A Priori Bounds in the First Boundary Value Problem in Elasticity

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In this paper we obtain bounds for the strain energy, and pointwise bounds for the solution of the first boundary value problem in the equations of elasticity. The inequalities derived are such that the Rayleigh-Ritz technique may be used to improve the bounds.

1. Introduction

In a recent paper [2] [2] the authors gave a method for obtaining bounds in the Dirichlet problem for second-order elliptic partial differential equations. Pointwise bounds were obtained which involved, in addition to the approximating function, only undifferentiated data of the problem. The inequalities were of a priori type, hence only regularity conditions were imposed on the approximating functions.

In this paper we obtain bounds for the strain energy, and pointwise bounds for the solution of the first boundary value problem in the equations of elasticity. Other methods for obtaining bounds, which are based on known variational principles, have been given in the literature (see, for instance, Prager and Synge [9], Diaz and Greenberg [3], Synge [10], and others).

A method based on a priori inequalities was proposed by Payne and Weinberger [8]. Their method involved the introduction of a somewhat complicated unit-tangent vector, and pointwise bounds involved differentiated data. We derive a method which yields pointwise bounds which involve the undifferentiated data of the problem.

In the problem under consideration the displacement vector with components u_i satisfies the following system of second-order equations in some N-dimensional bounded domain D,

$$L_i(u) = u_{i,jj} + \alpha u_{j,ji} = F_i \tag{1.1}$$

where α involves the elastic constants λ and μ ($\alpha = (\lambda + \mu)/\mu = (1 - 2\sigma)^{-1}$, σ denoting Poisson's ratio), and F_i is proportional to the body force. In (1.1) $_{,ji}$ denotes $\partial^2/\partial x_j \partial x_i$ and a repeated index indicates summation from 1 to N, where N denotes the number of dimensions. By specializing to two or three dimensions we can treat the two physically interesting cases. In addition to the differential equation (1.1) the displacement components satisfy the condition

$$u_i = f_i \tag{1.2}$$

on the boundary C of D. Regions which extend to infinity may also be treated if suitable conditions are imposed at infinity.

The strain energy $E(u,u)$ for problem (1.1), (1.2) is defined as:

$$2\mu^{-1}E(u,u) = \int_D \left[\frac{1}{2}(u_{i,j}+u_{j,i})(u_{i,j}+u_{j,i}) + (\alpha-1)(u_{j,j})^2 \right] dV. \tag{1.3}$$

Our problem then is that of obtaining upper and lower bounds for $E(u,u)$ and for the value of $u_i(P)$ at a point P in D. As we shall see, these bounds follow immediately once we have established certain a priori bounds. Let us suppose that we are able to establish the following inequality for sufficiently smooth functions w_i:

$$E(w,w) \leq K_1 \oint_C \eta_1 w_i w_i dS + K_2 \oint_C \eta_2 \, \mathrm{grad}_s w_i \cdot \mathrm{grad}_s w_i dS + K_3 \int_D L_i(w) L_i(w) dV \tag{1.4}$$

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where η_1 and η_2 are given positive functions on C, and K_1, K_2, and K_3 are explicitly determined. Here grad$_s$ denotes the tangential projection of the gradient. Now if we set

$$w_i = u_i - \phi_i, \tag{1.5}$$

the right hand side of (1.4) will involve only the data of (1.1) and (1.2) and the approximating function ϕ_i. We would then have an a priori bound for $E(u-\phi, u-\phi)$ which could be made small if the functions ϕ_i could be chosen to approximate the data closely in the mean square sense. But by the triangle inequality

$$\sqrt{E(\phi,\phi)} - \sqrt{E(u-\phi, u-\phi)} \leq \sqrt{E(u,u)} \leq \sqrt{E(\phi,\phi)} + \sqrt{E(u-\phi, u-\phi)}. \tag{1.6}$$

Hence knowledge of the a priori inequality (1.4) would give upper and lower bounds for $E(u,u)$. The inequality (1.4) with $w_i = u_i - \phi_i$ is suitable for application of the Rayleigh-Ritz technique for improving the bounds.

The pointwise bounds will also follow from appropriate a priori inequalities. We concern ourselves then in this paper with the establishment of the necessary a priori inequalities.

