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# Statistics of a Radio Wave Diffracted by a Random Ionosphere[1, 2, 3]

S. A. Bowhill

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For some purposes, particularly in connection with the study of the random structure of the lower ionosphere, using very low frequencies, it is necessary to find the detailed statistical properties of a random signal diffracting in free space. Mathematical tools for evaluating these parameters have been developed, and are applied in this paper. Allowance is made for the effect of sphericity of the wave incident on the ionosphere, and anisotropy of the irregular variations of signal is permitted. The case of oblique incidence of a wave on the ionosphere is also considered.

## 1. Introduction

There has of late been an increasing interest in the irregular structure of the lower ionosphere, particularly in connection with scatter propagation. In order to interpret irregular fluctuations in radio signals observed at the ground in terms of irregularities in the ionosphere, it is first necessary to find how much the nature of the signal changes as it propagates from the ionosphere to the earth. This propagation, which takes place in free space, will be termed "diffraction," in the sense used in a previous paper [Bowhill, 1957].

Most of the methods so far proposed for dealing with this statistical problem assume that the signal at the layer is like the signal produced when an infinite plane wave passes through a thin diffracting screen. The field at the screen is then resolved into a number of plane waves propagating in various directions, their amplitudes being given by the Fourier spectrum of the field variation. The signal at any distance from the screen is then given by the vector sum of the signals from the various plane waves, combined with the appropriate phases. Booker, Ratcliffe, and Shinn [1950] stated this principle, but Hewish [1952] was the first to show quantitatively how the magnitude of the irregular field variations changed with distance from the screen. He assumed a rectangular distribution of spatial wave numbers in the screen. Feinstein [1954] has used a rather different approach based on Huygens' principle, which has also been exploited by Chernov [1960]. Ratcliffe [1956] has summarized the physical principles involved in these approaches.

Pitteway [1958] has recently given an analytic method for calculating the field due to any particular assembly of irregularities; this, however, is difficult to apply in a statistically random medium.

It is important to distinguish at this point between the various types of random screen postulated. The two types most commonly considered are screens which modulate the phase or the amplitude of the signal. A phase screen is said to be "deep" or "shallow" depending on whether the phase excursion of the signal is substantially larger or smaller than one radian. The deep phase screen problem has been approached by Hewish [1951] by using a sinusoidal variation of phase with distance, but the extension of a detailed theory to this case presents considerable difficulties. The shallow phase screen, on the other hand, is exactly equivalent to a shallow amplitude screen [Bowhill, 1959]. This interchangeability of phase and amplitude

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is exploited in the phase-contrast microscope, in which variations in phase are changed into variations in amplitude by artificially retarding the phase of the specular component by $\pi/2$. It is well known that this only gives an improvement in visibility when the phase variations are small (i.e., the phase screen is shallow).

The results of the analysis in this paper will be applied principally to the random diffraction of very low frequency radio waves. The fading for these frequencies is usually shallow indicating that a shallowly modulated random field at the ionosphere must be responsible. Attention will therefore be confined to shallowly modulated screens.

To illustrate the method of analysis, it will first be applied, in section 2, to a one-dimensional random screen. The extension to two dimensions is made in section 3. Section 4 deals with the correlation between amplitude and phase in the signal, and section 5 with the effect of wave sphericity. The work is extended to oblique incidence in section 6.

## 2. One-Dimensional Diffraction Problem

In treating the spatial correlation function and spatial frequency spectrum of the signal variations, it is much more convenient to use purely real quantities, rather than the complex variables of amplitude and phase. This enables the use of complex Fourier transforms to relate field variations over the screen to the complex angular spectrum of plane waves which they produce.

For an amplitude screen, which alone will be considered (in view of the equivalence indicated above), the electric field $\underline{E}$ at the screen is given by

$$\underline{E}(x_0, 0) = A(x_0)$$

where $A(x_0)$ is a purely real quantity. The field at a distance $z$ from the screen is then defined to be

$$\underline{E}(x, z) = [E_1(x) + iE_2(x)] \exp(2\pi i z/\lambda) \tag{1}$$

where $E_1(x)$ and $E_2(x)$ are both purely real, and represent the phase components of $\underline{E}(x,z)$ in phase and in quadrature with the undiffracted portion of the incident plane wave. For a shallow screen, $E_1(x)$ gives the amplitude variation, and $E_2(x)$ the phase variation. In the analysis which follows, various statistical parameters of the quantities $E_1(x)$ and $E_2(x)$, which are actually observed in experiments at ground level, are evaluated in terms of the corresponding parameters of the original amplitude variation $A(x_0)$.

Using the notation of Booker, Ratcliffe, and Shinn [1950], we have for the angular spectrum at the screen, in terms of $s = \sin\theta$, where $\theta$ is the scattering angle,

$$P_0(s) = \int_{-\infty}^{\infty} A(x_0) \cdot \exp(-2\pi i s x_0/\lambda)\, dx_0 \tag{2}$$

where $\lambda$ is the wavelength. The diffracted field $\underline{E}(x,z)$ is given by the Fourier transform of this, together with the phase factor $\exp(2\pi i c z/\lambda)$, where $c = \cos\theta$.

$$\begin{aligned}\underline{E}(x,z) &= \frac{1}{\lambda}\int_{-\infty}^{\infty} \exp\{2\pi i(sx+cz)/\lambda\} ds \cdot \int_{-\infty}^{\infty} A(x_0)\cdot \exp(-2\pi i s x_0/\lambda) dx_0 \\ &= \frac{1}{\lambda}\iint_{-\infty}^{\infty} A(x_0)\cdot \exp\{2\pi i[(x-x_0)s+cz]/\lambda\} dx_0 ds. \end{aligned} \tag{3}$$

By the definition of $E_1(x)$ in equation (1),

$$\begin{aligned} E_1(x) &= \mathscr{R}\left\{\frac{1}{\lambda}\iint_{-\infty}^{\infty} A(x_0)\cdot \exp\{2\pi i[(x-x_0)s-(1-c)z]/\lambda\} dx_0 ds\right\} \\ &= \frac{1}{2\lambda}\iint_{-\infty}^{\infty} A(x_0)[\exp\{2\pi i[(x-x_0)s-(1-c)z]/\lambda\} + \exp\{-2\pi i[(x-x_0)s-(1-c)z]/\lambda\}] dx_0 ds \end{aligned}$$

since $A(x_0)$ is purely real. The spatial spectrum $P_1(\nu)$ of $E_1(x)$ is now defined by a similar expression to (2):

$$P_1(\nu)=\int_{-\infty}^{\infty} E_1(x)\cdot \exp(-2\pi i\nu x/\lambda)dx=\frac{1}{2\lambda}\iiint_{-\infty}^{\infty} A(x_0)\cdot[\exp\{2\pi i[(x-x_0)s-\nu x-z(1-c)]/\lambda\}$$
$$+\exp\{-2\pi i[(x-x_0)s+\nu x-z(1-c)]/\lambda\}]dx_0dxds.$$

Since the integrand is a well-behaved function, and all the limits of the integration are $-\infty$ to $+\infty$, the integration can be carried out in any order. In fact, it is convenient to integrate with respect to first $s$ and then $x$. With the approximation (valid for small $s$) that $c=1-s^2/2$, $P_1(\nu)$ is easily evaluated as

$$P_1(\nu)=\frac{1}{2}\int_{-\infty}^{\infty} A(x_0)[\exp\{2\pi i(-\nu x_0+z\nu^2/2)/\lambda\}+\exp\{2\pi i(-\nu x_0-z\nu^2/2)/\lambda\}]dx_0$$
$$=\cos(\pi z\nu^2/\lambda)\cdot\int_{-\infty}^{\infty} A(x_0)\cdot\exp(-2\pi i\nu x_0/\lambda)dx_0$$

or

$$P_1(\nu)=P_0(\nu)\cdot\cos(\pi z\nu^2/\lambda) \tag{4}$$

from equation (2). This equation gives the spatial spectrum of $E_1(x)$ at any distance $z$ from the screen. The *power* spectrum $W_1(\nu)$ of this variation is defined by

$$W_1(\nu)=\overline{P_1(\nu)\cdot P_1^*(\nu)} \tag{5}$$

the asterisk denoting the complex conjugate, and the bar denoting averaging of the result "over systems"—i.e., the mean for a number of statistically similar screens. This quantity is essentially real, and is a relatively smoothly varying function of $\nu$, because of the process of averaging. It can be related by equation (4) to the spatial power spectrum $W_0(\nu)$ at the screen:

$$W_1(\nu)=\overline{P_0(\nu)\cdot P_0^*(\nu)\cdot\cos^2(\pi z\nu^2/\lambda)}$$

or

$$W_1(\nu)=W_0(\nu)\cdot\cos^2(\pi z\nu^2/\lambda) \tag{6}$$

where

$$W_0(\nu)=\overline{P_0(\nu)\cdot P_0^*(\nu)}.$$

Similar quantities $P_2(\nu)$ and $W_2(\nu)$ can be defined for the quadrature component $E_2$ of the field (see equation (5)); a similar analysis to the preceding gives

$$\left.\begin{aligned}P_2(\nu)&=P_0(\nu)\cdot\sin(\pi z\nu^2/\lambda)\\ W_2(\nu)&=W_0(\nu)\cdot\sin^2(\pi z\nu^2/\lambda)\end{aligned}\right\} \tag{7}$$

