JOURNAL OF RESEARCH of the National Bureau of Standards—D. Radio Propagation
Vol. 65D, No. 5, September–October 1961

Theoretical Scattering Coefficient for Near Vertical Incidence From Contour Maps [1]

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(Received January 30, 1961; revised April 10, 1961)

In calculation of the theoretical scattering coefficient for a terrain, previous authors tentatively assumed the normalized autocovariance function $\rho(r)=e^{-Ar^2}$ for the ground elevation as a function of distance from a given point. Recently autocorrelation studies were made using maps with contours ranging from one to twenty-five feet. These resulted in curves of $\rho(r)$, which are approximated by exp $(-|r|/B)$. The theoretical scattering cross section (σ_0) of many such terrains can be expressed as

$$\sigma_0=4\sqrt{2}\,\frac{\pi B^2}{\lambda^2}\left(\frac{\theta}{\sin\theta}\right)e^{-4k^2\sigma^2\cos^2\theta}\sum_{n=1}^{\infty}\frac{(4k^2\sigma^2)^n(\cos^2\theta)^{n+1}}{(n-1)!\,(2k^2B^2\sin^2\theta+n^2)^{3/2}},\qquad (1)$$

where σ, λ, k, and θ are standard deviation of the target terrain, wavelength, wavenumber $(2\pi/\lambda)$ and the angle of incidence respectively. For the case where $1/B$ is small as compared to k, the above expression becomes

$$\sigma_0=\frac{4\sigma^2}{\lambda B}\,(\theta\cot^4\theta)\qquad \text{for } \theta\neq 0°.\qquad (2)$$

These expressions, when normalized, are in agreement with experimental results of other authors. It is also noteworthy that the results obtained with an acoustic simulator model compared very well with this theoretical expression. This work is based on the property that the ground is conducting and has random elevation variations. Theoretical results calculated on the basis of varying ground impedance rather than its elevation are also in agreement with this expression.

1. Introduction

In recent years, the calculation of backscatter from a rough surface, with very obvious extension and application to the radar return from the moon, has attracted considerable attention. Here we follow the basic approach of Davies [1954] as modified by Moore [1957] and Cooper [1958]. The modified Kirchhoff-Huygens' principle is employed in the calculation using modified spherical variables of integration. The ground model used assumes "facets," of variable size and height above mean ground level, whose position is described statistically.

Instead of assuming a correlation model for terrain, as is usually done, the present approach used contour maps of different terrain samples in the United States to calculate the terrain-elevation autocovariance and other statistical properties. The overall approach to the problem is an approximation, but the results so obtained are very reasonable indeed, insofar as the comparison with terrain return and moon-echo data reported in various publications [Briggs, 1960; Hughes, 1960; Nielson, 1960] is concerned.

[1] This work was sponsored by the Naval Ordnance Test Station, China Lake, Calif., under contract No. N123(60530)18138A.

2. Statistical Properties of Terrain

Various autocovariance [Ament, 1953; Davies, 1954; Moore, 1957b; Daniels, 1960] functions for rough terrain have so far been assumed. In some instances [Cohen, 1948] an effort was made to derive an approximate expression for the autocorrelation from actual data on terrain elevation. A general expression [Norton, 1960] for space correlation function ρ $(\bar{r})$ describing the random variation of the refractive index over space is given here:

$$\rho(\bar{r})=\left[\frac{2}{\Gamma(\mu)}\right]\left(\frac{r}{2l_0}\right)^{u}K_\mu\left(\frac{r}{l_0}\right).\qquad (2.1)$$

where

l_0=characteristic scale
μ=constant
$\Gamma(\mu)$=gamma function
$K_\mu\left(\frac{r}{l_0}\right)$=modified Bessel function of the second kind.

It seems to cover some of the most commonly assumed [Wheelon, 1959] expressions for correlation functions.