2. Auxiliary Inequalities

In this section we employ some generalized Green's identities to derive certain inequalities which will be used in sec. 3. We consider first a function h harmonic in D. As indicated in [8, eq 2.4] the function h satisfies the following identity:

$$\oint_C [f^k n_k h_{,i} h_{,i} - 2f^i n_j h_{,i} h_{,j}] dS = \int_D [f^k_{,k} \delta_{ij} - 2f^i_{,j}] h_{,i} h_{,j} dV \tag{2.1}$$

where the f^k are components of a piecewise continuous vector field and δ_{ij} denotes the Kronecker delta. The n_j denote the components of the unit normal directed outward from D on C. In [8, eq 2.4] we have taken $a^{ij} \equiv \delta_{ij}$ and used the fact that h is harmonic. By breaking up the derivatives of h in the boundary integral into normal and tangential components we obtain

$$\oint_C f^k n_k \left[|\mathrm{grad}_s h|^2 - \left(\frac{\partial h}{\partial n}\right)^2 \right] dS = \int_D [f^k_{,k} \delta_{ij} - 2f^i_{,j}] h_{,i} h_{,j} dV + 2\oint_C f^i \, \mathrm{grad}_s h \cdot \mathrm{grad}_s x_i \frac{\partial h}{\partial n} dS \tag{2.2}$$

where $\frac{\partial h}{\partial n}$ denotes the normal derivative of h on C. If we now choose the f^k in such a way that $t = f^k n_k > 0$ on C, we obtain the inequality

$$\oint_C t \left(\frac{\partial h}{\partial n}\right)^2 dS \leq \oint_C t \, |\mathrm{grad}_s h|^2 dS - 2\oint_C f^i \, \mathrm{grad}_s h \cdot \mathrm{grad}_s x_i \frac{\partial h}{\partial n} dS + aD(h,h). \tag{2.3}$$

In (2.3) we have used the notation

$$D(h,h) = \int_D h_{,i} h_{,i} dV \tag{2.4}$$

for the Dirichlet integral, and the inequality

$$\int_D [2f^i_{,j} - f^k_{,k} \delta_{ij}] h_{,i} h_{,j} dV \leq aD(h,h). \tag{2.5}$$

The constant a is clearly any upper bound for the largest eigenvalue of the coefficient matrix $2f^i_{,j} - f^k_{,k} \delta_{ij}$. Then from Green's first identity, i.e.,

$$D(h,h) = \oint_C h \frac{\partial h}{\partial n} dS, \tag{2.6}$$

we obtain from (2.3)

$$\oint_C t \left(\frac{\partial h}{\partial n}\right)^2 dS \leq \oint_C t \, |\text{grad}_s \, h|^2 dS - \oint_C [2f^i \, \text{grad}_s \, h \cdot \text{grad}_s \, x_i - ah] \frac{\partial h}{\partial n} dS. \tag{2.7}$$

Now from Schwarz's inequality we have

$$\oint_C t \left(\frac{\partial h}{\partial n}\right)^2 dS \leq \oint_C t \, |\text{grad}_s \, h|^2 dS + \left\{\oint_C t^{-1} [2f^i \, \text{grad}_s \, h \cdot \text{grad}_s \, x_i - ah]^2 dS \cdot \oint_C t \left(\frac{\partial h}{\partial n}\right)^2 dS\right\}^{1/2}, \tag{2.8}$$

from which we obtain, by completing the square,

$$\left\{\oint_C t \left(\frac{\partial h}{\partial n}\right)^2 dS\right\}^{1/2} \leq \frac{1}{2}\left\{\oint_C t^{-1}[2f^i \, \text{grad}_s \, h \cdot \text{grad}_s \, x_i - ah]^2 dS\right\}^{1/2}$$

$$+ \left\{\oint_C t \, |\text{grad}_s \, h|^2 dS + \frac{1}{4}\oint_C t^{-1}[2f^i \, \text{grad}_s \, h \cdot \text{grad}_s \, x_i - ah]^2 dS\right\}^{1/2}, \tag{2.9}$$

or, using the arithmetic-geometric mean inequality, the cruder inequality

$$\oint_C t \left(\frac{\partial h}{\partial n}\right)^2 dS \leq \oint_C t^{-1}[2f^i \, \text{grad}_s \, h \cdot \text{grad}_s \, x_i - ah]^2 dS + 2\oint_C t \, |\text{grad}_s \, h|^2 dS. \tag{2.10}$$

This inequality bounds the normal derivative of a harmonic function h in terms of h and its tangential derivatives on C. (Instead of (2.10) the somewhat better though more involved inequality [8, eq 3.17], with $a^{ij} = \delta_{ij}$, could be used.) In deriving (2.10) we have used the procedures employed in [8]. For completeness and because the case of the harmonic function was not given explicitly in the previous paper we have again presented the derivation of (2.10).