Equations (6) and (7) enable the spatial correlograms of $E_1(x)$ and $E_2(x)$ in the diffraction pattern to be determined. Assuming that the correlogram of $A(x)$ is given, the Wiener-Khintchine theorem can be used to give $W_0(\nu)$. The power spectrum $W_1(\nu)$ is then calculated from equation (20), and the correlogram of $E_1(x)$ found by the inverse Wiener-Khintchine theorem.

This one-dimensional analysis, in which the random variations in signal are taken to occur in the $x$-direction only, represents an unrealistic physical model. However, as some existing work [Hewish, 1951, 1952; Jones, Millman, and Nertney, 1953] has assumed this type of variation, some results worked out for this case will be quoted in the next section, for comparison with the two-dimensional results.

## 3. Two-Dimensional Diffraction Problem

In this section the diffraction pattern of a two-dimensional random screen is considered, using an analysis closely similar to that employed in the previous section for the one-dimensional case. The modulation is once again assumed to be in amplitude only, as there is again an equivalence between shallow phase and amplitude screens. The effect of anisotropy of the screen will be taken into consideration by defining a spatial power spectrum $W_0(\nu,\chi)$, related to a two-dimensional spatial correlogram $P_0(\alpha,\beta)$ of the field $A(x,y)$ at the screen by a two-dimensional form of the Wiener-Khintchine thereom. As the field $A(x_0,y_0)$ is purely real, the correlogram $\rho_0(\alpha,\beta)$ is given by

$$\rho_0(\alpha,\beta)=\frac{\overline{A(x,y)\,A(x+\alpha,\,y+\beta)}-\overline{A(x,y)}^2}{\overline{A^2(x,y)}-\overline{A(x,y)}^2}$$

The field $\overline{A(x,y)}$, the mean field over the whole screen, has a constant phase at all points since $A(x,y)$ is real, and propagates as a single plane wave. It is convenient to imagine it removed; $A(x,y)$ then has both positive and negative values, and will be defined to have a mean square value of unity; so

$$\left.\begin{aligned}\overline{A(x_0,y_0)}&=0\\ \overline{A^2(x_0,y_0)}&=1\end{aligned}\right\} \tag{8}$$

and the correlogram is given by

$$\rho_0(\alpha,\beta)=\overline{A(x_0,\,y_0)A(x_0+\alpha,\,y_0+\beta)}.$$

The single plane wave, which will be called the "specular component" (by analogy with specular reflection from a mirror, compared with irregular reflection from a rough surface), can be reintroduced when necessary into the diffraction pattern as a constant added to $E_1(x,y)$.

Proceeding exactly as in section 2, the two phase components of the diffracted field are defined by the vector field at $(x,y,z)$:

$$\underline{E}(x,y,z)=[E_1(x,y)+iE_2(x,y)]\ \exp\ (2\pi iz/\lambda). \tag{9}$$

The phases of the diffracted waves at $z$ are given by purely geometrical considerations. Let the plane of one wave intersect the $(x,z)$ plane at an angle $\sin^{-1}s$, and the $(y,z)$ plane at an angle $\sin^{-1}r$. Then the angle between the wave front and the $(x,y)$ plane is

$$\tan^{-1}\{(s^{-2}-1)^{-1}+(r^{-2}-1)^{-1}\}^{1/2}$$

and the phase of the wave at a distance $z$ from the screen is

$$2\pi iz\{1+(s^{-2}-1)^{-1}+(r^{-2}-1)^{-1}\}^{-1/2}/\lambda.$$

If $s$ and $r$ are small, this can be expanded to give, to a first order,

$$2\pi iz(s^2/2+r^2/2)/\lambda$$

The corresponding equation to (3) for $\underline{E}(x,z)$ is therefore

$$\underline{E}(x,y,z)\ \exp\ (-2\pi iz/\lambda),$$

$$=\frac{1}{2}\iiiint_{-\infty}^{\infty}A(x_0,y_0)\cdot\exp\{2\pi i[(x-x_0)s+(y-y_0)r-s^2z/2-r^2z/2]/\lambda\}\,ds\,dr\,dx_0\,dy_0 \tag{10}$$

and by similar steps to the previous analysis, it can be shown that

$$P_1(\nu,\chi)=P_0(\nu,\chi)\cdot\cos\{\pi z(\nu^2+\chi^2)/\lambda\} \tag{11}$$

where $P_0(\nu,\chi)$ is the two-dimensional spectrum of the field at the screen. Similarly

$$P_2(\nu,\chi)=P_0(\nu,\chi)\cdot\sin\{\pi z(\nu^2+\chi^2)/\lambda\}$$

and the spatial power spectra of $E_1(x,y)$ and $E_2(x,y)$ are given by

$$\left.\begin{aligned} W_1(\nu,\chi)&=W_0(\nu,\chi).\cos^2\{\pi z(\nu^2+\chi^2)/\lambda\} \\ W_2(\nu,\chi)&=W_0(\nu,\chi).\sin^2\{\pi z(\nu^2+\chi^2)/\lambda\}. \end{aligned}\right\} \tag{12}$$

These spectra are proportional to the area of the diffracting screen; this will be shown below in deriving (16). The spatial correlogram $\rho_1(\alpha,\beta)$ of $E_1(x,y)$ is given by the usual relation

$$\rho_1(\alpha,\beta)=\frac{\overline{E_1(x,y)E_1(x+a,y+\beta)}}{\overline{E_1^2(x,y)}} \tag{13}$$

since the mean value of $E_1(x,y)$ is zero. On comparing this definition with (9), it will be seen that (13) has the term $\overline{E_1^2(x,y)}$ in the denominator, whereas, from (8), $\overline{A^2(x_0,y_0)}=1$. In fact, the mean square fluctuation in $A(x_0,y_0)$ at the screen appears, at some distance away, as a fluctuation partly in $E_2(x,y)$. It is easily shown from (12) that

$$\overline{E_1^2(x,y)}+\overline{E_2^2(x,y)}=\overline{A^2(x_0,y_0)}=1, \tag{14}$$

the sum of the mean square values of $E_1(x,y)$ and $E_2(x,y)$ remaining constant at all distances from the screen. These mean square values represent, in the case of an amplitude screen, the amount of fluctuation of the amplitude and the phase respectively at a distance $z$ from the screen.

The autocovariance of $E_1$ can be calculated from the spectrum $W_1(\nu,\chi)$ in (12) by the Wiener-Khintchine theorem; $\overline{E_1^2(x,y)}$ is then its value for $\alpha=\beta=0$. Before this can be calculated, the power spectrum $W_1(\nu,\chi)$ of the signal $A(x_0,y_0)$ at the screen must be found. Some assumption is needed for the spatial characteristics of the signal. They will be assumed to be as follows:

i. $A(x,y)$ has a Gaussian correlogram in all directions.
ii. The contours of the two-dimensional correlogram are elliptical.
iii. The ellipses have their principal axes oriented along the $x$ and $y$ axes of the co-ordinate system.