A search of the literature shows that small-scale perturbations of the terrain elevation have not so far been used to calculate an experimental autocovari-

TABLE 1. *Location of terrain samples and the surface characteristic constant*, B

Sample	General location	Average elevation	Standard deviation	Sampling interval	Surface characteristic constant, (1/B)
		ft	*ft*	*ft*	
1	Lawrence, Kans	853.5	5.63	30	0.1193×10^{-3}
2	Tees Nos, Ariz	5254	49	5	1.3213×10^{-3}
3	Turtle Mountain, N. Dak	1990	8.2	5	0.7070×10^{-3}
4	Gila River, Ariz	2617	13.6	5	$.9236 \times 10^{-3}$
5	White River, Ariz	5028	5.18	5	1.3610×10^{-3}
6	Mountain Park, N. Mex	7530	1030	62.5	0.2729×10^{-3}
7	Sandia Park, N. Mex	6780	250	20	$.9860 \times 10^{-3}$

ance function. For this reason, contour maps for seven different terrains in the United States were selected, as listed in table 1. This set included relatively flat land, rolling plains and some rugged mountain areas. The average elevation (above sea level) and standard deviation of these samples varied from 850 to 7,530 ft and 5 to 1,030 ft, respectively. Three distinctly different but random lines were drawn on each of these contour maps. Along these lines, the elevation of the terrain was read to within one tenth of a foot (by interpolation) on 1-ft contour maps, and within 5 ft on 20- and 25-ft contour maps. Using a reasonable compromise between resolution and confidence level [Blackman and Tukey, 1958], and due to the unavailability of less-than-one-foot contour maps, horizontal sampling intervals of 5, 20, 30, and 62.5 ft were used. The number of points for a subsample varied from 58 to 610. The autocovariance was calculated using a CRC–102 computer. The resulting curves were averaged for each type of terrain. A simple theoretical approximation of these curves for high confidence level portions was found to be

$$\rho(r) \approx e^{-|r|/B} \qquad (2.2)$$

where

B=characteristic constant
r=distance between points

Seven cases for the experimental autocovariance are shown in figure 1.

In figure 1 the autocorrelation curves for samples 1 and 3 seem to slope off faster than the rest. In case of sample 1 this is caused by the lag distance for a given number of lags being greater than for samples 2 through 5 and 7. In view of this, if all these curves were plotted on a semilog paper (as in fig. 1), for a given lag distance rather than for a fixed number of lags, these would plot approximately as straight lines over limited distances. These two curves can also be approximated as $\rho(r)=e^{-|r|/B}\cos cr$, where c was found to be 0.2188 and 0.7931 deg/ft for samples 1 and 3, respectively. For near vertical incidence, the distances involved are rather short, and therefore $\cos cr$ is approximately unity. It is felt that the apparent cosine factor in the autocovariance appears because of the relatively short length of the sample used in this case. This seems to indicate that the autocovariance function $\rho(r)$ often varies as $e^{-|r|/B}$.

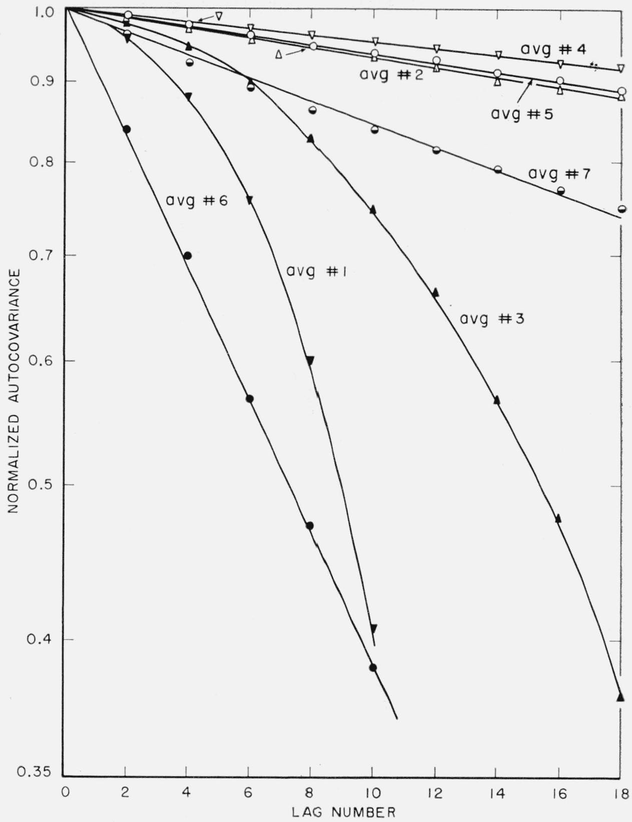

FIGURE 1. *Normalized autocovariance curves—(portions with highest confidence level).*

Sample No. and type of terrain	Length of data (points in each subsample)	Sampling interval
		ft.
1 (relatively flat)	58, 101, 079	30
2 (rolling and flat)	252, 276, 215	5
3 (rolling)	189, 194, 144	5
4 (irregular, sloping)	226, 196, 261	5
5 (flat, slight slope)	287, 264, 272	5
6 (very rugged mtn.)	216, 187, 187	62.5
7 (rugged mountain)	187, 187, 187	20

It is known [Cohen, 1948; Isakovitch, 1952] that the slope of $\rho(r)$ at r equal to zero should be zero unless the ground elevation function has infinite slopes. In this instance, $\rho(r)$ has a slope of $-1/B$ which is in most cases believed to be of the order of 10^{-3} (see table 1). Considering the minute size (in the general range of 10^{-3} to 10^{-4} m) of the earth particles, this may be quite a reasonable approximation to the exact description of the terrain roughness.