Now let ψ_i be the ith component of a vector function which possesses piecewise continuous second derivatives in D and vanishes on C. Then consider the following identity:

$$2\int_D f^k \psi_{i,k} L_i(\psi) dV = 2\oint_C f^k \psi_{i,k}[\psi_{i,j}n_j + \alpha\psi_{j,j}n_i] dS + \int_D f^k_{,k}[\psi_{i,j}\psi_{i,j} + \alpha\psi_{i,i}\psi_{j,j}] dV$$

$$- 2\int_D [f^k_{,j}\psi_{i,k}\psi_{i,j} + \alpha f^k_{,i}\psi_{i,k}\psi_{j,j}] dV - \oint_C f^k n_k[\psi_{i,j}\psi_{i,j} + \alpha\psi_{i,i}\psi_{j,j}] dS. \tag{2.11}$$

Since ψ_i vanishes on C the first and last integrals on the right combine to give (upon rearranging of (2.11))

$$\oint_C f^k n_k[\psi_{i,j}\psi_{i,j} + \alpha\psi_{i,i}\psi_{j,j}] dS = 2\int_D f^k \psi_{i,k} L_i(\psi) dV$$

$$- \int_D f^k_{,k}[\psi_{i,j}\psi_{i,j} + \alpha\psi_{i,i}\psi_{j,j}] dV + 2\int_D [f^k_{,j}\psi_{j,k}\psi_{i,j} + \alpha f^k_{,i}\psi_{i,k}\psi_{j,j}] dV. \tag{2.12}$$

Since f^i has bounded first derivatives we may easily obtain, for α positive,

$$2\int_D [f^k_{,j}\psi_{i,k}\psi_{i,j} + \alpha f^k_{,i}\psi_{i,k}\psi_{j,j}] dV - \int_D f^k_{,k}[\psi_{i,j}\psi_{i,j} + \alpha\psi_{i,i}\psi_{j,j}] dV \leq b\int_D [\psi_{i,j}\psi_{i,j} + \alpha\psi_{i,i}\psi_{j,j}] dV, \tag{2.13}$$

where the constant b may be easily determined. (The physically interesting values of α lie in the interval $(1/3, \infty)$.)

Note that for ψ_i vanishing on C the integral on the right of (2.13) is proportional to the strain energy $E(\psi, \psi)$. Also the quantity inside the square bracket on the left of (2.12) is proportional to the sum of the squares of the surface tractions. By the arithmetic-

geometric mean inequality and the fact that α is positive, we have then

$$\oint_C t[\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dS \leq \left[b+\frac{[f^k f^k]_{\max}}{\beta}\right]\int_D [\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dV+\beta\int_D L_i(\psi)L_i(\psi)dV \qquad (2.14)$$

for any positive β. Now since ψ_i vanishes on C we have

$$\int_D [\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dV=-\int_D \psi_i L_i(\psi)dV \qquad (2.15)$$

or by Schwarz's inequality

$$\frac{\int_D [\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dV}{\int_D \psi_i\psi_i dV} \leq \frac{\int_D L_i(\psi)L_i(\psi)dV}{\int_D [\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dV}. \qquad (2.16)$$

However, for $\alpha > 0$

$$\frac{\int_D [\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dV}{\int_D \psi_i\psi_i dV} \geq \frac{\int_D \psi_{i,j}\psi_{i,j}dV}{\int_D \psi_i\psi_i dV} \geq \lambda_1, \qquad (2.17)$$

where λ_1 is the first eigenvalue in the fixed membrane problem for D. Lower bounds for λ_1 are easily derivable from the Faber-Krahn inequality [5], [6], or from monotony principles. Thus from (2.16) and (2.17) we have

$$\int_D \psi_i\psi_i dV \leq \frac{1}{\lambda_1}\int_D [\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dV \leq \frac{1}{\lambda_1^2}\int_D L_i(\psi)L_i(\psi)dV. \qquad (2.18)$$

The insertion of (2.18) into (2.14) then yields

$$\oint_C t[\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dS \leq [b/\lambda_1+2\{[f^k f^k]_{\max}/\lambda_1\}^{\frac{1}{2}}]\int_D L_i(\psi)L_i(\psi)dV, \qquad (2.19)$$

where we have made the optimal choice for β.

