JOURNAL OF RESEARCH of the National Bureau of Standards—D. Radio Propagation
Vol. 65D, No. 4, July–August 1961

Effect of Multiple Atmospheric Inversions on Tropospheric Radio Propagation [1]

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(October 14, 1960; revised January 23, 1941)

Of the various mechanisms put forward in recent years to explain long range tropospheric radio propagation the most important seem to be scattering from atmospheric turbulence and partial reflexion from high level subsidence inversion layers. In this paper the writer extends earlier theory appropriate to the case of a single elevated inversion layer to cover the case of multiple layers. In some cases it is found that the theory and formulas can be applied, with very little modification, to the latter. The special cases in which this can be done are worked out in detail and it is found that, if certain conditions are satisfied, several weak high level inversions can produce a similar effect on the propagation to a single strong inversion.

1. Introduction

Of the various mechanisms put forward in recent years to explain long range tropospheric propagation the most important seem to be scattering from atmospheric turbulence and partial reflection from high level subsidence inversions. In 1952 the writer, after first reviewing earlier experimental work, showed theoretically that a well developed inversion could cause the phenomenon. In a three part paper in 1955, after first reviewing the major theories which had been put forward in the interim, he considered in greater detail the special characteristics of the field under a high level subsidence inversion (part II). It is now fairly generally agreed that the major mechanisms responsible for the long distance fields in tropospheric propagation are scattering from atmospheric turbulence and partial reflection from high level subsidence inversions.

Refractometer measurements have shown that several minor high level inversions can coexist at different levels and that this situation is probably present for most of the time. It would therefore seem desirable to make the previous investigations complete by extending the theory to take account of the case when several such inversions are present. This is what will be attempted in the present paper.

2. Propagation Under Several High Level Inversions

It is possible to show [Northover, 1952] that, provided the distance of the field point from the exciting source is small compared with the radius of the earth, the Hertzian function Π representing electromagnetic disturbances excited by a horizontal electric dipole (or system of dipoles) in a stratified atmosphere outside a spherical earth is expressible in the form

$$\frac{1}{r}\sum_{(n)}\Omega_n(kr)\,P_n(\cos\theta) \tag{1}$$

where $\Omega_n(kr)$ is a solution of the equation

$$\Omega_n^{||}(\xi)+\left\{\kappa(\xi)-\frac{n(n+1)}{\xi^2}\right\}\Omega_n(\xi)=0, \qquad \xi=2\pi r/\lambda \tag{2}$$

[1] Contribution from Department of Mathematics, Carleton University, Ottawa, Canada.

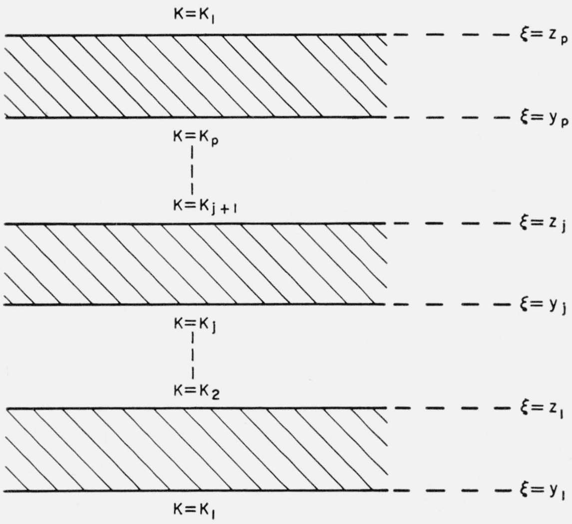
and that this representation is valid whether the dielectric constant κ or the refractive index μ is a single analytic function of height (as for a surface inversion) or has different forms for different regions (as when high level inversions are present). Further, it appears that, subject to the same restriction on range, the condition at the surface of a perfectly conducting earth can be taken as $\Pi=0$.

Consider the case when several high level inversions are present. Let there be p of them and let the boundaries of the jth be given by $\xi=y_j$, $\xi=z_j(z_j>y_j)$; let also

$$z_0(=x)<y_1<z_1<y_2<z_2 \ldots <y_p<z_p< \ldots \tag{3}$$

Let the dielectric constant below the lowest[2] be κ_1 and above the highest be unity; let also $\kappa=\kappa_2$ in $z_1<\xi<y_2$, $\kappa=\kappa_3$ in $z_2<\xi<y_3$ . . . $\kappa=\kappa_j$ in $z_{j-1}<\xi<y_j$ and $\kappa=\kappa_p$ in $z_{p-1}<\xi<y_p$.