The $\rho=0.61$ contour on the two-dimensional correlogram is used to define two structure sizes $D_1$ and $D_2$ of the screen in the $x$ and $y$ directions. The correlogram of $A(x,y)$ is then

$$\rho_0(\alpha,\beta)=\exp\{-(\alpha^2/2D_1^2)-(\beta^2/2D_2^2)\}. \tag{15}$$

The angular spectrum at the screen is given by an equation like (2)

$$P_0(s,r)=\iint_{-\infty}^{\infty} A(x_0,y_0)\cdot\exp\ \{-2\pi i(sx_0+ry_0)/\lambda\}dx_0dy_0$$

and the power spectrum is given by

$$\begin{aligned} W_0(s,r)=\overline{P_0(s,r)\cdot P_0^*(s,r)}&=\text{mean of}\left\{\iint_{-\infty}^{\infty} A(x_0,y_0)\cdot\exp\{-2\pi i(sx_0+ry_0)/\lambda\}dx_0dy_0\right. \\ &\left.\times\iint_{-\infty}^{\infty} A(x_0',y_0')\cdot\exp\{2\pi i(sx_0'+ry_0')/\lambda\}dx_0'dy_0'\right\} \\ &=\iiiint_{-\infty}^{\infty} A(x_0,y_0)\cdot A(x_0',y_0')\cdot\exp\{2\pi i[(x_0'-x_0)s+(y_0'-y_0)r]/\lambda\}dx_0dy_0dx_0'dy_0' \end{aligned}$$

where $x_0'$ and $y_0'$ are variables of integration, analogous to $x_0$ and $y_0$, but differentiated from them to avoid confusion. So

$$W_0(s,r)=\iiiint_{-\infty}^{\infty} A(x_0,y_0)\cdot A(x_0+\alpha,y_0+\beta).\exp\ \{2\pi i(\alpha s+\beta r)/\lambda\}dx_0dy_0d\alpha\ d\beta$$

where $\alpha=x_0'-x_0$, $\beta=y_0'-y_0$. By (14), it is evident that

$$\overline{A(x_0,y_0)\cdot A(x_0+\alpha,\ y_0+\beta)}=\rho_0(\alpha,\beta)$$

and therefore

$$W_0(s,r)=\iiiint_{-\infty}^{\infty} \rho_0(\alpha,\beta)\cdot\exp\{2\pi i(\alpha s+\beta r)/\lambda\}dx_0dy_0d\alpha\ d\beta.$$

Now neither $x_0$ nor $y_0$ appears in the integrand; so the integration with respect to $x_0$ and $y_0$ will give a $W_0(s,r)$ which is proportional to the total area $S$ of the diffracting screen:

$$W_0(s,r)=S\cdot\iint_{-\infty}^{\infty} \rho_0(\alpha,\beta)\cdot\exp\{2\pi i(\alpha s+\beta r)/\lambda\}d\alpha d\beta. \tag{16}$$

Substituting for $\rho_0(\alpha,\beta)$ from (15) and integrating,

$$W_0(s,r)=2\pi SD_1D_2.\exp\{-2\pi^2(D_1{}^2s^2+D_2{}^2r^2)/\lambda\}\cdot \tag{17}$$

The spatial power spectra of $E_1(x,y)$ and $E_2(x,y)$ are found by substituting from (17) into (12):

$$\left.\begin{aligned} W_1(\nu,\chi)&=2\pi SD_1D_2.\exp\{-2\pi^2(D_1{}^2\nu^2+D_2{}^2\chi^2)/\lambda^2\}.\cos^2\{\pi z(\nu^2+\chi^2)/\lambda\} \\ W_2(\nu,\chi)&=2\pi SD_1D^2.\exp\{-2\pi^2(D_1{}^2\nu^2+D_2{}^2\chi^2)/\lambda^2\}.\sin^2\{\pi z(\nu^2+\chi^2)/\lambda\} \end{aligned}\right\} \tag{18}$$

It is now possible to find the numerator of (13); the covariance of $E_1(x,y)$ and $E_1\ (x+\alpha,y+\beta)$. The inverse transform corresponding to (16) is

$$\overline{E_1(x,y)\,E_1(x+\alpha,y+\beta)}=\frac{1}{S\lambda^2}\iint_{-\infty}^{\infty} W_1(\nu,\chi)\cdot\exp\{-2\pi i(\nu\alpha+\chi\beta)/\lambda\}d\nu\,d\chi \tag{19}$$

with the value of $W_1(\nu,\chi)$ given by (18). The integral is evaluated by putting

$$\cos^2\{\pi z(\nu^2+\chi^2)/\lambda\}=1/2+(1/4)\exp\{2\pi iz(\nu^2+\chi^2)/\lambda\}+(1/4)\exp\{-2\pi iz(\nu^2+\chi^2)/\lambda\}. \tag{20}$$

As (19) is the inverse of (16), the term 1/2 gives simply $(1/2)\rho_0(\alpha,\beta)$. The two remaining terms in (40) are identical except that the sign of $z$ is reversed. The integral can be simplified further by separating completely the integrals with respect to $\nu$ and $\chi$. This gives

$$\overline{E_1(x,y)\cdot E_1(x+\alpha,y+\beta)}=\frac{\pi D_1D_2}{4\lambda^2}\{I(\alpha,D_1,z)I(\beta,D_2,z)+I(\alpha,D_1,-z)I(\beta,D_2,-z)\}+(1/2)\rho_0(\alpha+\beta) \tag{21}$$

where

$$\begin{aligned} I(\alpha,D_1,z)&=\int_{-\infty}^{\infty}\exp\ \{2\pi i[\alpha\nu+(z+\pi iD_1^2/\lambda)\nu^2]/\lambda\}d\nu \\ &=\frac{\lambda(1+a_1^2)^{-1/4}}{\sqrt{2\pi D_1}}\exp\ \{-(1/2)\ \tan^{-1}a_1-(1-ia_1)\alpha^2/2D_1^2(1+a_1^2)\} \end{aligned} \tag{22}$$

where the important substitutions

$$a_1=\frac{\lambda z}{\pi D_1^2},\ a_2=\frac{\lambda z}{\pi D_2^2}$$

have been made. These parameters will appear as the measures of distance on all the graphs depicting the behavior of the various statistical parameters of the diffraction pattern.

Finally, on substituting from (22) and (21),

$$\overline{E_1(x,y)\cdot E_1(x+\alpha,y+\beta)}=(1/2)\exp\{-\alpha^2/2D_1{}^2-\beta^2/2D_2{}^2\}$$
$$\underset{*}{+}(1/2)(1+a_1{}^2)^{-1/4}(1+a_2{}^2)^{-1/4}\exp\{-\alpha^2/2D_1{}^2(1+a_1{}^2)-\beta/2D_2^2(1+a_2^2)\}$$
$$\times\cos\{a_1\alpha^2/2D_1{}^2(1+a_1{}^2)+a_2\beta^2/2D_2{}^2(1+a_2{}^2(1+a_2{}^2)-(\tan^{-1}a_1+\tan^{-1}a_2)/2\}. \tag{23}$$

By putting $\alpha=\beta=0$

$$\overline{E_1{}^2(x,y)}=1/2\underset{*}{+}(1/2)(1+a_1{}^2)^{-1/4}\cdot(1+a_2{}^2)^{-1/4}\cdot\cos\{(\tan^{-1}a_1+\tan^{-1}a_2)/2\}\cdot \tag{24}$$

The correlogram $\rho_1(\alpha,\beta)$ is given by the ratio of (23) to (24). The expressions for the auto-covariance of $E_2(x,y)$ are identical, except that the positive signs, denoted above with asterisks, become negative. Some special cases will now be investigated.

(i) *Near the screen* $a_1=a_2=0$, and

$$\overline{E_1(x,y)\cdot E_1(x+\alpha,y+\beta)}=\exp(-\alpha^2/2D_1{}^2-\beta^2/2D_2{}^2)=\rho_0(\alpha,\beta)$$
$$\overline{E_1{}^2(x,y)}=1,\ \overline{E_2{}^2(x,y)}=0.$$

as would be expected, since $E_1(x,y)=A(x_0,y_0)$ at the screen.

(ii) *Very far from the screen* $a_{1,2}\to\infty$, $\tan^{-1}a_1=\tan^{-1}a_2=\pi/2$, and

$$\overline{E_1(x,y)E_1(x+\alpha,y+\beta)}=(1/2)\exp\{-\alpha^2/2D_1{}^2-\beta^2/2D_2{}^2\}$$
$$\overline{E_1{}^2(x,y)}=\overline{E_2{}^2(x,y)}=1/2$$
$$\rho_1(\alpha,\beta)=\rho_2(a,\beta)=\exp\{-\alpha^2/2D_1{}^2-\beta^2/2D_2{}^2\}. \tag{25}$$

The second result in (25) means that the space correlation of both amplitude and phase at a great distance from an amplitude screen is just the same as the correlation at the screen. This is *not* true at intermediate distances, as we shall shortly see. The first result in (25) shows that the fluctuations occur equally in amplitude and phase under these conditions. The phase-amplitude locus of the instantaneous signal at large distances therefore forms a circular "cloud" around the vector denoting the mean carrier signal, as in figure 1.