3. A Priori Inequalities

Let w_i have piecewise continuous second derivatives in D. We seek an a priori bound for $E(w,w)$. To this end we decompose w_i as

$$w_i=\overline{w}_i+w_i' \qquad (3.1)$$

where

$$L_i(\overline{w})=0 \qquad (3.2)$$

in D and

$$\overline{w}_i=w_i \qquad (3.3)$$

on C. The function w_i' then satisfies

$$L_i(w')=L_i(w) \qquad (3.4)$$

in D and

$$w'_i=0 \qquad (3.5)$$

on C. Then clearly

$$E(w,w)=E(\overline{w},\overline{w})+E(w_i'w') \qquad (3.6)$$

since $\overline{w}$ and w' are orthogonal in E-norm (i.e., $E(\overline{w},w')=0$). We derive first a bound for

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$E(w',w')$.   From the fact that $w'_i=0$ on C we have from (2.18), with ψ_i replaced by w'_i,

$$2\mu^{-1}E(w',w')\leq\frac{1}{\lambda_1}\int_D L_i(w')L_i(w')\,dV=\frac{1}{\lambda_1}\int_D L_i(w)L_i(w)\,dV. \tag{3.7}$$

Just as one establishes the Dirichlet principle for harmonic functions it is possible in bounding $E(\overline{w},\overline{w})$ to show that

$$E(\overline{w},\overline{w})\leq E(H,H) \tag{3.8}$$

where H is any vector which satisfies the same boundary values as $\overline{w}$ on C.   Let H_i be the components of the harmonic vector which takes on the same boundary values as $\overline{w}_i$.   Then

$$E(\overline{w},\overline{w})\leq\overline{\mu}\int_D H_{i,j}H_{i,j}\,dV, \qquad \overline{\mu}=\begin{cases}\mu[2+N(\alpha-1)], & \alpha>1 \\ 2\mu & , \alpha\leq 1\end{cases}$$

$$=\overline{\mu}\oint_C H_i\frac{\partial}{\partial n}H_i\,dS, \tag{3.9}$$

the latter equality following from Green's theorem and the fact that H_i is harmonic.   Now choose f^i in such a way that $t=f^in_i>0$ on C.   Then by the arithmetic-geometric mean inequality we have

$$E(\overline{w},\overline{w})\leq\frac{\overline{\mu}}{\gamma}\oint_C t^{-1}H_iH_i\,dS+\mu\gamma\oint_C t\frac{\partial H_i}{\partial n}\frac{\partial H_i}{\partial n}\,dS \tag{3.10}$$

for any positive γ.   Now since the H_i are harmonic we may use inequality (2.10) for vectors to obtain

$$\oint_C t\frac{\partial H_i}{\partial n}\frac{\partial H_i}{\partial n}\,dS\leq\oint_C t^{-1}[2f^j\operatorname{grad}_s H_i\cdot\operatorname{grad}_s x_j-aH_i][2f^k\operatorname{grad}_s H_i\cdot\operatorname{grad}_s x_k-aH_i]\,dS$$

$$+2\oint_C t\operatorname{grad}_s H_i\cdot\operatorname{grad}_s H_i\,dS. \tag{3.11}$$

Now since $H_i=\overline{w}_i=w_i$ on C we have upon combining (3.6), (3.7), (3.10), and (3.11) the desired bound

$$2\overline{\mu}^{-1}E(w,w)\leq\frac{1}{\gamma}\oint_C t^{-1}w_iw_i\,dS+\gamma\Big\{2\oint_C t\operatorname{grad}_s w_i\cdot\operatorname{grad}_s w_i\,dS$$

$$+\oint_C t^{-1}[2f^j\operatorname{grad}_s w_i\cdot\operatorname{grad}_s x_j-aw_i][2f^k\operatorname{grad}_s w_i\cdot\operatorname{grad}_s x_k-aw_i]\,dS\Big\}+\frac{\mu/\overline{\mu}}{\lambda_1}\int_D L_i(w)L_i(w)\,dV \tag{3.12}$$

which can clearly be put in the form (1.4) if so desired.