These values of κ refer to the space between the layers; throughout a given layer κ is supposed to change smoothly, e.g., through the second layer κ changes smoothly from κ_2 at the bottom ($\xi=y_2$) to κ_3 at the top ($\xi=z_2$). See figure.



Remembering (1) we assume that the secondary field below all inversions (i.e., in the space $x<\xi<y_1$) is of the form

$$-\frac{i\beta}{rb_1}\Sigma(2n+1)\{F_n\zeta_n(\xi\sqrt{\kappa_1})+G_n\eta_n(\xi\sqrt{\kappa_1})\}P_n(\cos\theta) \tag{4}$$

where

$$\zeta_n(x)=\left(\frac{1}{2}\pi x\right)^{1/2}H^{(2)}_{n+1/2}(x);\ \eta_n(x)=\left(\frac{1}{2}\pi x\right)^{1/2}H^{(1)}_{n+1/2}(x) \tag{4A}$$

and we are assuming the time factor $e^{i\omega t}$, and the primary contribution from the dipole is

$$\left.\begin{aligned}&-\frac{i\beta}{rb_1}\sum(2n+1)\zeta_n(b_1)\psi_n(\xi\sqrt{\kappa_1})P_n(\cos\theta),\text{ when }\xi\sqrt{\kappa_1}<b_1\\ &\text{and}\\ &-\frac{i\beta}{rb_1}\sum(2n+1)\psi_n(b_1)\zeta_n(\xi\sqrt{\kappa_1})P_n(\cos\theta),\text{ when }\xi\sqrt{\kappa_1}>b_1\end{aligned}\right\} \tag{5}$$

where

$$b_1=2\pi(R+H)\sqrt{\kappa_1}/\lambda;\ 2\psi_n(x)=\zeta_n(x)+\eta_n(x). \tag{5A}$$

[2] The normal slow lapse rate of μ will be approximately allowed for by taking an "effective" value for the earth's radius. This is a reasonable procedure if the inversions are not too high [cf Norton, 1959; Bremmer, 1960; Wait, 1959].

The constant β is proportional to the square root of the power radiated: (see reference 1, page 111), R and H are, respectively, the earth's radius and antenna height.

Let the total field in the space $z_{j-1}<\xi<y_j$, i.e., the region where $\kappa=\kappa_j$ be

$$-\frac{i\beta}{rb_1}\sum(2n+1)\{A_n^{(j)}\zeta_n(\xi\sqrt{\kappa_j})+B_n^{(j)}\eta_n(\xi\sqrt{\kappa_j})\}P_n(\cos\theta)\ldots \tag{6}$$

and within the jth inversion layer $(y_j<\xi<z_j)$ let it be

$$-\frac{i\beta}{rb_1}\sum(2n+1)\{C_n^{(j)}u_n^{(j)}(\xi)+D_n^{(j)}v_n^{(j)}(\xi)\}P_n(\cos\theta) \tag{7}$$

where $u_n^{(j)}(\xi)$, $v_n^{(j)}(\xi)$ are any pair of fundamental solutions of (2) within the jth layer $(y_j<\xi<z_j)$.

The field above the complete set of inversion layers is of the form

$$-\frac{i\beta}{rb_1}\sum(2n+1)E_n\zeta_n(\xi)P_n(\cos\theta)\cdot \tag{8}$$

It will be useful to note here that, in view of the continuity of the rising and descending field contributions at the lower boundary of the first layer, we have the formulas

$$\begin{aligned}A_n^{(1)}&=F_n+\psi_n(b_1)\\ B_n^{(1)}&=G_n\end{aligned} \tag{9}$$

for $b_1<\xi<y_1$, and also, that

$$A_n^{(p+1)}=E_n;\qquad B_n^{(p+1)}=0\ \ldots \tag{10}$$

in view of the radiation condition at infinity.

