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Graphical Determination of Radio Ray Bending in an Exponential Atmosphere *

C. F. Pappas, L. E. Vogler, and P. L. Rice

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This paper presents a simple engineering method for calculating the amount of bending undergone by a radio ray passing through an exponential model atmosphere. For any initial takeoff angle and for values of the surface refractivity ranging from 200 to 450, the bending angle τ may be determined as a function of height above the earth's surface, using a few graphs and a few calculations. Indications of the accuracy of the method are given at the end of the paper.

1. Introduction

The course of a radiofrequency electromagnetic wave traveling through the atmosphere is altered by variations of the atmospheric index of refraction, n. These variations, due to changes in vapor pressure, air pressure and temperature, are extremely complex in detail; however, mathematical models of refraction can be constructed which represent an average picture of the variations. This paper considers an "exponential" model (the CRPL Exponential Reference Atmosphere [1][1]), in which the refractivity $(n-1)\times 10^6$ decreases exponentially with height, causing the radio wave to be bent away from its initial direction. The amount of bending is measured by the refraction angle, τ, and is important in such problems as the accurate determination by radar of the range and height of flying objects, the location of extra-terrestrial radio noise sources in radio astronomy, and the analysis of radio communication systems.

Mathematically, τ may be expressed in the following integral form [2,3]

$$\tau=-\cos\theta_0\int_{n_0}^{n_1}(dn/n)[(nr/n_0r_0)^2-\cos^2\theta_0]^{-1/2} \quad (1)$$

where n is the atmospheric index of refraction, n_0 is the value of n at the surface of the earth and r, r_0, and θ_0 are defined in figure 1. The CRPL Exponential Reference Atmosphere is characterized by an index of refraction of the form

$$n=1+(n_0-1)e^{-c_eh}$$

where c_e is the decay constant and h is the altitude above the surface of the earth. For this model atmosphere eq (1) is not integrable in closed form; it can be expanded in series, but the resulting expression is quite complicated for hand calculations. A numerical integration method has been used to compute values of τ by Bean and Thayer; these are listed in reference 1. This method is only practical through the use of a large scale computer.

It might be noted that when θ_0 is large τ may be calculated by a formula which is quite simple and very accurate [4]:

$$\tau=\left(\frac{n_0-1}{n_0}\right)ctn\theta_0(1-e^{-c_eh})\ \text{(radians)}. \quad (2)$$

However, for small θ_0 no simple expression is available for calculations; thus, an engineering method was developed to provide a quick and practical means to obtain τ in this case. This method has the added advantage over the tables in ref [1] in that the N_s of τ is not limited to those listed.

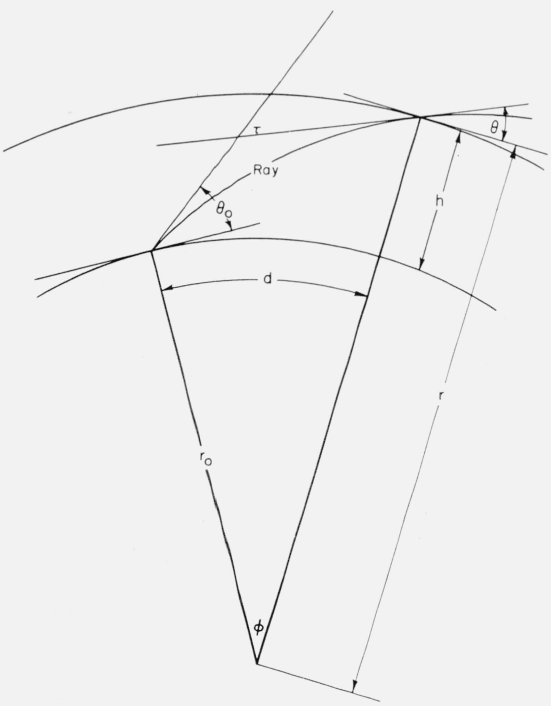

FIGURE 1. *Geometry of radio-ray refraction.*

*Contribution from Central Radio Propagation Laboratory, National Bureau of Standards, Boulder, Colorado.

[1] Figures in brackets indicate the literature references at the end of this paper.