We now derive a bound for the integral of the sum of the squares of the displacement components w_i over D.   This inequality will be used in sec. 5 to derive pointwise bounds for w_i.   We seek then a bound for $\int_D w_iw_i\,dV$.

To this end we introduce the auxiliary vector χ with components χ_i defined by

$$L_i(\chi)=w_i \tag{3.13}$$

in D and

$$\chi_i=0 \tag{3.14}$$

on C.   Then

$$\int_D w_iw_i\,dV=\int_D w_iL_i(\chi)\,dV=\oint_C w_i[\chi_{i,j}n_j+\alpha\chi_{i,i}n_i]\,dS+\int_D \chi_iL_i(w)\,dV. \tag{3.15}$$

By Schwarz's inequality for vectors we have

$$\int_D w_i w_i dV = \int_D L_i(\chi) L_i(\chi) dV \le \left\{ \oint_C t^{-1}[w_i w_i + (w_i n_i)^2] dS \oint_C t[\chi_{i,i}\chi_{i,i} \right.$$

$$\left. + \alpha\chi_{i,i}\chi_{i,i}] dS \right\}^{\frac{1}{2}} + \left\{ \int_D \chi_i \chi_i dV \int_D L_i(w) L_i(w) dV \right\}^{\frac{1}{2}}. \quad (3.16)$$

Since χ_i vanishes on C we may use (2.18) and (2.19) with ψ_i replaced by χ_i to obtain

$$\left[\int_D w_i w_i dV \right]^{\frac{1}{2}} \le \left\{ [b/\lambda_1 + 2([f^k f^k]_{\max}/\lambda_1)^{\frac{1}{2}}] \oint_C t^{-1}[w_i w_i \right.$$

$$\left. + (w_i n_i)^2] dS \right\}^{\frac{1}{2}} + \left\{ \frac{1}{\lambda_1^2} \int_D L_i(w) L_i(w) dV \right\}^{\frac{1}{2}}. \quad (3.17)$$

If we should now set $w_i = u_i - \phi_i$ where u_i is the solution of (1.1), (1.2) and ϕ_i is an approximating function then the right hand side of (3.17) would involve u_i only through its data. We employ this inequality in the next section for obtaining pointwise bounds for w_i.

4. Pointwise Bounds

In order to obtain pointwise bounds we again decompose w_i as in eq (3.1). Pointwise bounds for $\overline{w}_i$ are then derived by making use of a mean-value theorem for the solution of the homogeneous equations of elasticity. Such a mean-value theorem was given in the three-dimensional case by Aquaro [1] and Synge [10]. The analogous N-dimensional result was obtained by Diaz and Payne [4].

Let $S(r)$ be a sphere of radius r and surface $\sum(r)$ with center at the point at which pointwise bounds are sought. We choose the origin of a spherical-coordinate system at this point, and require that $S(r)$ be contained in D.

In what follows whenever a Greek index is repeated the summation convention is not to be used, i.e., $u_{\alpha,i} u_{\alpha,i}$ will denote the square of the gradient of a single-displacement component u_α.

From the mean-value theorem [4, eq 3.9]

$$r^{N+1}\overline{w}_\beta(0) = \frac{N}{2(N+\alpha)w_N} \oint_{\Sigma(r)} [(2-\alpha)\delta_{\beta j} r^2 + (N+2)\alpha x_\beta x_j]\overline{w}_j dS. \quad (4.1)$$

If we now integrate (4.2) with respect to r from the origin to R, with R so chosen that $S(R)$ is still contained in D, we obtain

$$\overline{w}_\beta(0) = \frac{N(N+2)}{2(N+\alpha)R^{N+2}w_N} \int_{S(R)} [(2-\alpha)\delta_{\beta j} r^2 + (N+2)\alpha x_\beta x_j]\overline{w}_j dV. \quad (4.2)$$

An application of Schwarz's inequality then yields

$$|\overline{w}_\beta(0)|^2 \le \frac{N}{N+4} \left[\frac{(N+2)}{2(N+\alpha)} \right]^2 \left\{ \frac{N(2-\alpha)^2 + 2\alpha(2-\alpha)(N+2) + (N+2)^2\alpha^2}{w_N R^N} \right\} \int_D \overline{w}_j \overline{w}_j dV$$

$$\equiv A(N)R^{-N} \int_D \overline{w}_j \overline{w}_j dV. \quad (4.3)$$

By our construction we see that R is actually the distance from the origin to the boundary.