It was also found that the ground elevations generally are normally distributed. For a certain type of terrain, the ground elevation data along a random line is most probably a random function belonging to a large, approximately stationary ensemble. It is further suspected that the expressions obtained for the scales of roughness covered in this study when reduced by a scale factor, might well describe extremely minute variations of ground roughness as seen by very high frequency waves.

3. Scattering Coefficient

The expressions for vector waves, in radar return, although easy to set up are very difficult to evaluate. For this reason the Kirchhoff-Huygens' principle has been applied to scalar waves. This leads to difficult integration problems and other such complexities, so certain simplifying assumptions were made. Similar assumptions were made by others [Davies, 1954; Moore, 1957b]. These are as follows:

(1) No portion of the ground is shielded from the incident radiation.

(2) The ground is considered to be a perfect conductor.

(3) The magnitudes of the surface currents are of the same order as those of a plane reflector, but the phase varies in a random manner, depending on the height of a particular point.

(4) The reradiation from a particular small area on the ground is isotropic.

(5) The antenna gain G is essentially uniform for $-\theta_0<\theta<\theta_0$, and is zero outside of this range.

(6) The results of this scalar wave approach are an approximation to those for the vector waves, as most field vectors would be nearly parallel to the surface involved

The variables ϕ and ϕ' were replaced with ψ and ψ', where ψ' is the sum of ψ and α as shown in figure 2. This change of variable was made following Davies' [1954] work, in order to make the integration a little less complex.

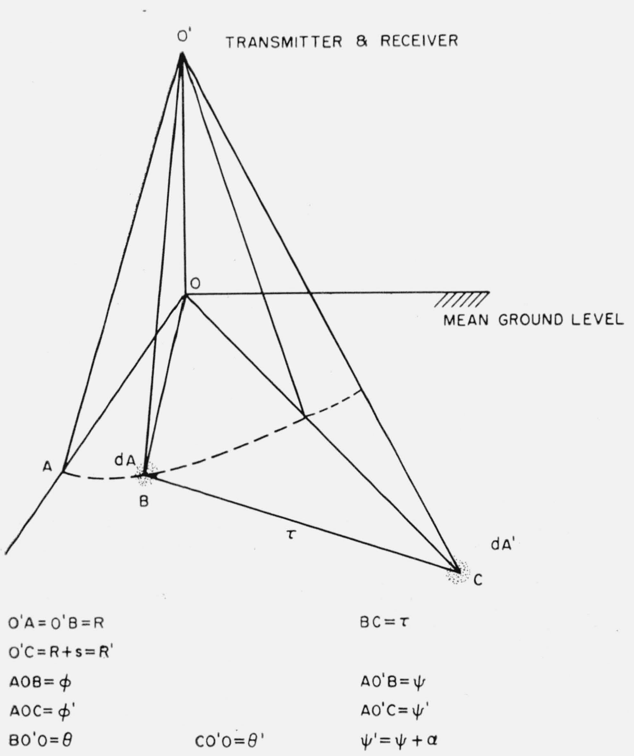

FIGURE 2. *Geometry of the problem, showing interrelation between various variables of integration.*

The radar equation as applied to pulse radar [Moore and Williams, 1957a], gives the average received power $\overline{P}_r$ from many scatterers as

$$\overline{P}_r=\frac{\lambda^2}{32\pi^2}\int\frac{P_T\left(t-\frac{2R}{c}\right)G^2(\theta)\sigma_0(\theta,\lambda,\ldots)}{R^3}dR. \quad (3.1)$$

where

P_T=power transmitted
R=range
σ_0=scattering coefficient per unit area (assumed independent of ϕ)
t=time
c=velocity of propagation (velocity of light)
G=antenna gain (assumed independent of ϕ)
λ=wavelength
θ, ϕ=angles as shown on figure 2.