3. Boundary Conditions

These conditions, expressing the continuity of the field and of its normal derivative at the interfaces of the successive inversion layers, read as follows:

At $\xi=z_{j-1}$

$$\left.\begin{aligned}A_n^{(j)}\zeta_n(z_{j-1}\sqrt{\kappa_j})+B_n^{(j)}\eta_n(z_{j-1}\sqrt{\kappa_j})&=C_n^{(j-1)}u_n^{(j-1)}(z_{j-1})+D_n^{(j-1)}v_n^{(j-1)}(z_{j-1})\\ \sqrt{\kappa_j}\{A_n^{(j)}\zeta_n'(z_{j-1}\sqrt{\kappa_j})+B_n^{(j)}\eta_n(z_{j-1}\sqrt{\kappa_j})\}&=C_n^{(j-1)}\{u_n^{(j-1)}(z_{j-1})\}'+D_n\{v_n^{(j-1)}(z_{j-1})\}'\ldots,\end{aligned}\right\} \tag{11}$$

where $j=2, 3, \ldots, p$.

At $\xi=y_{j-1}$

$$\left.\begin{aligned}C_n^{(j-1)}u_n^{(j-1)}(y_{j-1})+D_n^{(j-1)}v_n^{(j-1)}(y_{j-1})&=A_n^{(j-1)}\zeta_n(y_{j-1}\sqrt{\kappa_{j-1}})+B_n^{(j-1)}\eta_n(y_{j-1}\sqrt{\kappa_{j-1}}),\\ C_n^{(j-1)}\{u_n^{j(-1)}(y_{j-1})\}'+D_n^{(j-1)}\{v_n^{(j-1)}(y_{j-1})\}'&=\sqrt{\kappa_{j-1}}\{A_n^{(j-1)}\zeta_n'(y_{j-1}\sqrt{\kappa_{j-1}})\\ &\quad+B_n^{(j-1)}\eta_n'(y_{j-1}\sqrt{\kappa_{j-1}})\}\ldots.\end{aligned}\right\} \tag{12}$$

where $j=3, 4, \ldots, p+1$.

If we let

$$B_n^{(j)}/A_n^{(j)}=-H_j\ \ldots \tag{13}$$

we find, after somewhat involved algebra

$$H_{j-1}=\frac{Q_{j-1}-H_j\bar{P}_{j-1}}{P_{j-1}-H_j\bar{Q}_{j-1}}\qquad(2\leq j\leq p+1) \tag{14}$$

where

$$Q_{j-1}=N^{(j-1)}\zeta_n(y_{j-1}\sqrt{\kappa_{j-1}})\zeta_n(z_{j-1}\sqrt{\kappa_j})+L^{(j-1)}\zeta_n'(y_{j-1}\sqrt{\kappa_{j-1}})\zeta_n'(z_{j-1}\sqrt{\kappa_j})$$

$$-M_1^{(j-1)}\zeta_n'(y_{j-1}\sqrt{\kappa_{j-1}})\zeta_n(z_{j-1}\sqrt{\kappa_j})-M_2^{(j-1)}\zeta_n(y_{j-1}\sqrt{\kappa_{j-1}})\zeta_n'(z_{j-1}\sqrt{\kappa_j})$$

$$P_{j-1}=N^{(j-1)}\eta_n(y_{j-1}\sqrt{\kappa_{j-1}})\zeta_n(z_{j-1}\sqrt{\kappa_j})+L^{(j-1)}\eta_n'(y_{j-1}\sqrt{\kappa_{j-1}})\zeta_n'(z_{j-1}\sqrt{\kappa_j})$$

$$-M_1^{(j-1)}\eta_n'(y_{j-1}\sqrt{\kappa_{j-1}})\zeta_n(z_{j-1}\sqrt{\kappa_j})-M_2^{(j-1)}\eta_n(y_{j-1}\sqrt{\kappa_{j-1}})\zeta_n'(z_{j-1}\sqrt{\kappa_j}) \tag{14a}$$

the bar means that we interchange the ζ and η functionality symbols in these formulas, and

$$L^{(j)}=\sqrt{\kappa_{j+1}}[u_n^{(j)}(z_j)v_n^{(j)}(y_j)-u_n^{(j)}(y_j)v_n^{(j)}(z_j)]$$

$$M_1^{(j)}=\{u_n^{(j)}(z_j)\}'v_n^{(j)}(y_j)-u_n^{(j)}(y_j)\{v_n^{(j)}(z_j)\}' \tag{14b}$$

$$M_2^{(j)}=\sqrt{\frac{\kappa_{j+1}}{\kappa_j}}\{u_n^{(j)}(z_j)\{v_n^{(j)}(y_j)\}'-\{u_n^{(j)}(y_j)\}'v_n^{(j)}(z_j)\}$$

$$N^{(j)}=\frac{1}{\sqrt{\kappa_j}}[\{u_n^{(j)}(z_j)\}'\{v_n{}^{(j)}(y_j)\}'-\{u_n{}^{(j)}(y_j)\}'\{v_n{}^{(j)}(z_j)\}']. \tag{14B}$$