(iii) *Isotropic screen* $D_1=D_2=D$, and $a_1=a_2=a$.

$$\overline{E_1(x,y)E_1(x+\alpha,y+\beta)}=(1/2)\exp\{-(a^2+\beta^2)/2D^2\}$$
$$+(1/2)(1+a^2)^{-1/2}\cdot\exp\{-(\alpha^2+\beta^2)/2D^2(1+a^2)\}\cdot\cos\{a(\alpha^2+\beta^2)/2D^2(1+a^2)-\tan^{-1}a\}. \tag{26}$$

The quantities $\alpha$ and $\beta$ appear in this expression only as $(\alpha^2+\beta^2)$, so the contours of correlation are circles at all distances, and the diffraction pattern is never anisotropic. Also

$$\overline{E_1{}^2(x,y)}=1/2\underset{*}{+}(1/2)(1+a^2)^{-1}$$

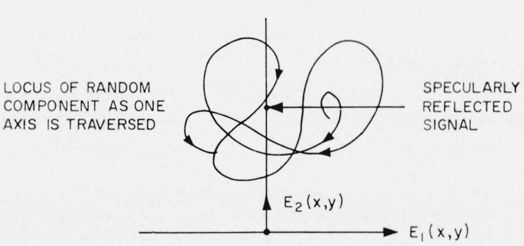


FIGURE 1. *A possible phase—amplitude diagram for the diffracted signal a long way from a random screen.*

the asterisk signifying, as previously, the sign which is to be reversed to give the corresponding value for $\overline{E_2^2(x,y)}$. $\overline{E_1^2(x,y)}$ and $\overline{E_2^2(x,y)}$ are plotted on figure 2. It can be seen that the mean square values of $E_1$ and $E_2$ are virtually equal at distances from the screen greater than that corresponding to about $a=4$. It will be shown later that, under these circumstances, the diffraction pattern is not correlated with the signal at the screen.

Slight rearrangement of (26) gives

$$\overline{E_1(x,\ y)\ E_1(x+\alpha,\ y+\beta)}=(1/2).\ \exp\{-(\alpha^2+\beta^2)/2D^2\}$$

$$\underset{*}{+}(1/2)(1+a^2)^{-1}.\ \exp\{-(\alpha^2+\beta^2)/2D^2(1+a^2)\}\ [\cos\{a(\alpha^2+\beta^2)/2D^2(1+a^2)\}$$

$$+\ a\ \sin\ \{a(\alpha^2+\beta^2)/2D^2(1+a^2)\}]. \quad (27)$$

(iv) *One-dimensional screen* $\quad D_1=D;\ D_2\rightarrow\infty.\quad$ So $a_2\rightarrow 0$.
From (24),

$$\overline{E_1^2(x,\ y)}=1/2\underset{*}{+}(1/2)(1+a^2)^{-1/4}\cdot\cos\{(\tan^{-1}a)/2\}$$
$$=1/2\underset{*}{+}(1/2)\{[1+(1+a^2)^{1/2}][1+a^2]^{-1}/2\}^{1/2}.$$

This also is plotted in figure 2, for comparison with the curve for the two-dimensional case. Evidently the mean square values of $E_1(x,y)$ and $E_2(x,y)$ become nearly equal much closer to a two-dimensionally irregular screen than a screen which is irregular in one dimension only. This illustrates that a one-dimensional model may give a quite inadequate representation of two-dimensional diffraction phenomena. For this case,

$$\overline{E_1(x,\ y)\ E_1(x+\alpha,\ y+\beta)}=(1/2)\ \exp(-\alpha^2/2D^2)$$
$$\underset{*}{+}(1/2)(1+a^2)^{-1/4}\ \exp\ \{-\alpha^2/2D^2(1+a^2)\}\cdot\cos\{a\alpha^2/2D^2(1+a^2)$$
$$-(\tan^{-1}a)/2\}.$$

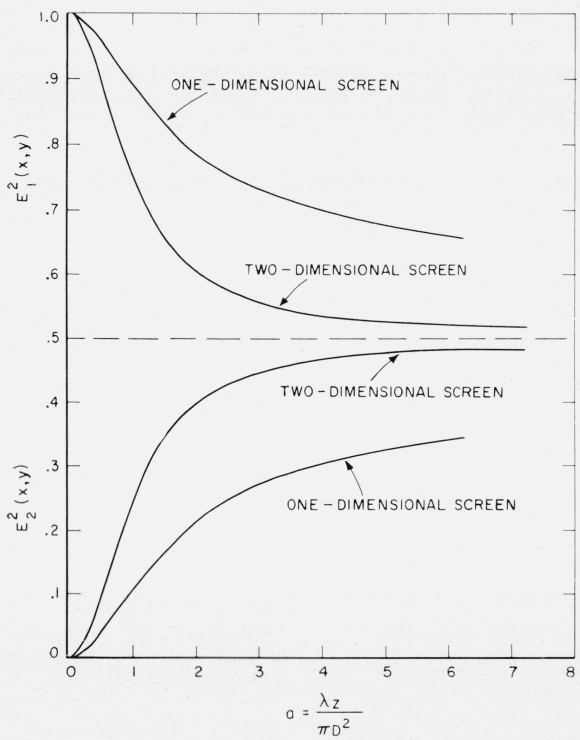


FIGURE 2. *Variation of mean square diffracted signal with distance from two kinds of random screen.*

It will be noted from (27) that the correlogram is not Gaussian in form, though $E_1(x,y)$ has a Gaussian correlogram for $a=0$, and both $E_1(x,y)$ and $E_2(x,y)$ have Gaussian correlograms as $a \rightarrow \infty$ (see equation (25)). This introduces some difficulty as to what is to be called the "structure size" $d$ of the diffraction pattern, for comparison with experiment. The following definition is adopted

$$\begin{aligned} d &= \text{Structure size in } x \text{ direction} = (-\partial^2\rho/\partial\alpha^2)^{-1/2}_{\alpha=\beta=0} \\ &= D_1 \text{ for the correlogram } \rho_0(\alpha,\beta). \end{aligned}$$

This definition bases the structure size on the Gaussian curve which is the best fit to the correlogram, near the origin of $\alpha$. It can best be found by expanding (23) in powers of $\alpha^2$ and $\beta^2$, and neglecting terms of order higher than the second; using equations (13) and (24),

$$\begin{aligned} d_1^2/D_1^2 &= \frac{1 \underset{*}{+} (1+a_1^2)^{-1/4}(1+a_2^2)^{-1/4} \cos\{(\tan^{-1} a_1 + \tan^{-1} a_2)/2\}}{1 \underset{*}{+} (1+a_1^2)^{-5/4}(1+a_2^2)^{-1/4}[\cos(\tan^{-1} a_1 + \tan^{-1} a_2)/2 - a_1 \sin\{(\tan^{-1} a_1 + \tan^{-1} a_2)/2\}]} \\ d_2^2/D_2^2 &= \frac{1 \underset{*}{+} (1+a_1^2)^{-1/4}(1+a_2^2)^{-1/4} \cos\{(\tan^{-1} a_1 + \tan^{-1} a_2)/2\}}{1 \underset{*}{+} (1+a_1^2)^{-1/4}(1+a_2^2)^{-5/4}[\cos(\tan^{-1} a_1 + \tan^{-1} a_2)/2 - a_2 \sin\{(\tan^{-1} a_1 + \tan^{-1} a_2)/2\}]}. \end{aligned} \tag{28}$$

To help in understanding the significance of these results, a number of special cases are examined below.

(v) *Near the screen*, $a_1$ and $a_2 \rightarrow 0$, and $d_1 \rightarrow D_1$, $d_2 \rightarrow D_2$ for $E_1(x,y)$, as found previously under (i). For the quadrature component $E_2(x,y)$ however, it is easily shown that

$$\left.\begin{aligned} d_1 &= D_1 \left\{\frac{K^4 + 2K^2/3 + 1}{K^4 + 2K^2 + 5}\right\}^{1/2} \\ d_2 &= D_2 \left\{\frac{K^4 + 2K^2/3 + 1}{5K^4 + 2K^2 + 1}\right\}^{1/2} \end{aligned}\right\} \tag{29}$$

where $K = D_1/D_2$.