The field E at $0'$ is obtained by applying Huygens' principle to the modified geometry of figure 2, and can be written as

$$E=\int\frac{1}{\lambda R}(I_s\eta)^{1/2}\cos\theta e^{-j4\pi R/\lambda}dA. \quad (3.2)$$

where

$I_s=P_TG/(4\pi R^2)$
η=intrinsic impedance of free space
dA=area element on the ground $\left(=RdR\frac{d\psi}{\sin\theta}\right)$.

One arrives at the following expression for E by applying (3.2) to a rough surface,

$$E=\frac{1}{\lambda}\int_{\frac{-c\tau}{4}}^{\frac{c\tau}{4}}\int_{-\theta_0}^{\theta_0}\frac{\cot\theta}{R}\left(\frac{P_TG\eta}{4\pi R^2}\right)^{1/2}\exp\left[j2k(R-\delta\cos\theta)\right]d\psi dR \quad (3.3)$$

where

k=wave number $\left(\frac{2\pi}{\lambda}\right)$
τ=pulse width
$2\theta_0$=beam width of the antenna
$\delta=\delta(R_0+s,\psi+\alpha)$=elevation of ground height above mean ground level at a point located at range R_0+s and modified azimuth angle $\psi+\alpha$.
R_0=range to the projection of a point at range R, on the mean ground level (see fig. 3).

Equation (3.3) takes into account the variations in phase and neglects the other effects of the changes in range. This is based on the approximation that the percentage changes in range are not appreciable for near-vertical incidence. The received power is

$$P_r=\frac{1}{2}\mathscr{R}\mathrm{e}\left[\frac{EE^*}{\eta}\frac{G\lambda^2}{4\pi}\right] \quad (3.4)$$

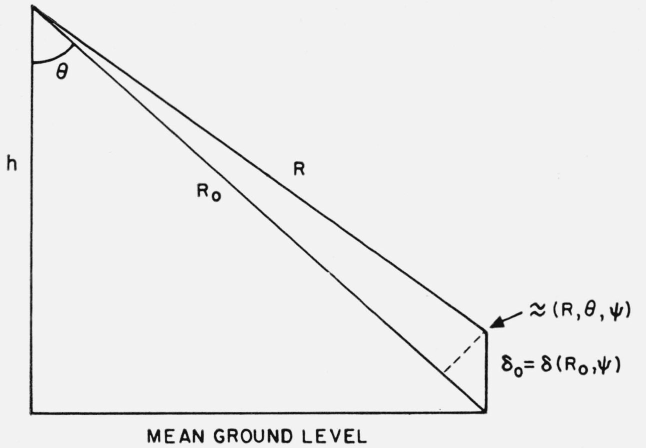

FIGURE 3. *A typical point, at a random height δ_0 above mean ground level.*

where E^* stands for complex conjugate of E and $\mathscr{R}$e is the symbol for 'real part of' the expression following it. The difference in ranges of two points on ground located at (R,θ,ψ) and (R',θ',ψ') can be expressed as follows with the help of figure 3,

$$R-R'=[R_0-\delta(R_0,\psi)\cos\theta]-[R_0'-\delta(R_0',\psi')\cos\theta']$$

$$=-s+[\delta(R_0+s,\psi+\alpha)\cos\theta'-\delta(R_0,\psi)\cos\theta]$$

$$=-s+(\delta-\delta_0)\cos\theta \tag{3.5}$$

where $\theta\approx\theta'$ (assumed)

$$\delta=\delta(R_0+s,\psi+\alpha)$$

$$\delta_0=\delta(R_0,\psi)$$

$$R_0'\cong R_0+s$$

The substitution of (3.3) in (3.4) results in

$$P_r=\frac{1}{32\pi^2}\mathscr{R}e\int\frac{P_TG^2\cos\theta\cos\theta'}{R^2R^{2\prime}}e^{-j2k(R-R')}dAdA'. \tag{3.6}$$

where the primed quantities refer to a point at (R', θ', ψ').

The average received power can now be expressed in terms of R, s, ψ, and α by substituting (3.5) in (3.6) as

$$\overline{P}_r=\frac{1}{32\pi^2}\int_{-\pi}^{\pi}\int_{-2\theta_0}^{2\theta_0}\int_{-(R_0-h)}^{\infty}\int_{R_0-\frac{c\tau}{2}}^{R_0}\frac{P_TG^2\cot\theta\cot\theta'}{RR'}$$

$$\overline{\exp[j2k\{s+(\delta_0-\delta)\cos\theta\}]}dRdsd\psi d\alpha \tag{3.7}$$

where primed quantities refer to values at a point different from that for unprimed quantities. The limits $-2\theta_0$ to $2\theta_0$ on the variable ψ are used to cover the entire 360° of the azimuth angle.