2. Calculation of τ_a

The approximation of τ is denoted by τ_a; the formula is given by

$$\tau_a = f(n_0) \cos \theta_0 q e^{Bf(n_0)+pC} \tag{3}$$

where the terms are explained as follows:

τ_a = the bending approximation in milliradians.

$f(n_0)$ = read from figure 2 for a given N_s. If more accuracy is desired, this value can be computed by

$$f(n_0) = \left[\left(\frac{\pi}{2}\right)\left(\frac{n_0-1}{n_0}\right)(k-1)\right]^{1/2} \times 10^3.$$

n_0 = the index of refraction at the earth's surface.

k = the effective earth's radius factor.

N_s = the surface refractivity.

θ_0 = the initial elevation angle expressed in milliradians.

h_m = the height above the surface of the earth in meters.

q = read from figure 3 for a given h_m and θ_0.

B = read from figure 4 or 5 for a given h_m and θ_0.

p = a correction factor for N_s which is read from figure 6 for a given N_s.

C = a height correction factor which is obtained from figure 7 for a given h_m.

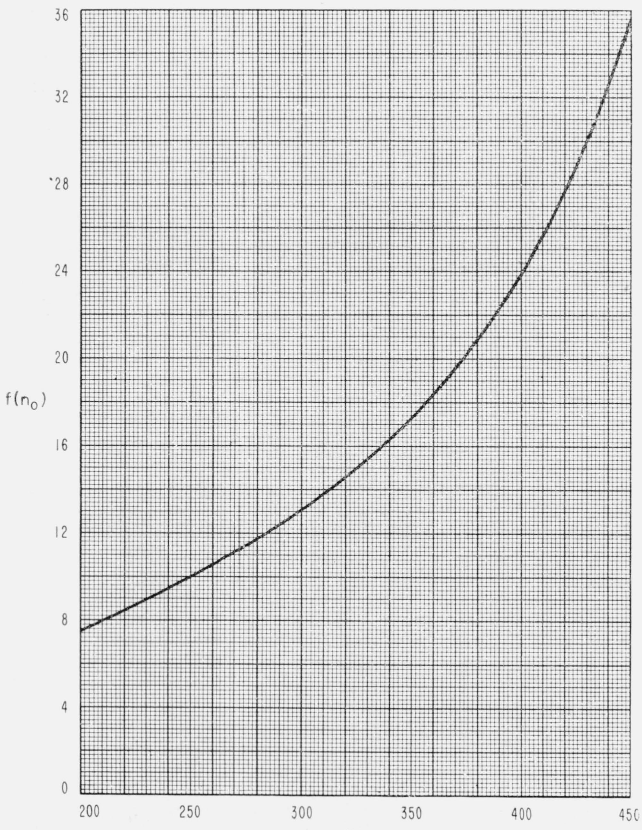

FIGURE 2. f(n_0) *versus* N_s.

Two examples of the computation of τ_a are included to illustrate the use of the τ_a formula. One example is for an N_s of 252.9 and the second is an example using an N_s of 404.9 in which the pC correction factor has an effect.

Example 1. (Calculation of τ_a with $N_s \leq 344.5$)

Given: $N_s = 252.9$
$\theta_0 = 40$ milliradians
$h_m = 500$ meters

Find: $\tau_a = f(n_0) \cos \theta_0 q e^{Bf(n_0)+pC}$

$f(n_0) = 10.06$ (fig. 2)
$\cos \theta_0 = .99920$
$q = .0385$ (fig. 3)
$B = .00077$ (fig. 4)
$Bf(n_0) = .0077$
$p = 0$ (fig. 6)
$pC = 0$
$e^{Bf(n_0)+pC} = 1.0077$
$\tau_a = .39$ milliradians
$\tau = .38$ milliradians[2]

Example 2. (Calculation of τ_a with $N_s > 344.5$)

Given: $N_s = 404.9$
$\theta_0 = 200$ milliradians
$h_m = 30$ meters

Find: $\tau_a = f(n_0) \cos \theta_0 q e^{Bf(n_0)+pC}$
$f(n_0) = 24.70$ (fig. 2)
$\cos \theta_0 = .98007$
$q = .000472$ (fig. 3)
$B = .00117$ (fig. 4)