(vi) *Isotropic screen* $D_1 = D_2 = D$, $K=1$.

Near the screen, $d=D$ for $E_1(x,y)$ as found previously.

From (29) $d = D/\sqrt{3}$ for $E_2(x,y)$.

This result implies that the structure size of the shallow phase variations *near* an amplitude screen (or of the amplitude variations near the phase screen) is $1/\sqrt{3}$ times the structure of the screen itself. In addition, the correlogram of $E_2(x,y)$ is far from Gaussian; it can easily be shown by expanding (23) in powers of $a$ that, near the screen,

$$\rho_2(\alpha,\beta) = \exp\{-(\alpha^2+\beta^2)/2D^2\} \cdot \{1 - (\alpha^2+\beta^2)/D^2 + (\alpha^2+\beta^2)^2/8D^4\} \tag{30}$$

exactly. This is plotted in figure 3, and has an oscillating form. At any distance from an isotropic screen, (28) reduces to for $E_1(x,y)$:

$$d = D\left\{\frac{(1+a^2)(2+a^2)}{a^4+a^2+2}\right\}^{1/2}$$

for $E_2(x,y)$:

$$d = D\left\{\frac{1+a^2}{3+a^2}\right\}^{1/2}.$$

These relations are shown plotted in figure 4. The structure size of $E_1(x,y)$ actually has a maximum at intermediate values of $z$.

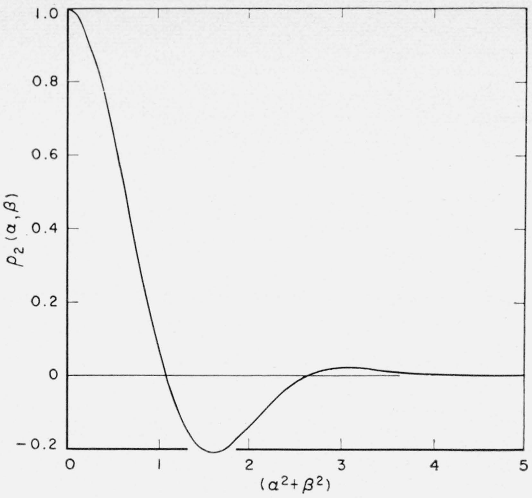


FIGURE 3. *Correlogram of the shallow phase variations near a two-dimensional amplitude screen.*

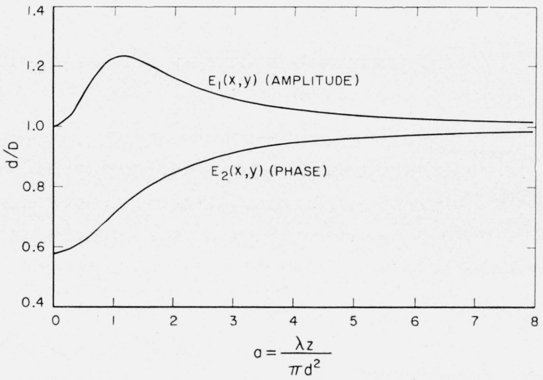


FIGURE 4. *Variation of structure size* d *with distance from a two-dimensional amplitude screen.*

(vii) *One-dimensional screen* $D_1=D$; $D_2\to\infty$, $a_2\to 0$, $K\to 0$. Near the screen,

$$d=D \text{ for } E_1(x,y),\ d=D/\sqrt{5} \text{ for } E_2(x,y),$$

showing the effect described in (vii) to an even greater extent. At any distance, for $E_1(x,y)$:

$$d=D\left\{\frac{1+(1+a^2)^{-1/4}\cos\{(\tan^{-1}a)/2\}}{1+(1+a^2)^{-5/4}\,[\cos\{(\tan^{-1}a)/2\}-a\sin\{(\tan^{-1}a)/2\}]}\right\}^{1/2}$$

for $E_2(x,y)$:

$$d=D\left\{\frac{1-(1+a^2)^{-1/4}\cos\{(\tan^{-1}a)/2\}}{1-(1+a^2)^{-5/4}\,[\cos\{(\tan^{-1}a)/2\}-a\sin\{(\tan^{-1}a)/2\}]}\right\}^{1/2}.$$

In the general case of an anisotropic screen, (28) can be used to find the structure sizes $d_1$ and $d_2$ in the $x$ and $y$ directions. To clarify the kind of behavior that may occur, figure 5 has been prepared. This shows how the contours of the correlation ellipse change shape with increasing distance from a random amplitude screen. The two screens shown are

(a) isotropic, with $D_1=D_2=1$

(b) anisotropic, with $D_1=2$, $D_2=1$.

The distances from the screen were chosen as 0, $\infty$, and $\pi D_1^2/\lambda$. It is interesting to note that the phase correlation ellipse for the anisotropic screen does not keep the same shape as the screen is approached—its minor axis is reduced by a greater factor than its major axis. This is because as it approaches the limiting case of an anisotropic screen—namely, a one-dimensional screen—the major axis of the phase correlation ellipse near the screen approaches the true structure size $D_1$ of the screen, while the minor axis is reduced by the factor $\sqrt{5}$ established above in (vii).

The reason for dealing with this topic in such detail is to demonstrate that the structure size of the irregularities in the diffraction pattern produced by a shallow phase or amplitude

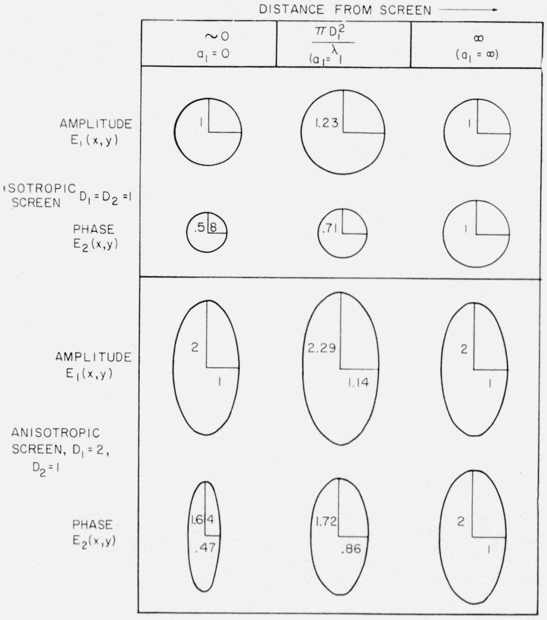


FIGURE 5. *Contours of the spatial correlogram of the wave diffracted from two kinds of amplitude screen, at different distances.*

modulated screen is not always the same as that of the screen itself. The preceding lengthy mathematical argument is necessary to establish the precise effect of the diffraction process, so that existing experimental data may be interpreted with confidence.

## 4. Cross-Correlation Relationships for $E_1$ and $E_2$

A good deal of attention has been paid by various workers (for instance, Jones, Millman, and Nertney [1953]) to the question of whether the signal in the diffraction pattern is the same as the signal at a random screen directly opposite to it. It must be emphasized that the question is not whether the signals are *statistically* similar, i.e., have the same structure size or fading time, but whether the *detailed* changes in signal are similar at the two planes. Expressed in more exact language, the cross-correlation is to be evaluated between, for instance, the amplitude of the signal at the screen and the amplitude at a distant plane.

Another cross-correlation coefficient of considerable importance is that between the amplitude and the phase of the diffracted signal [Bowhill, 1957]. Jones, Millman, and Nertney [1953] assumed that this correlation arises as a result only of correlation between the amplitude and phase of the signal at the screen. It is shown here, however, that they may be correlated even for a screen which changes the phase only or the amplitude only.

The theory is developed for a shallow amplitude modulating screen, and is extended to the case of a phase screen. The quantity $E_1(x,y)$ therefore represents the amplitude of the signal in the diffraction pattern (with the specular component removed) and $E_2(x,y)$ represents the phase of the signal. The quantities

$$\overline{A(x,y)E_1(x,y)} \text{ and } \overline{E_2(x,y)E_1(x,y)} \tag{31}$$

are to be calculated; the known values of $\overline{E_1^2(x,y)}$, $\overline{E_2^2(x,y)}$ and $\overline{A^2(x,y)}$ then give the cross-correlation coefficients $\rho_{11}$ between $A(x,y)$ and $E_1(x,y)$, and $\rho_{12}$ between $E_1(x,y)$ and $E_2(x,y)$.