It is apparent from the statistical studies of contour maps that the probability density function for terrain elevation above the mean (say δ) can be written as

$$p(\delta)=\frac{1}{(2\pi\sigma^2)^{1/2}}e^{-\delta^2/(2\sigma^2)}, \tag{3.8}$$

The conditional (normal) density function for the elevations δ given δ_0 at two points on the ground, a distance r apart, is given by

$$p(\delta;\delta_0,r)=\frac{1}{[2\pi\sigma^2(1-\rho^2)]^{1/2}}e^{-\frac{(\delta-\delta_0\rho)^2}{2\sigma^2(1-\rho^2)}}. \tag{3.9}$$

The expression $\exp[j2k\{s+(\delta_0-\delta)\cos\theta\}]$ in (3.7) is first averaged over δ using (3.8) and the result is

$$\exp[j2ks-a^2k^2\cos^2\theta+j2k\delta_0(1-\rho)\cos\theta], \tag{3.10}$$

where

$$a^2=2\sigma^2(1-\rho^2).$$

Now (3.10) is averaged over δ_0 using (3.9), and it becomes,

$$\exp[j2ks-4k^2\sigma^2(1-\rho)\cos^2\theta]. \tag{3.11}$$

One can deduce from Davies' [1954] approximation $r^2\approx R^2\alpha^2+s^2\operatorname{cosec}^2\theta$ (see fig. 2) that

$$r\approx R\alpha+\frac{s^2\operatorname{cosec}^2\theta}{2R\alpha}. \tag{3.12}$$

It can be now shown that the integration of (3.7) after substituting (3.11), (3.12), and (2.2) results in an expression, which when rearranged, becomes

$$\overline{P}_r=\frac{\lambda^2}{32\pi^2}\int\left(\frac{P_TG^2}{R3}\right)\left[\frac{4\sqrt{2}\pi}{B\lambda^2}\left(\frac{\theta}{\sin\theta}\right)\right.$$

$$\left.e^{-4k^2\sigma^2\cos^2\theta}\sum_{n=1}^{\infty}\frac{(4k^2\sigma^2)^n(\cos^2\theta)^{n+1}}{(n-1)!\left[2k^2\sin^2\theta+\frac{n^2}{B^2}\right]^{3/2}}\right]dR. \tag{3.13}$$

Here it has again been assumed that R' and θ' are approximately equal to R and θ respectively, for near-vertical incidence. From the comparison of (3.1) with (3.13) it is clear that the scattering coefficient, σ_0, is given by

$$\sigma_0=\frac{4\sqrt{2}\pi B^2}{\lambda^2}\left(\frac{\theta}{\sin\theta}\right)$$

$$e^{-4k^2\sigma^2\cos^2\theta}\sum_{n=1}^{\infty}\frac{(4k^2\sigma^2)^n(\cos^2\theta)^{n+1}}{(n-1)![2k^2B^2\sin^2\theta+n^2]^{3/2}}. \tag{3.14}$$

The value of $1/B$ is far less than k for *nearly smooth* surfaces, and hence the scattering coefficient for such surfaces can be approximated from (3.14) as

$$\sigma_0 \text{ (for nearly smooth surfaces)} \approx \frac{4\sigma^2}{B\lambda} (\theta \cot^4 \theta) \quad (3.15)$$

for $\theta \neq 0°$.

4. Experimental Verification

For *nearly* smooth surfaces, the surface characteristic constant $1/B \approx 0$, and (3.15) gives the scattering coefficient. This result compares very closely with published results [Nielson, 1960] for new ice as shown in table 2.

For rough (not *nearly* smooth) surfaces, (3.14) describes the relationship of the scattering coefficient σ_0 and other variables such as the angle of incidence θ, wavelength λ, standard deviation σ and surface covariance constant B, etc. Two curves of the scattering coefficient σ_0 versus θ for each of the three values of λ/B, 0.1, 0.5, and 1.0 for σ/λ equal to 0.05, and 0.1 are shown in figure 4. It may be noticed that as the surface becomes rougher, or as λ/B increases for a specified λ, the scattering coefficient curve becomes flatter, showing the relative importance of the contribution of the power return from the surface at angles other than those near zero. As expected, when the surface becomes smoother or $1/B$ decreases, the received power seems to come primarily from near-zero angles. These curves are quite similar to those recently published [Campbell, 1959; Dye, 1959; Edison, 1960]. The experimental data [Nielson, 1960] on desert and new ice also seems to follow the pattern of these theoretical curves described above.

