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A Formula For Radio Ray Refraction in An Exponential Atmosphere[1]

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A formula for the radio ray refraction angle, τ, is derived by integration of the approximate differential equation for the case where the refractivity, $(n-1)\times 10^6$, decreases exponentially with height above the surface of a smooth, spherical earth. The solution is in terms of the widely tabulated exponential and error functions, and is accurate to within 4 percent over the useful range of the variables employed.

1. Background

In problems where the direction of radio propagation is important, e.g., tracking radar, it is convenient to be able to compute the refraction of radio rays as they pass through the atmosphere. Assuming that the refractive index, n, is a function only of height above the surface of a smooth, spherical earth, the usual integral for the ray refraction angle, usually called the "bending", τ, is [1, 2] [2]

$$\tau=-\int_0^{h_\tau}\frac{\cos\theta_0[dN/dh]dh\times 10^{-6}}{\left\{\sin^2\theta_0+\frac{2h}{r_0}-2(N_s-N)\cos^2\theta_0\times 10^{-6}\right\}^{1/2}}, \tag{1}$$

where h_τ is the "target" height to which the bending is to be calculated, N is the radio refractivity, $(n-1)\times 10^6$, with N_s the surface value, and θ_0 is the initial elevation angle of the radio ray. The geometry involved in the problem is shown on figure 1.

The integral (1) is to be evaluated under the assumption that

$$N=N_s\exp\{-ch\}, \tag{2}$$

where c is a constant determining the gradient of N with respect to height; this is the exponential model atmosphere recently recommended for international use by the CCIR [3].

2. Development

The integral (1) with N given by eq (2) is put in readily integrable form by means of the approximation

$$N_s\times 10^{-6}[1-\exp\{-ch\}]\sim\gamma h \tag{3}$$

made where N_s-N appears in the denominator of (1). The parameter γ is defined by

$$\gamma=\frac{N_s\times 10^{-6}[1-\exp\{-cH\}]}{H},\ 0<H<h_\tau$$

where H is an effective integration height which was determined from published tabulations of refraction variables in model atmospheres of exponential form [4] for $\theta_0=0$, and is given by

$$H=4.75\,[\exp\{-0.01158cN_s\}]\,[1-\exp\{-ch_\tau\}]. \tag{4}$$

With (3), the integral for τ in an exponential atmosphere is approximated by

$$\tau\simeq cN_s\times 10^{-6}\cos\theta_0\int_0^{h_\tau}\frac{\exp\{-ch\}dh}{\left\{\sin^2\theta_0+2h\left[\frac{1}{r_0}-\gamma\cos^2\theta_0\right]\right\}^{1/2}} \tag{5}$$

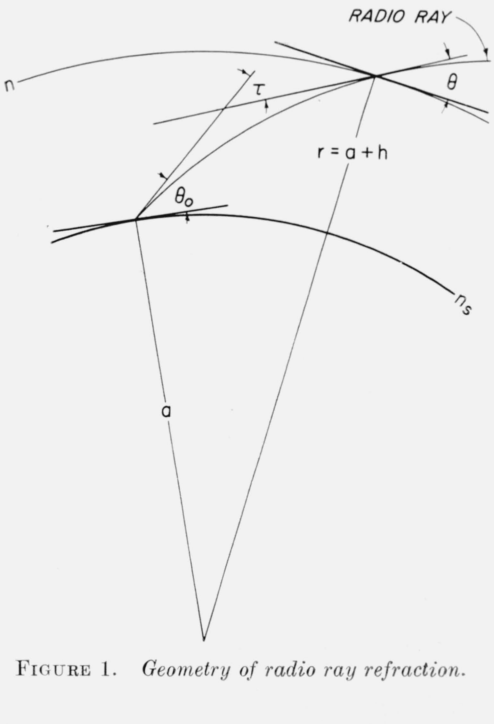

FIGURE 1. *Geometry of radio ray refraction.*

[1] Contribution from Central Radio Propagation Laboratory, National Bureau of Standards, Boulder, Colo.
[2] Figures in brackets indicate the literature references at the end of this paper.