The quantities (31) are computed from the relations

$$\left.\begin{aligned}\overline{A(x,y)E_1(x,y)} &= (1/2)\overline{E_1^2(x,y)} + (1/2)\overline{A^2(x,y)} - (1/2)\overline{[A(x,y)-E_1(x,y)]^2}\\ \overline{E_1(x,y)E_2(x,y)} &= (1/2)\overline{E_1^2(x,y)} + (1/2)\overline{A^2(x,y)} - (1/2)\overline{[E_1(x,y)-E_2(x,y)]^2}.\end{aligned}\right\}\tag{32}$$

The last terms in these equations represent the areas under the power spectra of $[A(x,y)-E_1(x,y)]$ and $[E_1(x,y)-E_2(x,y)]$, since the mean square value of any random variable with zero mean is given by the integral of its power spectrum with respect to frequency.

The power spectrum of $[A(x,y)-E_1(x,y)]$ can be found from (11);

$$P_1(\nu,\chi)=P_0(\nu,\chi)\cdot\cos\{\pi z(\nu^2+\chi^2)/\lambda\}. \tag{11}$$

The spectrum of $[A(x,y)-E_1(x,y)]$ is therefore

$$P_0(\nu,\chi)-P_1(\nu,\chi)=P_0(\nu,\chi)[1-\cos\{\pi z(\nu^2+\chi^2)/\lambda\}]$$

and its power spectrum is given, exactly as in (6), by

$$W_0(\nu,\chi)[1-\cos\{\pi z(\nu^2+\chi^2)/\lambda\}]^2=4\,[W_2(\nu,\chi)]_{z/2}-[W_2(\nu,\chi)]_z$$

and substituting in (32),

$$\overline{A(x,y)E_1(x,y)}=(1/2)[\overline{E_1^2(x,y)}]_z+(1/2)-2\,[\overline{E_2^2(x,y)}]_{z/2}+(1/2)[\overline{E_2^2(x,y)}]_z=2\,[\overline{E_1^2(x,y)}]_{z/2}-1$$

a relation which holds for any autocorrelation function at the screen. Using $\overline{E_1^2(x,y)}$ from (24), the correlation $\rho_{11}$ is given by

$$\rho_{11}=\overline{A(x,y)E_1(x,y)}/\overline{E_1^2(x,y)}=\frac{\sqrt{2}\cdot(1+a_1^2/4)^{-1/4}(1+a_2^2/4)^{-1/4}\cos\{[\tan^{-1}(a_1/2)+\tan^{-1}(a_2/2)]/2\}}{\{1+(1+a_1^2)^{-1/4}(1+a_2^2)^{-1/4}\cos\{[(\tan^{-1}a_1+\tan^{-1}a_2)/2]\}\}^{1/2}}$$

for a screen with a Gaussian correlogram.

For an isotropic screen, $a_1=a_2=a$, and this reduces to

$$\rho_{11}=\{(1+a^2)/(1+a^2/2)\}^{1/2}(1+a^2/4)^{-1}.$$

This function is plotted as the full curve in figure 6. The correlation is perfect near the screen, since then $E_1(x,y)=A(x,y)$. It decreases progressively to zero with increasing distance, reaching a value of 0.1 when $a=7.2$.

The corresponding function for a one-dimensional screen, $a_1=a$, $a_2=0$, is given by

$$\rho_{11}=\frac{2(1+a^2/4)^{-1/4}\cos\{[\tan^{-1}(a/2)]/2\}}{\{1+(1+a^2)^{-1/4}\cos\{(\tan^{-1}a_1+\tan^{-1}a_2)\}\}^{1/2}}$$

This function is plotted as the broken curve on figure 6. The correlation decreases much less rapidly with increasing distance from the screen than for the two-dimensional case.

To find the correlation $\rho_{12}$, (32) is rearranged to give

$$\overline{E_1(x,y)E_2(x,y)}=1/2-(1/2)\overline{[E_1(x,y)-E_2(x,y)]^2}$$

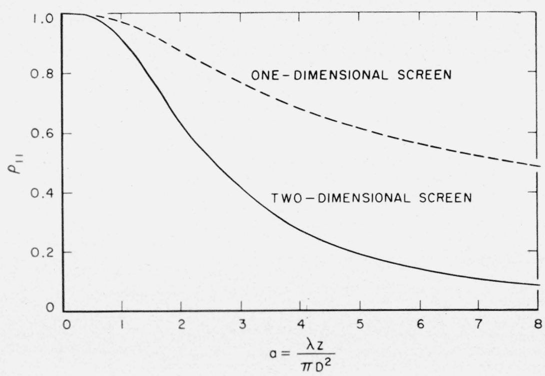


FIGURE 6. *Variation of the correlation of the diffraction pattern amplitude with the amplitude at the screen as a function of distance for one- and two-dimensional screens.*

and the power spectrum of $[E_1(x,y)-E_2(x,y)]$ is given from (11) by

$$\overline{[P_1(\nu,\chi)-P_2(\nu,\chi)][P_1^*(\nu,\chi)-P_2^*(\nu,\chi)]}$$

$$=W_0(\nu,\chi)[\cos\{\pi z(\nu^2+\chi^2)/\lambda\}-\sin\{\pi z(\nu^2+\chi^2)/\lambda\}]^2=W_0(\nu,\chi)-W_0(\nu,\chi)\sin\{2\pi z(\nu^2+\chi^2)/\lambda\}.$$

So

$$\overline{E_1(x,y)E_2(x,y)}=(1/2\lambda^2S)\iint_{-\infty}^{\infty}W_0(\nu,\chi)\sin\{2\pi z(\nu^2+\chi^2)/\lambda\}d\nu d\chi$$

$$=(\pi D_1D_2/2i\lambda^2)\iint_{-\infty}^{\infty}[\exp\{2\pi iz(\nu^2+\chi^2)/\lambda\}-\exp\{-2\pi iz(\nu^2+\chi^2)/\lambda\}]$$

$$\cdot\exp\{-2\pi^2(D_1^2\nu^2+D_2^2\chi^2)/\lambda^2\}d\nu d\chi.$$

$$=(\pi D_1D_2/2i\lambda^2)\left\{\int_{-\infty}^{\infty}\exp\{2\pi i\nu^2(z+\pi iD_1^2/\lambda)/\lambda\}d\nu\int_{-\infty}^{\infty}\exp\{2\pi i\chi^2(z+\pi iD_2^2/\lambda)/\lambda\}d\chi\right.$$

$$\left.-\int_{-\infty}^{\infty}\exp\{2\pi i\nu^2(-z+\pi iD_1^2/\lambda)/\lambda\}d\nu\int_{-\infty}^{\infty}\exp\{2\pi i\chi^2(-z+\pi iD_2^2/\lambda)/\lambda\}d\chi\right\}$$

and on integrating this expression by the usual methods

$$\overline{E_1(x,y)E_2(x,y)}=(1/2)(1+a_1^2)^{-1/4}(1+a_2^2)^{-1/4}\sin\{(\tan^{-1}a_1+\tan^{-1}a_2)/2\}\cdot$$

Also, from (24)

$$4\cdot\overline{E_1^2(x,y)}\cdot\overline{E_2^2(x,y)}=1-(1+a_1^2)^{-1/2}(1+a_2^2)^{-1/2}\cos^2\{(\tan^{-1}a_1+\tan^{-1}a_2)/2\}$$

and the correlation $\rho_{12}$ between $E_1(x,y)$ and $E_2(x,y)$ is given by

$$\rho_{12}=\frac{(1+a_2^2)^{-1/4}(1+a_2^2)^{-1/4}\sin\{(\tan^{-1}a_1+\tan^{-2}a_2)/2\}}{[1-(1+a_1^2)^{-1/2}(1+a_2^2)^{-1/2}\cos^2\{(\tan^{-1}a_1+\tan^{-1}a_2)/2\}]^{1/2}}. \tag{33}$$