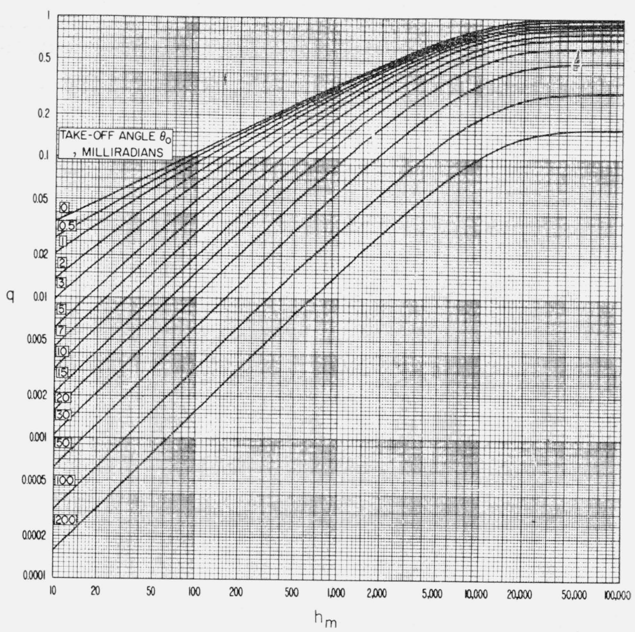

FIGURE 3. q *versus* h_m.

[2] Obtained by numerical integration.

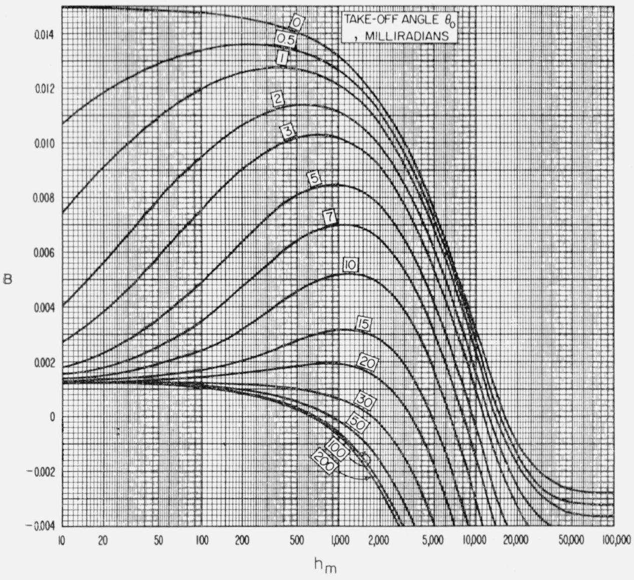

FIGURE 4. B *versus* h_m.

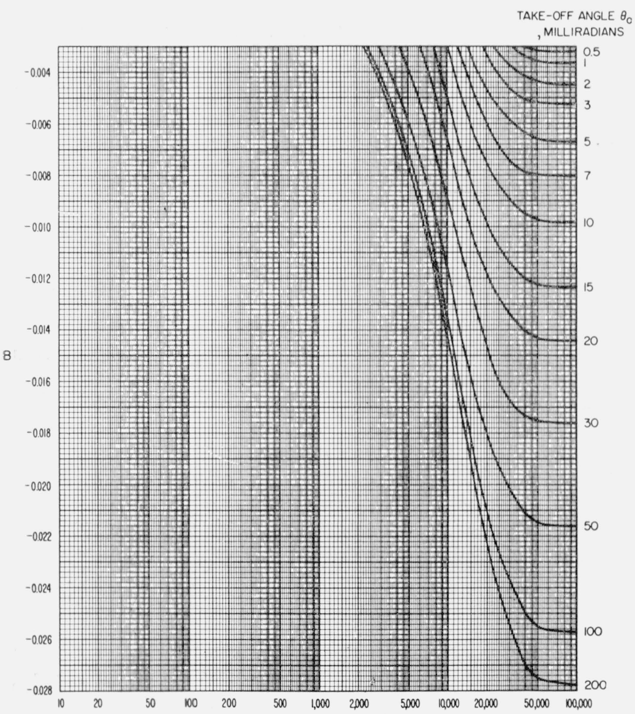

FIGURE 5. B *versus* h_m.