In cases where the earth behaves effectively as a perfect conductor the surface condition is, as already mentioned, $\Pi=0$, by (4) and (5) this amounts to

$$\left\{F_n+\frac{1}{2}\zeta_n(b_1)\right\}\zeta_n(x\sqrt{\kappa_1})+\left\{G_n+\frac{1}{2}\zeta_n(b_1)\right\}\eta_n(x\sqrt{\kappa_1})=0. \tag{15}$$

Hence, by some rather involved algebra, it is possible to derive an expression for the total Hertzian function at points below the lowest inversion similar in form to that developed in the writer's 1952 paper, namely,

$$\Pi=-\frac{i\beta}{2rb_1}\sum_0^\infty(2n+1)\frac{\zeta_n(b_1)-H_1\eta(b_1)}{\zeta_n(x_1)-H_1\eta_n(x_1)}\{\zeta_n(x_1)\eta_n(\xi_1)-\eta_n(x_1)\zeta_n(\xi_1)\}P_n(\cos\theta) \tag{16}$$

in which

$$\xi_1=\xi\sqrt{\kappa_1};\ x_1=x\sqrt{\kappa_1};\ x=2\pi R/\lambda \tag{16a}$$

and H_1 is the value of H_j when $j=1$: it is expressible in terms of F_n and G_n by (9).

We have now arrived at the mathematical solution of the problem, for, since H_1 can now be determined (see the argument immediately following), then (9) and (15) now amount to two simultaneous linear equations for F_n and G_n. Determination of H_1:
We have, from (10)

$$H_{p+1}=\frac{B_n^{(p+1)}}{A_n^{(p+1)}}$$

$$=0\text{ by (10)} \tag{17}$$

we have

$$H_p=Q_p/P_p \tag{17a}$$

and then all the H's from H_{p-1} down to H_1 are given in succession by (14).

4. Approximate Evaluation of the Field

The field series (16) cannot, as it stands, be used to discuss the behavior of the field below the inversions because of its slowness of convergence. To make further progress, it is necessary to transform it into a more rapidly convergent series (series of residues) by complex contour integration (Watson's transformation), as was done in the writer's 1952 paper on the single high level inversion. As in that paper, the contribution of the imaginary axis to the complex contour integral involved is zero, and it appears that the series of residues giving the field is of the same general form as that obtained therein. It is as follows:

$$\Pi=\frac{\pi i\beta}{rb}\sum_{j=1}^{\infty}\nu_j\frac{\zeta_{\nu_j-1/2}(b)\eta_{\nu_j-1/2}(x_1)-\zeta_{\nu_j-1/2}(x_1)\eta_{\nu_j-1/2}(b)}{2i-\{\eta_{\nu_j-1/2}(x_1)\}^2[\partial H_1/\partial s]_{(s=\nu_j)}}$$
$$\times\{\zeta_{\nu_j-1/2}(x_1)\eta_{\nu_j-1/2}(\xi_1)-\eta_{\nu_j-1/2}(x_1)\zeta_{\nu_j-1/2}(\xi_1)\}\sec\nu_j\pi P_{\nu_j-1/2}(-\cos\theta) \quad (18)$$

where the summation is over the roots ν_j of the equation

$$\zeta_{s-1/2}(x_1)/\eta_{s-1/2}(x_1)=H_1 \quad (19)$$

considered as an equation in s.

5. Thin Layers—Meter Wave Propagation

It can be shown that, as in the case of a single inversion layer, the high level inversions have their maximum effect upon the propagation when they are sufficiently thin to act effectively as dielectric discontinuities. The condition that this may be so is that $I\sqrt{2p}$ must be small compared with unity for each inversion, where $I=2\pi\tau/\lambda$, $p=h/R$ being the layer thickness and h the inversion height. We shall, therefore, confine ourselves to a discussion of this case only.