For an isotropic screen, $a_1=a_2=a$; this reduces to

$$\rho_{12}=(1+a^2)^{-1/2}$$

which is plotted in figure 7. The correlation between $E_1(x,y)$ and $E_2(x,y)$ has a maximum value of $1/\sqrt{2}$ at the screen, and decreases to zero with increasing distance. For an anisotropic screen, (33) must be expanded in powers of $a_1$ and $a_2$. The correlation as $a_1$ and $a_2\to0$ is given by

$$\rho_{12}=\frac{a_1+a_2}{[3a_1^2+2a_1a_2+3a_2^2]^{1/2}}=\frac{1+K^2}{[3K^4+2K^2+3]^{1/2}} \tag{34}$$

where $K=D_1/D_2$ expresses the degree of anisotropy of the screen, as in (29). Whatever the value of $K$, (34) shows that this maximum value of the correlation $\rho_{12}$ lies between the values

$$\rho_{12}=1/\sqrt{2}\text{ for an isotropic screen }a_1=a_2$$

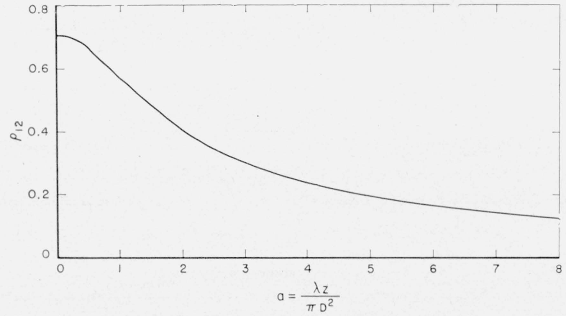


FIGURE 7. *Variation of amplitude-phase cross-correlation with distance from a two-dimensional amplitude screen.*

and

$$\rho_{12}=1/\sqrt{3} \text{ for a one-dimensional screen } a_1 \gg a_2. \tag{35}$$

The cross-correlation coefficients $\rho_{11}$ and $\rho_{12}$ between the phase components $E_1(x,y)$ and $E_2(x,y)$ must now be related to the amplitude and phase characteristics of the diffraction pattern. For an amplitude screen, it is apparent from figure 1 that for a shallow screen,

$$\text{Amplitude in pattern} \triangleq (\text{specular signal}) + E_1(x,y)$$
$$\text{Phase in pattern} \triangleq -E_2(x,y)/(\text{specular signal})$$

since the positive phase direction is opposite to the positive $E_2$ direction, taking an increase in phase to mean a *retarding* of the wave. This means that:

Correlation between amplitude in pattern and at screen $=\rho_{11}$, and correlation between amplitude and phase in pattern $=-\rho_{12}$. Similarly, for a *phase modulating screen*

$$\text{Amplitude in pattern} \triangleq (\text{specular signal}) + E_2(x,y)$$
$$\text{Phase in pattern} \triangleq E_1(x,y)/(\text{specular signal})$$

and also

Correlation between phase in pattern and at screen $=\rho_{11}$

Correlation between phase and amplitude in pattern $=\rho_{12}$.

It has therefore been established that the correlation between the amplitude and phase in a diffracted signal is zero at large distances, but at smaller distances is positive for a phase modulating screen and negative for an amplitude screen. This occurs at a distance from the screen given by $a\sim 1$, or $a\sim D^2/\lambda$. This is closely related to the Rayleigh distance for an optical system of aperture equal to the structure size of the screen irregularities.

This result has been derived by setting up the wave solution. It may be partially confirmed by a physical-optical argument, as follows. Consider a one-dimensional phase-modulating screen, with a phase profile $\phi(x)$. The signal at a point near the screen is determined only by the properties of the screen in its immediate neighborhood. In fact, any small portion of the screen centered about $x_0$ acts as a positive lens of focal length

$$F=-2\pi/\lambda\left[\frac{d^2\phi}{dx^2}\right]_{x=x_0}. \tag{36}$$

The plane wave incident on the screen can be imagined as coming from a point source of waves, very far away. The image formed by the effective "lens" of the phase screen appears to have angular magnification $=(1-z/F)^{-1}$ where $z$ is the distance of the point of observation from the screen. The image of the source appears the same brightness as the source itself, so the signal power arriving at $z$ is proportional to the angular magnification. So the signal amplitude is proportional to $(1-z/F)^{-1/2}$; if $z$ is assumed much less than $F$, approximately

$$\text{signal amplitude} \propto 1+z/2F=1-(\lambda z/4\pi)\left[\frac{d^2\phi}{dx^2}\right]_{x=x_0}$$

on substitution from (36). Any correlation between the amplitude and phase in the pattern must therefore arise through correlation between $\phi(z)$ and $(-d^2\phi/dx^2)$.

In fact, it can easily be shown that if $\phi(x)$ is a Gaussianly distributed random function, there is a correlation between these two quantities of $[\rho''(0)/\rho''''(0)]^{1/2}$, where $\rho(\alpha)$ is the correlogram of $\phi(x)$. For a Gaussian correlogram, which has been assumed throughout this section, this reduces to $1/\sqrt{3}$; a result which is identical with that given for the same case in (35). It should be emphasized, however, that geometric optics can *only* be used near the screen, where the diffraction effects have not yet developed appreciably.

## 5. Effect of a Point Source of Waves

In all the previous sections it has been assumed that there is no systematic variation of phase across the random screen. However, if the screen is illuminated with radiation from a point source a distance $z'$ from the screen, instead of from a plane wave source, the phase of the wave before passing through the screen varies with distance. The signal emerging from the screen is now given by

$$E_0(x_0,y_0)=A(x_0,y_0)\ \exp\ \{2\pi i(z'^2+x_0^2+y_0^2)^{1/2}/\lambda\}.$$

If the cone angle of the diffracted waves is small, the important regions of the screen are for $x_0,\ y_0 \ll z'$. So the approximation

$$(z'^2+x_0^2+y_0^2)^{1/2} \simeq z'+x_0^2/2z'+y_0^2/2z' \tag{37}$$

may be used.

Exactly as in section 3, the vector signal $\underset{\sim}{E}(x,y,z)$ at a distance $z$ from the screen can be found and the phase components $E_1(x,y,z)$ and $E_2(x,y,z)$ evaluated by a relationship similar to (9). In that equation, the phase of the specular component is independent of $x$ and $y$ since the specular component is a plane wave. In the present case, however, the specular wave is spherical and so the analogous equation to (9) is

$$\underline{E}(x,y,z)=[E_1(x,y,z)+iE_2(x,y,z)]\ \exp\ \{2\pi i[(z+z')^2+x^2+y^2]^{1/2}/\lambda\}$$

and for $x,y \ll z$,

$$[(z+z')^2+x^2+y^2]^{1/2} \simeq z+z'+x^2/2(z+z')+y^2/2(z+z'). \tag{38}$$

The corresponding equation to (10) can then be written down as

$$\underline{E}(x,y,z)\ \exp\ \{-2\pi i\ [(z+z')^2+x^2+y^2]^{1/2}/\lambda\}=(1/\lambda^2)\iiiint_{-\infty}^{\infty} A(x_0,y_0)\cdot \exp\{2\pi i\ [(x-x_0)s$$
$$+(y-y_0)r+(z'^2+x_0^2+y_0^2)^{1/2}+z(1-s^2/2-r^2/2)]/\lambda\}\ dx_0dy_0dsdr$$

or

$$\underline{E}(x,y,z,)=(1/\lambda^2)\iiiint_{-\infty}^{\infty} A(x_0,y_0)\ \exp\ \{2\pi iG/\lambda\}\ dx_0dy_0dsdr$$

where

$$G=(x-x_0)s+(y-y_0)r+(z'^2+z_0^2+y_0^2)^{1/2}+z(1-s^2/2-r^2/2)-[(z+z')^2+x^2+y^2]^{1/2}.$$

By substitution from (37) and (38), approximately

$$G=(x-x_0)s+(y-y_0)r+z'+x_0^2/2z'+y_0^2/2z'+z-zs^2/2-zr^2/2-z-z'-x^2/2(z+z')-y^2/2(z+z')$$
$$=(x-x_0)s+(y-y_0)r+x_0^2/2z'+y_0^2/2z'-zr^2/2-zs^2/2-x^2/2(z+z')-y^2/2(z+z').$$