$$
\begin{aligned}
Bf(n_0) &= .0289 \\
p &= .30 && \text{(fig. 6)} \\
C &= -.125 && \text{(fig. 7)} \\
pC &= -.0375 \\
Bf(n_0)+pC &= -.0086 \\
e^{Bf(n_0)+pC} &= .9914 \\
\tau_a &= .0113 \text{ milliradians} \\
\tau &= .0113 \text{ milliradians}^2
\end{aligned}
$$

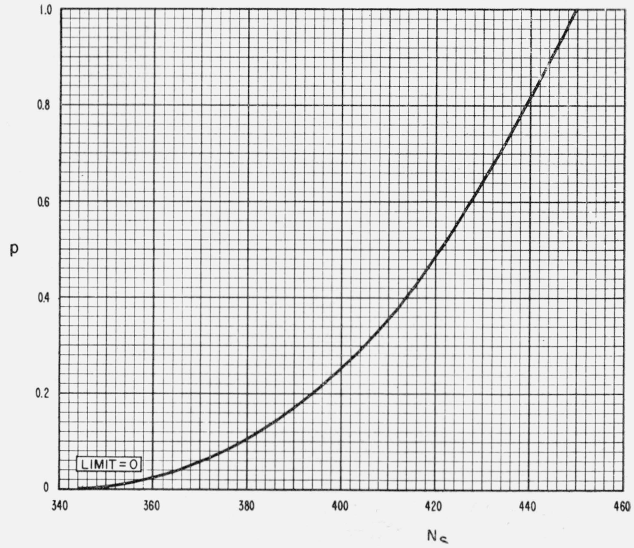

FIGURE 6. N_s *correction factor.*

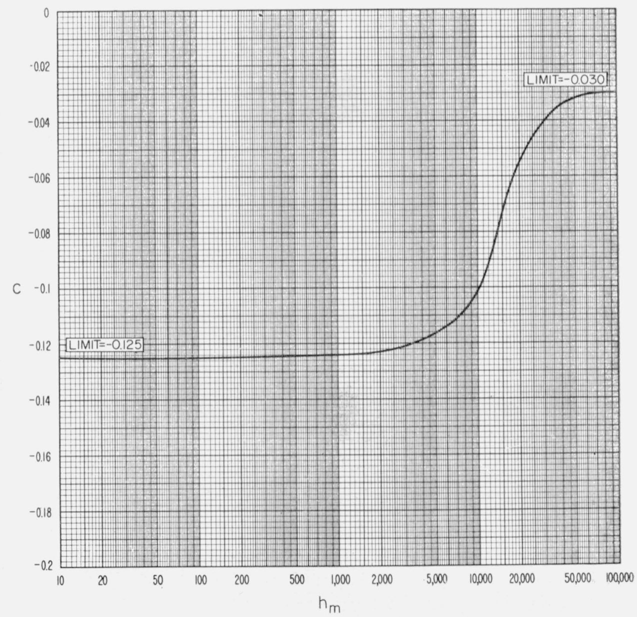

FIGURE 7. *Height correction factor.*

3. Derivation of τ_a formula

By plotting τ versus h for many different values of θ_0 and n_0, it was decided that the simplest form that could be assumed for τ to obtain the accuracy desired was

$$\tau = f(n_0) \cos \theta_0 e^{A+Bf(n_0)} \tag{4}$$

The range of N_s considered lies between 200 and 450 since values outside this range rarely occur in actual practice.

Using two values of N_s, 200 and 344.5, and τ's obtained by the numerical integration procedure of ref [1], a least squares fit of

$$\ln\left(\frac{\tau}{f(n_0)\cos\theta_0}\right)$$

was made to obtain values of A and B for a given h_m and θ_0.

Values of e^A, denoted by q, were computed and graphed for h_m and given θ_0 (see fig. 3). The B values were also plotted versus h_m for given θ_0 (see figs. 4 and 5). It may be noted that q and B are not graphed for height values less than 10 m; at the upper range of height, a limit for q and B is approached and reached for a given θ_0.