When the layer thickness may be neglected, the preceding results simplify; we have

$$L^{(j-1)}=0;\ N^{(j-1)}=0$$

$$M_1^{(j-1)};M_2^{(j-1)}=-\sqrt{\kappa_{j-1}}:\sqrt{\kappa_j} \quad (20)$$

and since only the mutual ratios of $L^{(j-1)}$, $M_1^{(j-1)}$, $M_2^{(j-1)}$ and $N^{(j-1)}$ are significant, we may take

$$Q_{j-1}=\sqrt{\kappa_{j-1}}\,\zeta'_{\nu-1/2}\left(y_{j-1}\sqrt{\kappa_{j-1}}\right)\zeta_{\nu-1/2}\left(z_{j-1}\sqrt{\kappa_j}\right)-\sqrt{\kappa_j}\,\zeta_{\nu-1/2}\left(y_{j-1}\sqrt{\kappa_{j-1}}\right)\zeta'_{\nu-1/2}\left(z_{j-1}\sqrt{\kappa_j}\right)$$

$$P_{j-1}=\sqrt{\kappa_{j-1}}\,\eta'_{\nu-1/2}\left(y_{j-1}\sqrt{\kappa_{j-1}}\right)\zeta_{\nu-1/2}\left(z_{j-1}\sqrt{\kappa_j}\right)-\sqrt{\kappa_j}\,\eta_{\nu-1/2}\left(y_{j-1}\sqrt{\kappa_{j-1}}\right)\zeta'_{\nu-1/2}\left(z_{j-1}\sqrt{\kappa_j}\right)$$

$$\overline{Q}_{j-1}=\sqrt{\kappa_{j-1}}\,\eta'_{\nu-1/2}\left(y_{j-1}\sqrt{\kappa_{j-1}}\right)\eta_{\nu-1/2}\left(z_{j-1}\sqrt{\kappa_j}\right)-\sqrt{\kappa_j}\,\eta_{\nu-1/2}\left(y_{j-1}\sqrt{\kappa_{j-1}}\right)\eta'_{\nu-1/2}\left(z_{j-1}\sqrt{\kappa_j}\right)$$

$$\overline{P}_{j-1}=\sqrt{\kappa_{j-1}}\,\zeta'_{\nu-1/2}\left(y_{j-1}\sqrt{\kappa_{j-1}}\right)\eta_{\nu-1/2}\left(z_{j-1}\sqrt{\kappa_j}\right)-\sqrt{\kappa_j}\,\zeta_{\nu-1/2}\left(y_{j-1}\sqrt{\kappa_{j-1}}\right)\eta'_{\nu-1/2}\left(z_{j-1}\sqrt{\kappa_j}\right) \quad (21)$$

It is possible to make further progress only when the heights of the high level inversions are large compared with $R^{1/3}\lambda^{2/3}$. The inversions are then to be supposed as situated, above a well-known critical height which is of great importance in all diffraction theories for propagation along a sphere; asymptotic approximations may be applied there to all height gain functions. Fortunately, this condition is usually satisfied in practice, even for meter length VHF waves. It is then possible to show that the only roots ν which could possibly give important contributions to the field series (18) by reason of smallness of $-\text{Im}\ (\nu/x^{1/3})$ fall into the following types:

(A) those near [3] $\xi=x$

(B) those near $\xi=y_j\ (j=1, 2, \ldots, p)$

(A) corresponds to waves nearly grazing the horizontal level at the receiver and (B) to waves nearly grazing the lower boundaries of the successive inversion layers.

It can further be shown that the contribution of roots of the group (B) to the field series is unimportant compared with that of group (A). Although limitations of space forbid the

[3] By "ξ near x" we mean $|\xi-x|=O(x^{1/3})$.

inclusion here of the exact analysis, it may be remarked that the reason for this is to be found, not in any wide discrepancies in the magnitudes of their imaginary parts, but in the fact that for ν in group (B), $\{\eta_{\nu-1/2}(x_1)\}^2\ (\partial H_1/\partial s)_{(s=\nu)}$ becomes large of order $\mathrm{Exp}\left\{\frac{4}{3}\sqrt{2}x\left(\epsilon-\frac{\partial_0}{2}\right)^{3/2}\right\}$ i.e., of order $\mathrm{Exp}\left(\frac{4}{3}\sqrt{2}xp_j^{3/2}\right)$

where

$p_j=(y_j-x)/x$, by virtue of the factor $\{\eta_{\nu-1/2}(x_1)\}^2$ {we have written $\epsilon=(\nu-x)/x$ in the above}.