TABLE 2. *Comparison of theoretical versus experimental scattering coefficient*
(Normalized)

θ^0	σ_0 Theoretical	σ_0 Experimental
30	1.000	1.000
40	0.291	0.308
50	.088	.089
60	.022	.021
70	.004	.004

Similar results [2] obtained by an acoustic simulator at the University of New Mexico also verify these theoretical conclusions. It is interesting to note that an experimental expression ($\sigma_0 = \sigma_1 e^{-10\theta}$ for θ in radians) for the scattering coefficient [Hughes, 1960] of the Moon, verified by photographic astronomical [Briggs, 1960] calculations, is a very close fit to the graph of eq (3.14) for $\sigma/\lambda = 0.1$ and $\lambda/B = 1.0$ in the range of incidence angles of 3 to 14 deg. The authors believe that other values of B and σ may well be appropriate, as only a limited set has been tried to date.

[2] A. R. Edison and R. K. Moore, Preliminary report on an acoustic simulator for investigation of backscatter of E.M. waves, unpublished report of the University of New Mexico, Albuquerque, N. Mex. (1961).

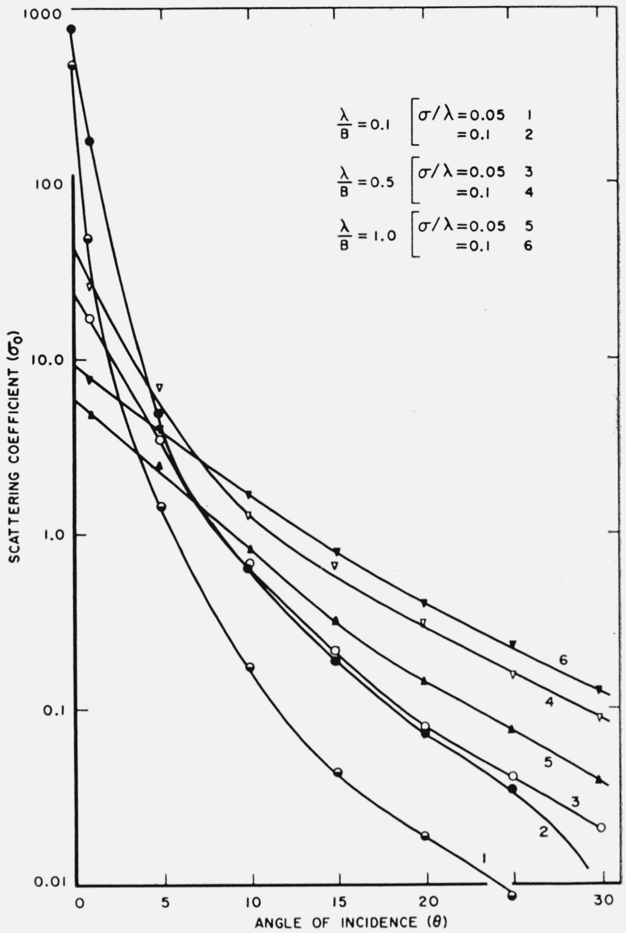

FIGURE 4. *Scattering coefficient versus the angle of incidence.*

5. Conclusions

The scattering coefficient (σ_0) for nearly smooth surfaces is inversely proportional to the wavelength, but varies directly with (σ^2), ($\theta \cot^4 \theta$) and $1/B$, where σ, θ, B are standard deviation, angle of incidence, and the terrain characteristic constant respectively. For rough surfaces it has a negative exponential factor, where the exponent is made up of $\frac{\sigma^2 \cos^2 \theta}{\lambda^2}$ times a constant. The surface characteristic constants B and σ can be calculated from the radar return data. Although approximate, the theoretical results agree well with the experimental data; and therefore, suggest the usefulness of the approach. The application of these results may be extended to the moon-echo data, with proper corrections for Faraday and liberation effects, etc. This investigation has established that for near-vertical incidence, the normalized autocovariance for the terrain elevation is more often of the exponential form exp $(-|r|/B)$ rather than the Gaussian form, exp $(-r^2/B)$. The former may well be more appropriate for finer terrain irregularities than those considered in this paper. It may also be representative of the normalized autocovariance of the moon surface. An exact theoretical, but usable, expression for the scattering coefficient has been very reasonably approximated by the results of this study.

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(Paper 65D5-147)