for a dipole if the reflecting layer (at $z=h$) was neglected. Numerical values of W were presented on previous occasions for the intermediate range where $k\rho$ is unrestricted but $z=z_0=0$ [Wait and Carter, 1960].

As can be seen from (4), the height dependence of the individual modes is described entirely by the function $f_n(z)$. When $z=0$, we see that

$$[f_n(0)]^2=\frac{C_n^2}{C_n^2-\Delta_g^2}. \tag{15}$$

For most applications to ELF propagation, $|C_n^2| \gg |\Delta_g^2|$, and thus $f_n(0)$ can be replaced by unity. This approximation is contained implicitly in earlier work. For present purposes it appears to be preferable to define a "height-gain function" in the manner

$$F_n(z)=\frac{f_n(z)}{f_n(0)} \tag{16}$$

Using (10) this can be rewritten in the form

$$F_n(z)=\frac{e^{ikC_nz}+R_g(C_n)e^{-ikC_nz}}{1+R_g(C_n)}$$

$$=1+i\Delta_g kz-\frac{(kC_nz)^2}{2}$$

$$+\text{terms in } (kz)^3,\ (kz)^4, \text{ etc.} \tag{17}$$

Since

$$C_0^2 \cong i\frac{\Delta_i+\Delta_g}{kh} \cong i\frac{\Delta_i}{kh} \tag{18}$$

it is seen that for the zero mode

$$F_0(z)=1+i\Delta_g kz-\frac{ik^2z^2\Delta_i}{2kh}+\ \ldots\ . \tag{19}$$

For most ELF applications $F_0(z)$ departs only slightly from unity. Also, it should be noted that if the ground was a perfect conductor the linear term in height is absent. Similar remarks apply to the higher modes. In fact, if $\Delta_g=0$, we have

$$F_n(z)=\cos kC_nz$$

for all n.

It is seen from (6) that the height-gain functions for the H_ϕ component is identical to $F_n(z)$. In the case of the E_ρ field it is desirable to define a height factor $G_n(z)$ as follows

$$G_n(z)=\frac{g_n(z)}{f_n(0)}. \tag{20}$$

For a given mode, this quantity is proportional to the ratio of the horizontal electric field at height z to the vertical electric field at height zero. From (11), it is seen that

$$G_n(z)=\frac{g_n(z)}{f_n(0)}=\frac{C_n(e^{ikC_nz}-R_ge^{-ikC_nz})}{1+R_g}$$

$$=\Delta_g+ikC_n^2z-\frac{(kC_nz)^2}{2}\Delta_g+\ \ldots\ . \tag{21}$$

For mode zero

$$G_0(z) \cong \Delta_g-\Delta_i\frac{z}{h}-i\Delta_i\Delta_g\frac{kz^2}{2h}+\ \ldots\ . \tag{22}$$

Again, under most practical conditions, only the first term of this series need be retained. In fact, if $z=0$, we have exactly that

$$G_n(0)=\Delta_g \tag{23}$$

which is simply a consequence of the Leontovich boundary condition imposed at $z=0$. If the ground is perfectly conducting

$$G_n(z)=iC_n \sin kC_nz. \tag{24}$$

5. Excitation of the Modes

We now consider the excitation coefficient δ_n defined by (7). Since

$$\frac{\partial}{\partial C}R_gR_i=R_i\frac{\partial R_g}{\partial C}+R_g\frac{\partial R_i}{\partial C}$$

$$=R_i\frac{2\Delta_g}{(C+\Delta_g)^2}+R_g\frac{2\Delta_i}{(C+\Delta_i)^2}, \qquad (25)$$

it readily follows that

$$\delta_n=\frac{1}{1+\frac{i\Delta_g}{kh(C_n^2-\Delta_g)^2}+\frac{i\Delta_i}{kh(C_n^2-\Delta_i^2)}} \qquad (26)$$

If $|\Delta_g|\ll|\Delta_i|$, which is the usual case, the second term in the denominator can be neglected. Then to the same approximation

$$\delta_n=\frac{1}{1+\frac{\sin 2khC_n}{2khC_n}} \qquad (27)$$

where use has been made of $R_i=e^{2ikC_nh}$. Actually, (27) is exact if $\Delta_g=0$ and it can be regarded as an excellent approximation provided $|\Delta_g|\ll|\Delta_i|$. For practical applications at ELF, this may be further approximated to

$$\delta_0\simeq\frac{1}{2}\text{ since }khC_0\ll 1$$

and

$$\delta_n\simeq 1\text{ since }2khC_n\simeq 2n\pi.$$

6. Further Extensions

The extension to a vertical magnetic dipole source is very simple. In view of the symmetrical properties of Maxwell's equations we can write down the corresponding field expressions. Thus