The change of variable:

$$z(h)=\sqrt{z_0^2+ch},$$

$$z_0^2=\frac{ckr_0 \sin^2\theta_0}{2},$$

where

$$k=\frac{1}{1-\gamma r_0 \cos^2\theta_0},$$

results in:

$$\tau \cong N_s \times 10^{-6} \cos\theta_0 \sqrt{2ckr_0} \exp\{z_0^2\} \int_{z_0}^{z(h_\tau)} \exp\{-z^2\}\, dz.$$

This can be given in terms of the error function, [5]

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

as:

$$\tau \cong N_s \times 10^{-4} \cos\theta_0 \sqrt{ck} \exp\{z_0^2\} [\operatorname{erf}\{z(h_\tau)\}-\operatorname{erf}\{z_0\}], \quad (6)$$

where $\sqrt{\pi r_0/2}$, r_0 in kilometers, has been absorbed as a factor of 100 in the constant 10^{-4}.

Equation (6) has the following form for the case of total bending, $h_\tau \geqq 70$ km:

$$\tau(h_\tau \rightarrow \infty) \cong N_s \times 10^{-4} \cos\theta_0 \sqrt{ck} \exp\{z_0^2\}[1-\operatorname{erf}\{z_0\}], \quad (7)$$

which reduces to an interestingly short form for $\theta_0=0$,

$$\tau(h_\tau \rightarrow \infty, \theta_0=0) \cong N_s \times 10^{-4} \sqrt{ck}.$$

3. Accuracy of the Solution

Equation (6) approximates the true value of τ with increasing accuracy as the initial elevation angle approaches $\pi/2$; this can be shown by substituting the asymptotic expansion of the error function [5] in eq (7), with the result that

$$\tau(h_\tau \rightarrow \infty) \cong N_s \times 10^{-6} \cot\theta_0 \left[1-\frac{1}{2z_0^2}+\frac{3}{4z_0^4}-\frac{15}{8z_0^6}+\cdots\right].$$

If z_0 is large enough so that $\frac{1}{2} z_0^2$ may be neglected (approximately for $\theta_0 > 0.3$ radians) then the familiar approximation to the total bending at large elevation angles results [1, 2],

$$\tau(h_\tau \rightarrow \infty) \cong N_s \times 10^{-6} \cot\theta_0, \ \theta_0 > 0.3,$$

which is known to be convergent to the true value of τ as θ_0 approaches $\pi/2$. The extension to the case of $h_\tau < 70$ km yields similar results.

The accuracy of eq (6), for $\theta_0 < 0.3$ radians, has been checked against published results obtained for the CRPL Exponential Reference Atmospheres [4]. The largest errors found, using (4) to obtain H, were about 4 percent, and these were for a profile with a very strong N-gradient; the largest errors for profiles of average N-gradient were about 1.5 percent.

For ease of calculation H may be set constant for all profiles and all h_τ; a value of $H \cong 1$ km results in errors no larger than about 10 percent.

On the other hand if higher accuracy is desirable H may be made a function of initial elevation angle also; adding $f(\theta_0)$ to the value of H as given by (4), where

$$f(\theta_0) \sim \frac{185\,\theta_0}{1+24\,\theta_0}, \ \theta_0 \text{ in radians},$$

will limit the errors to about 1 percent for the CRPL exponential profiles.

In summary, eq (6) represents a relatively accurate and concise formulation for the ray bending in an exponential atmosphere. It has the advantage of being in terms of elementary, well-tabulated functions, and so can be easily programed for solution on digital or analogue computers, and in addition the solution is asymptotic to the correct values for large initial elevation angles.

4. References

[1] W. M. Smart, Spherical astronomy, Chapter III (Cambridge University Press, London, 1931).

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[4] B. R. Bean and G. D. Thayer, CRPL exponential reference atmosphere, NBS Mono. 4, (1959).

[5] National Bureau of Standards, Appl. Math. Series 41, Tables of the error function and its derivative.

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