This makes the leading terms of the series (18) which correspond to values of ν in this group exponentially small.

We shall therefore confine ourselves to consideration of values of ν for which $|\nu-x_1|$ is $O(x^{1/3})$.[4] For such values, we may apply well known asymptotic properties of Bessel functions (cf Watson's treatise or the writer's previous paper, 1) and we obtain, from (21)[5]

$$\kappa_j^{-1/2}Q_{j-1}\doteq i\left[\{2(p_{j-1}-\epsilon)+\partial_j\}^{1/2}-\left(\frac{\kappa_{j-1}}{\kappa_j}\right)^{1/2}\{2(p_{j-1}-\epsilon)+\partial_{j-1}\}^{1/2}\right]\cdot\frac{i}{\{2(p_{j-1}-\epsilon)\}^{1/2}}\mathrm{Exp}\left\{-\frac{4}{3}\sqrt{2}x(p_{j-1}-\epsilon)^{3/2}i\right\}$$

$$\kappa_j^{-1/2}P_{j-1}\doteq i\left[\{2(p_{j-1}-\epsilon)+\partial_j\}^{1/2}+\left(\frac{\kappa_{j-1}}{\kappa_j}\right)^{1/2}\{2(p_{j-1}-\epsilon)+\partial_{j-1}\}^{1/2}\right]\cdot\frac{1}{\{2(p_{j-1}-\epsilon)\}^{1/2}}$$

$$\kappa_j^{-1/2}\overline{Q}_{j-1}\doteq -i\left[\{2(p_{j-1}-\epsilon)+\partial_j\}^{1/2}-\left(\frac{\kappa_{j-1}}{\kappa_j}\right)^{1/2}\{2(p_{j-1}-\epsilon)+\partial_{j-1}\}^{1/2}\right]\cdot\frac{-i}{\{2(p_{j-1}-\epsilon)\}^{1/2}}\mathrm{Exp}\left\{\frac{4}{3}\sqrt{2}x(p_{j-1}-\epsilon)^{3/2}i\right\}$$

$$\kappa_j^{-1/2}\overline{P}_{j-1}=-i\left[\{2(p_{j-1}-\epsilon)+\partial_j\}^{1/2}+\left(\frac{\kappa_{j-1}}{\kappa_j}\right)^{1/2}\{2(p_{j-1}-\epsilon)+\partial_{j-1}\}^{1/2}\right]\cdot\frac{1}{\{2(p_{j-1}-\epsilon)\}^{1/2}} \qquad (22)$$

where

$$k_j=1+\partial_j,\ \text{etc.} \qquad (22A)$$

Using these approximations in (14), we obtain, since ν/x may be taken as effectively unity

$$H_{j-1}=\frac{\left[\begin{array}{l} i\left[\{2(p_{j-1}-\epsilon)+\partial_j\}^{1/2}-\{2(p_{j-1}-\epsilon)+\partial_{j-1}\}^{1/2}\right]\mathrm{Exp}\left\{-\frac{4}{3}\sqrt{2}xi(p_{j-1}-\epsilon)^{3/2}\right\}\\ +H_j\left[\{2(p_{j-1}-\epsilon)+\partial_j\}^{1/2}+\{2(p_{j-1}-\epsilon)+\partial_{j-1}\}^{1/2}\right]\end{array}\right]}{\left[\begin{array}{l}\left[\{2(p_{j-1}-\epsilon)+\partial_j\}^{1/2}+\{2(p_{j-1}-\epsilon)+\partial_{j-1}\}^{1/2}\right]-iH_j\left[\{2(p_{j-1}-\epsilon)+\partial_j\}^{1/2}\right.\\ \left.-\{2(p_{j-1}-\epsilon)+\partial_{j-1}\}^{1/2}\right]\mathrm{Exp}\left\{\frac{4}{3}\sqrt{2}xi(p_j-\epsilon)^{3/2}\right\}\end{array}\right]} \qquad (23)$$

It transpires that Im (ϵ) {which is always negative, so that $p_{j-1}-\epsilon$ is situated in the first quadrant} is never very small compared with $x^{-2/3}$, but $p_j\gg x^{-2/3}$ and hence

$$\left|\mathrm{Exp}\left\{-\frac{4}{3}\sqrt{2}xi(p_{j-1}-\epsilon)^{3/2}\right\}\right|\gg 1;\ \left|\mathrm{Exp}\ \frac{4}{3}\sqrt{2}xi(p_{j-1}-\epsilon)^{3/2}\right|\ll 1.$$

Again, it appears that H_j is never large compared with unity. The denominator may then be effectively replaced by

$$\{2(p_{j-1}-\epsilon)+\partial_j\}^{1/2}+\{2(p_{j-1}-\epsilon)+\partial_{j-1}\}^{1/2}$$

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[4] See also note 2.