$$H_z=W^*H_0 \qquad (28)$$

$$H_\rho=-S^*H_0 \qquad (29)$$

$$E_\phi=T^*H_0\eta \qquad (30)$$

where W^*, S^* and T^* are dimensionless factors and

$$H_0=i(1/\eta\lambda)Kds(e^{-ik\rho})/\rho \qquad (31)$$

where K is the magnetic current in the vertical magnetic dipole of length ds. The explicit expressions for W^*, S^*, and T^* are identical in form to those given by W, S, and T (i.e., (4) to (6)). Now, however, the modes are solutions of the equation

$$R_g^h(C)R_i^h(C)e^{-i2khC}=e^{-i2\pi m} \qquad (32)$$

where m is an integer and

$$R_g^h(C)=\frac{C-\Delta_g^h}{C+\Delta_g^h} \qquad (33)$$

and

$$R_i^h(C)=\frac{C-\Delta_i^h}{C+\Delta_i^h} \qquad (34)$$

where $\Delta_g^h\simeq\frac{1}{\Delta_g}$ and $\Delta_i^h\simeq\frac{1}{\Delta_i}$. The superscript h is to denote the horizontally polarized nature of these reflection coefficients.

In this case we note that

$$[f_m^h(0)]^2=\frac{C_m^2}{C_m^2-(\Delta_g^h)^2}\simeq-\Delta_g^2C_m^2 \qquad (35)$$

which is a very small quantity compared with unity. Consequently, the excitation of ELF waves by a vertical magnetic dipole on the ground plane is very poor. However, when the dipole is raised the situation is greatly improved. The height-gain function of the modes for this case is

$$F_m^h(z)=\frac{e^{ikC_mz}+R_g^h(C_m)e^{-ikC_mz}}{1+R_g^h(C_m)}$$

$$=1+ikz/\Delta_g-\frac{(kC_mz)^2}{2}+\ldots. \qquad (36)$$

Since Δ_g is a small quantity, the second term of this series may be quite large even for small heights.

The height-gain function $F_m^h(z)$ describes the behavior of both the H_z and E_ϕ components of the vertical magnetic dipole. The appropriate height-gain function for the horizontal magnetic field component H_ρ is defined by the function

$$G_m^h(z)=\frac{g_m^h(z)}{f_m^0(0)}=\frac{C_m(e^{ikC_mz}-R_g^he^{-ikC_mz})}{1+R_g^h} \qquad (37)$$

in complete analogy to (21) for the electric dipole case.

The excitation of the waveguide modes for a horizontal electric dipole was treated previously [Wait, 1960a]. For an electric dipole of moment Ids situated at $z=z_0$ and oriented in the direction $\phi=0$, it was shown that

$$E_z=\cos\phi E_0S \qquad (38)$$

where E_0 and S are given by (3b) and (5), respectively. Also

$$\eta_0H_z=-\sin\phi E_0T^* \qquad (39)$$

where, explicitly,

$$T^*=-\frac{\pi\rho}{h}e^{ik\rho}\sum_{m=0}^{\infty}\delta_mS_mH_1^{(2)}(kS_m\rho)f_m^h(z_0)f_m^h(z). \qquad (40)$$

For most applications the fields derived from S (i.e., the vertically polarized waves) are only significant.

The fields associated with T^* (i.e., the horizontally polarized waves) are hardly excited at all when $kz_0 \ll 1$ which is the usual case.

The field of a horizontal *magnetic* dipole of moment Kds has the same structure as the fields of the horizontal electric dipole. Explicitly

$$H_z = \cos \phi H_0 S^* \tag{41}$$

and

$$E_z = \sin \phi H_0 T / \eta_0. \tag{42}$$

The solution for the horizontal magnetic dipole has been discussed recently by Galejs [1961]. He obtains his results by an application of the reciprocity theorem as outlined by Wait [1960a].

The remarks in the preceding section concerning the height-gain functions for the vertical dipoles can be carried over to the horizontal dipoles without modification.

The generalization of the foregoing results to include the effects of earth curvature is not difficult. Provided one stays away from the antipode of the source dipole it is only necessary to multiply the field expressions by

$$\left(\frac{\theta}{\sin \theta}\right)^{\frac{1}{2}}$$

where θ is the central angle subtended by the source and observer at the center of the earth. The justification for this step was given previously. It might be added, however, that in addition to the restriction in the neighborhood of the antipode it is also necessary that

$$C_n^2 \gg h/a \tag{43}$$

and

$$\left(\frac{ka}{2}\right)^{\frac{1}{3}} C_n \gg 1. \tag{44}$$

It is not difficult to verify that these conditions are well justified at ELF.

7. References

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