Using the A and B values it was found that for an N_s greater than 344.5 a correction factor was needed to modify $qe^{Bf(n_0)}$ (or $e^{A+bf(n_0)}$) so that the $\ln(\tau/f(n_0)\cos\theta_0)$ and, consequently, the τ error were within an acceptable range. The height correction C was obtained from the difference between $\ln(\tau/f(n_0)\cos\theta_0)$ and $qe^{Bf(n_0)}$ for an N_s of 450 and plotted for graphical use (see fig. 7). Through further calculations and comparisons the relationship of N_s to the height correction C was determined for N_s greater than 344.5, and less than 450. This relationship was the basis for the N_s correction p which was plotted versus N_s (see fig. 6). Inclusion of this additional correction element results in an expression $qe^{Bf(n_0)+pC}$ for an N_s greater than 344.5. Since p equals zero when N_s is less than or equal to 344.5,

$$\tau_a = f(n_0)\cos\theta_0\, qe^{Bf(n_0)+pC}$$

becomes the general form for the simplified calculation of τ.

4. Accuracy of τ_a Method

In checking out the simplified calculation method various τ_a were compared with values of the CRPL exponential reference atmosphere τ for the same h_m and θ_0. Of the values computed τ_a showed the smallest absolute error at the lower heights and smaller N_s. The largest errors calculated were for N_s of 450, the maximum absolute error being 0.53 milliradians, with a maximum relative error of 6.8 percent.

Below is a table of computed values which gives indication of the range of absolute and percent error for several values of N_s. It will be noted that error values for an N_s of 200 and 344.5 were not included in this list since these values of N_s were used for the least squares fit and were considered to have less error than the N_s listed.

Only error values for τ_a greater than 1.0 milliradian were (arbitrarily) included in table 1. For τ_a less than 1.0 milliradian the absolute error is quite low, but the percent error can be high since τ_a is so small. This can give a somewhat distorted picture since a small τ_a may have an error of only 0.0001 milliradian and still be in error by greater than 3 percent.

TABLE 1.—*Range of error for $\tau_a > 1.0$ mr*

N_s	Range of absolute error	Range of percent error
252.9	0.0006 to 0.0739	0.01% to 2.11%
313.0	0 to .0918	0 to 1.91%
377.2	.0013 to .1507	.01% to 1.53%
404.9	0 to .2539	0 to 1.86%
450.0	.0024 to .5313	.22% to 6.79%

5. Explanation of Symbols

B=figures 4 and 5.

C=height correction factor; figure 7,

c_e=decay constant; see ref [1],

$f(n_0)$=figure 2,
$$=\left[\left(\frac{\pi}{2}\right)\left(\frac{n_0-1}{n_0}\right)(k-1)\right]^{1/2}\times 10^3,$$

h=altitude above the surface of the earth,

h_m=altitude above the surface of the earth in meters,

k=effective earth's radius factor
$$=\frac{n_0}{n_0-r_0c_e(n_0-1)},$$

n=atmospheric index of refraction
$=1+(n_0-1)e^{-c_e h}$ (CRPL Exponential Reference Atmosphere),

n_0=index of refraction at the earth's surface $=n(h=0)$,

N_s=surface refractivity $=(n_0-1)\times 10^6$,

p=N_s correction factor; figure 6,

ϕ=angle at center of the earth (see fig. 1) $=\theta+\tau-\theta_0$.

q=figure 3,

r=radial distance from the center of the earth,

r_0=distance from the center to the surface of the earth,

τ=bending
$$=-\cos\theta_0\int_{n_0}^{n_1}(dn/n)\,[(nr/n_0r_0)^2-\cos^2\theta_0]^{-1/2}$$

τ_a=bending approximation in milliradians,
$=f(n_0)\cos\theta_0\, q\, e^{Bf(n_0)+pC}$

θ=local elevation angle
$$=\cos^{-1}\left(\frac{n_0r_0\cos\theta_0}{nr}\right), \text{ (using Snell's law)}$$

θ_0=initial elevation or takeoff angle.

6. References

[1] B. R. Bean and G. D. Thayer, CRPL exponential reference atmosphere, NBS Monograph 4 (October 29, 1959).

[2] D. E. Kerr, Propagation of short radio waves, Radiation Laboratories Series, Vol 13, p. 49, (McGraw-Hill Book Co., New York, N.Y., 1951).

[3] B. R. Bean and G. D. Thayer, Models of the atmospheric radio refractive index, Proc. IRE **47**, No. 5, 740 (May 1959).

[4] W. M. Smart, Spherical astronomy, ch. III, Cambridge University Press, London (1931).

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