[5] Since $xp_j^{3/2}>>1$. See note 3.

and then (22) and (23) give

$$H_{j-1}=k_{j-1}+H_j \tag{24}$$

which yields at once

$$H_1=k_1+k_2+\ldots+k_p \tag{25}$$

where

$$k_j=i\,\frac{\{2(p_j-\epsilon)+\eth_{j+1}\}^{1/2}-\{2(p_j-\epsilon)+\eth_j\}^{1/2}}{\{2(p_j-\epsilon)+\eth_{j+1}\}^{1/2}+\{2(p_j-\epsilon)+\eth_j\}^{1/2}}\,\mathrm{Exp}\left\{-\frac{4}{3}\sqrt{2}\,xi(p_j-\epsilon(^{3/2}\right\}$$

$$\doteq Q_j/P_j,\text{ by (22).} \tag{25a}$$

But k_j is the value which H_1 would have in the field series (18) if the jth inversion only were present. We obtain, therefore, the following important result:

Addition Theorem. "When the inversion layers are at heights large, or moderately large compared with $R^{1/3}\lambda^{2/3}$, and when also they are sufficiently thin to act effectively as dielectric discontinuities, then the value of H_1 to be used in the field series (18) and in the mode equation (19), is $\sum_1^p k_j$ where k_j is the value which H_1 would have in (18) if the jth inversion only were present."

6. Discussion of Results

Even when the conditions of the above theorem are fulfilled, the resulting equation (19) for ν is, unfortunately, still too complicated to allow of much further progress being made.

There is, however, a special case of the theorem in which the characteristic properties of the propagation can be clearly perceived. It occurs when the layers are relatively close together.[6] As far as is known, this case is likely to occur in practice. It might, therefore, be useful to conclude by considering it.

7. Case When Inversions Are Close Together

In practice the dielectric constant differences [7] $\eth_j$ are all small compared with p_j. Since by hypothesis $xp_j^{3/2}\gg 1$ and ϵ is to be $O(x^{-2/3})$, (25A) gives the following approximate formula for k_j:

$$k_j=-i\,\frac{\eth_j-\eth_{j+1}}{8(p_j-\epsilon)}\,\mathrm{Exp}\left\{-\frac{4}{3}\sqrt{2}xi(p_j-\epsilon)^{3/2}\right\} \tag{26}$$

When the inversions are close together, we may take p_j in (26) to be effectively independent of j; that is, we may take p_j equal to an "effective" value p_E. It would be appropriate to call Rp_E the "cluster height" or "group height" of the system and this height would not differ much from the heights of any of the inversions in the case we now have in mind. Thus we have, in this case

$$H_1=-\frac{i\Delta}{8(p_E-\epsilon)}\,\mathrm{Exp}\left\{-\frac{4}{3}\sqrt{2}xi(p_E-\epsilon)^{3/2}\right\} \tag{27}$$

where Δ is the total drop in dielectric constant across all the inversion layers.

Hence, in this case, the inversions behave like a single inversion layer across which the refractive index drop is equal to the algebraic sum of all the differences of refractive index across the separate inversions of the group.

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[6] I.e., their widest separation is small compared with the height of the lowest.

[7] These are twice the corresponding refractive index differences.

8. Conclusion

In the writer's 1952 paper, it was found that a sharp high level inversion whose dielectric constant difference was about 2.58×10^{-5} could cause considerable anomalous propagation of meter radio waves. Inversions of this size do not, perhaps, often occur, but under settled weather conditions it is more likely that sharp high level inversions of much smaller dielectric constant difference would occur much more frequently. If several such small inversions occurred simultaneously {e.g., four such that $\partial_j-\partial_{j+1}=0.6\times10^{-5}$ would do}, the above theory shows that the propagation would be effectively the same as that characteristic of a single large inversion. A mechanism has therefore been found which is capable of explaining the long distance fields without having to invoke the existence of a *single strong* elevated high level inversion.