The phase components $E_1(x,y,z)$ and $E_2(x,y,z)$ are given by

$$\left.\begin{aligned} E_1(x,y,z)&=\mathscr{R}\{\underline{E}(x,y,z)\}=(1/2\lambda^2)\iiiint_{-\infty}^{\infty} A(x_0,y_0)\{\exp\ (2iG/\lambda)+\exp\ (-2iG/\lambda)\}\ dx_0dy_0dsdr \\ E_2(x,y,z)&=\mathscr{I}\{\underline{E}(x,y,z)\}=(1/2\lambda^2)\iiiint_{-\infty}^{\infty} A(x_0,y_0)\{\exp\ (2iG/\lambda)-\exp\ (-2iG/\lambda)\}dx_0dy_0dsdr \end{aligned}\right\} \tag{39}$$

and the spatial spectra of $E_1(x,y,z)$ and $E_2(x,y,z)$ are given, as before, by

$$\left.\begin{aligned} P_1(\nu,\chi)&=\iint_{-\infty}^{\infty} E_1(x,y,z)\cdot \exp\ \{-2\pi i(\nu x+\chi y)/\lambda\}dxdy \\ P_2(\nu,\chi)&=\iint_{-\infty}^{\infty} E_2(x,y,z)\cdot \exp\ \{-2\pi i(\nu x+\chi y)/\lambda\}dxdy. \end{aligned}\right\} \tag{40}$$

On substituting from (39) into (40) and integrating with respect to $s$, $r$, $x$, and $y$, an analogous expression to (30) is obtained:

$$P_1(\nu,\chi)=\frac{z+z'}{z'}\cos\{\pi(\nu^2+\chi^2)z(z+z')/\lambda z'\}\cdot\iint_{-\infty}^{\infty}A(x_0,y_0)\exp\{-2\pi i(\nu x_0+\chi y_0)(z+z')/\lambda z'\}\,dx_0dy_0. \tag{41}$$

The differences between (41) and (11) can be accounted for by supposing that there are apparent values of $z$ and $\lambda$ which, when substituted in (11) *and in all the results derived from it*, just allow for the fact that there is a point source of waves at a distance $z'$ from the screen.

In comparing (41) and (11), the double integral will first be considered. Both $\nu$ and $\chi$ in the exponent are multiplied by the factor $(z+z')/z$; so the angular spectrum must be scaled down in width by the reciprocal of this factor, relative to its width in the plane wave case.

The "generalized autocorrelation function" of Booker, Ratcliffe, and Shinn (1950) given by the Fourier transform of the angular power spectrum, is invariant with distance in the plane wave case. With point source illumination, its width increases proportionally to $(1+z/z')$, and finally increases proportionally to $z$. In the case of vertical incidence of waves on the ionosphere $z=z'$, and the apparent structure size of the screen irregularities is just twice the actual structure size.

These results can be appreciated physically by considering the behavior of the diffraction pattern as $\lambda\to 0$, and an analogy can be drawn with geometrical optics. If an amplitude screen (see fig. 8) is illuminated with a point source of light at a distance $z'$ from it, the scale of the amplitude variations a distance $z$ away from the screen will just be $(1+z/z')$ times the scale of the amplitude variations at the screen.

The other respect in which (41) and (11) differ is in the argument of the cosine. The quantity $(\nu^2+\chi^2)z(z+z')/z'$ replaces $(\nu^2+\chi^2)z$. But the quantities $\nu$ and $\chi$ have already been found to be multiplied by $(1+z/z')$; so $z$ in (30) must be replaced by

$$\frac{z(z+z')}{z'}\,\frac{1}{(1+z/z')^2}=\frac{zz'}{z+z'}=z_{\mathrm{eff}},$$

where $z_{\mathrm{eff}}$ is the effective value of $z$. In the case of vertical incidence on the ionosphere, $z=z'$, and $z_{\mathrm{eff}}=z/2$.

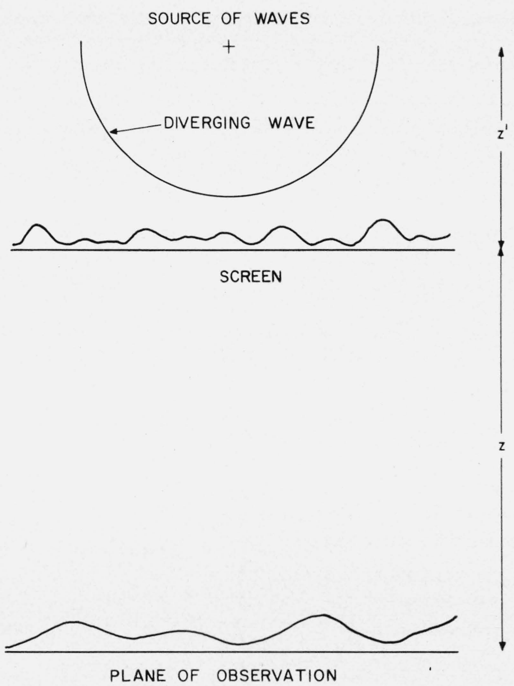


FIGURE 8. *Diagram showing the increase in structure size of the amplitude variations with increasing distance from a point source of waves.*

It is interesting to note that

$$\frac{1}{z_{\text{eff}}}-\frac{1}{z'}=\frac{1}{z}$$

which is exactly comparable with the object and image distance relations for a zone plate, the theory of which is closely related to that given above.

## 6. Effect of Oblique Incidence

If a plane wave is incident at an angle $i$ to a diffracting screen, the equations corresponding to those of section 3 can be deduced by an exactly similar method. If $y$ axis is in the plane of the incident wave, we may write $s=\sin i+s_1$, where $(\sin^{-1} s-\sin^{-1} s_1)$ is the scattering angle. The phase of the specular signal at a distance $z$ from the screen is now

$$(2\pi iz \cos i)/\lambda$$

and (9) must therefore be written

$$\underline{E}(x,y,z)=[E_1(x,y)+iE_2(x,y)] \exp [(2\pi iz \cos i)/\lambda]. \tag{42}$$

The phase of the scattered wave at distance $z$ is again given by

$$2\pi iz\{1+(s^{-2}-1)^{-1}+(r^2-1)^{-1}\}^{-1/2}/\lambda.$$

Putting $s=\sin i+s_1$ and expanding in powers of $s_1$ and $r$, this becomes

$$2\pi iz(\cos i-s_1 \tan i-\tfrac{1}{2}s_1^2 \sec^3 i-\tfrac{1}{2}r^2 \cos^3 i)/\lambda$$

and the exponent in the resulting equation for $(E_1+jE_2)$, analogous to (10), is

$$2\pi i\left\{(x-x_0-z \tan i)s_1+(y-y_0)r-\frac{1}{2}s_1^2 z \sec^3 i-\frac{1}{2}r^2 z \cos^3 i\right\}/\lambda.$$

Comparing this with the exponent in (10), it is evident that

(i) the origin of $x$ has been effectively displaced a distance $z \tan i$ in the plane of propagation—namely, to the "mid-point of the path", in propagation terminology.

(ii) the effective distance of the observing point from the screen has been multiplied by factors of $\sec^3 i$ and $\cos^3 i$ respectively for the $x$ and $y$ variations in the screen.

To make this more easily comprehensible, consider the case of oblique incidence on a diffracting screen with an isotropic spatial correlation function, Gaussian in form. The structure size of the interference pattern is found by putting

$$\left.\begin{aligned} a_1&=a \sec^3 i \\ a_2&=a \cos^3 i \end{aligned}\right\} \qquad K=\sec^6 i$$

in (28). Near the screen, from (29),

$$d_1=D\left\{\frac{\sec^{24} i+(2/3) \sec^{12} i+1}{\sec^{24} i+2 \sec^{12} i+5}\right\}^{1/2}$$

$$d_2=D\left\{\frac{\sec^{24} i+(2/3) \sec^{12} i+1}{5 \sec^{24} i+2 \sec^{12} i+1}\right\}^{1/2}.$$

Another important effect of oblique incidence, which does not enter explicitly into this analysis, is the following. If the random screen is composed of a "buckled specular reflector", it will act as a phase screen for normally incident waves. However, at oblique incidence, the phase change it causes in the wave is scaled by a factor $\cos i$. This is related to the familiar phenomenon that an optical surface becomes more nearly a specular reflector at grazing incidence. The same effect is observed for radio waves.



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## 7. Conclusion

Detailed calculations have been given for the statistics of a radio wave diffracted by a random screen. The results, some of which have been used in a previous paper [Bowhill, 1957], now take account of anisotropy of the diffracting screen, and oblique incidence of the waves

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