NOTE 1. *The influence of scattering from atmospheric turbulence.* The present paper has shown that when several high level inversions are simultaneously present their effect on propagation can be equivalent to that of a single well-marked high level inversion aloft, and Saxton [1951] in his experimental work on 9 and 45 Mc/s waves has found that the effect of scattering is, in general, likely to be unimportant compared with reflection effects from (the equivalent of) a single inversion aloft, at any rate over the path considered by him. On the other hand, experiments elsewhere seem to indicate that scattering from turbulence can be at least as important a factor in the propagation [Northover, 1955].

The basic meteorological question is whether departures from the standard atmosphere structure, which are significant from the radio propagation point of view, occur in a stratified form (which may be viewed as anisotropic turbulence), or as isotropic turbulence. The present paper has attempted to make the theory sufficiently general to cover the most important possibilities of the first case. The best type of radio measurement for deciding between these alternatives appears to be space diversity measurement in mutually perpendicular dimensions. The difference in the angular scattering characteristics of the two models could be ascertained, in theory, from such measurements. Experimental work along these lines has been carried out and results which have been obtained so far seem to favor the stratified model.

NOTE 2. *On the location of the roots ν.* When several high level inversions are present simultaneously we have found that there is one case only which is tractable mathematically, namely, that in which the inversions behave like a single high level inversion. This, therefore, is the only case which has been worked out in detail in the text and the disposition of ν for it is known from the writer's 1952 paper dealing with a single inversion layer. It was shown therein that the significant values of ν were those "near" x_1

$$\{\text{i.e., } |\nu-x_1|=O(x^{1/3})\}.$$

NOTE 3. *Asymptotic approximations to Hankel-Nielson functions.* These are summarized on page 116 of the writer's 1952 paper. Using them we obtain, since

$$xp^{3/2}>>1;\ \partial_{j-1}<<p_{j-1};\ x\epsilon^{3/2}=0\ (1)$$

$$2^{1/4}e^{-\frac{i\pi}{4}}\zeta_{\nu-1/2}(y_{j-1}\sqrt{\kappa_{j-1}})\sim\left\{p_{j-1}-\epsilon+\frac{1}{2}\partial_{j-1}\right\}^{-1/4}\exp\left\{-\frac{2}{3}\sqrt{2}ix\left(p_{j-1}-\epsilon+\frac{1}{2}\partial_{j-1}\right)^{3/2}\right\}$$

$$2^{1/4}e^{-\frac{i\pi}{4}}\zeta'_{\nu-1/2}(y_{j-1}\sqrt{\kappa_{j-1}})\sim\left\{p_{j-1}-\epsilon+\frac{1}{2}\partial_{j-1}\right\}^{1/4}\sqrt{2}\exp\left\{-\frac{2}{3}\sqrt{2}ix\left(p_{j-1}-\epsilon+\frac{1}{2}\partial_{j-1}\right)^{3/2}\right\}$$

$$2^{1/4}e^{-\frac{i\pi}{4}}\zeta_{\nu-1/2}(z_{j-1}\sqrt{\kappa_j})\sim\left(p_{j-1}+\frac{\tau_{j-1}}{R}-\epsilon+\frac{1}{2}\partial_j\right)^{-1/4}\exp\left\{-\frac{2}{3}\sqrt{2}ix\left(p_{j-1}+\frac{\tau_{j-1}}{R}-\epsilon+\frac{1}{2}\partial_j\right)^{3/2}\right\}$$

$$2^{1/4}e^{-\frac{i\pi}{4}}\zeta'_{\nu-1/2}(z_{j-1}\sqrt{\kappa_j})\sim\left\{p_{j-1}+\frac{\tau_{j-1}}{R}-\epsilon+\frac{1}{2}\partial_j\right\}^{1/4}\sqrt{2}\exp\left\{-\frac{2}{3}\sqrt{2}ix\left(p_{j-1}+\frac{\tau_{j-1}}{R}-\epsilon+\frac{1}{2}\partial_j\right)^{3/2}\right\}$$

in which τ_j is the thickness of the jth inversion. The corresponding expressions in η, η' are obtained from the above by changing the sign of the coefficients of i in the above.

This enables us to obtain (22) from (21) neglecting the term τ_{j-1}/R and the terms ∂_{j-1}, ∂_j in comparison with p_{j-1} in the exponentials.

9. References

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(Paper 65D4–142)