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Now from (3.17), since $L_i(\overline{}) \equiv 0$ it follows that

$$|\overline{w}_\beta(0)|^2 \le A(N)K_1^2 R^{-N} \oint_C t^{-1}[\overline{w}_i\overline{w}_i + (\overline{w}_i n_i)^2]dS$$

$$= A(N)K_1^2 R^{-N} \oint_C t^{-1}[w_i w_i + (w_i n_i)^2]dS \tag{4.4}$$

where

$$K_1 = [b/\lambda_1 + 2([f^k f^k]_{\max}/\lambda_1)^{1/2}]. \tag{4.5}$$

In order to obtain bounds for $w'_\beta(0)$ we introduce the Kelvin fundamental solution $z^k_\beta(O,Q)$ which denotes the components of displacement in the kth direction corresponding to a point load at the origin in the βth direction, i.e.,

$$z^k_\beta = \begin{cases} \dfrac{1}{4\pi\mu(\alpha+1)}[(\alpha+2)\log(1/r)\delta_{\beta k} + \alpha x_\beta x_k r^{-2}], & N=2 \\[2ex] \dfrac{1}{2\mu(N-2)(\alpha+1)w_N}[(\alpha+2)r^{-(N-2)}\delta_{\beta k} + (N-2)\alpha x_\beta x_k r^{-N}], & N>2. \end{cases} \tag{4.6}$$

For $N=2, 3$ these expressions are given in Love [7]. By the analogue of Green's third identity for the equations of elasticity we have

$$w'_\beta(0) = \oint_C z^k_\beta[w'_{k,j}n_j + \alpha w'_{j,j}n_k]dS + \int_D z^k_\beta L_k(w')dV. \tag{4.7}$$

Since $L_k(w') \equiv L_k(w)$ in D, for the boundary-value problem under consideration $L_k(w')$ will be known. Hence by Schwarz's inequality

$$|w'_\beta(0)|^2 \le \left\{\left[\oint_C t^{-1}(z^k_\beta z^k_\beta + [z^k_\beta n_k]^2)dS \oint_C t(w'_{i,j}w'_{i,j} + \alpha w'_{i,i}w'_{j,j})dS\right]^{1/2} + \left|\int_D z^k_\beta L_k(w)dV\right|\right\}^2. \tag{4.8}$$

Now since w'_i vanishes on C we may use inequality (2.19) to bound the second boundary integral on the right of (4.8); i.e.,

$$|w'^1_\beta(0)|^2 \le \left\{\left[K_1\oint_C t^{-1}(z^k_\beta z^k_\beta + [z^k_\beta n_k]^2)dS \int_D L_j(w)L_j(w)dV\right]^{1/2} + \left|\int_D z^k_\beta L_k(w)dV\right|\right\}^2. \tag{4.9}$$

Now since

$$|w_\beta(0)|^2 \le 2\{|\overline{w}_\beta(0)|^2 + |w'_\beta(0)|^2\} \tag{4.10}$$

we have by inequalities (4.4) and (4.10) a bound for any one of the displacement components at the origin.

We may also obtain pointwise bounds for the derivatives of u_β at points in D. Again we first obtain an a priori bound for the derivative of a general function w_β and interpret our results for $w_\beta = u_\beta - \phi_\beta$. As before w_β is decomposed as $w_\beta = \overline{w}_\beta + w'_\beta$. The pointwise bounds for the derivatives of $\overline{w}_\beta$ are again obtained by use of mean-value inequalities while the fundamental solution (4.6) is used to obtain derivatives of w'_β. Diaz and Payne [4, eq 4.14] have shown that the stress component $\overline{\tau}_{\beta\gamma}$ defined as

$$\overline{\tau}_{\beta\gamma} = \mu[(\overline{w}_{\beta,\gamma} + \overline{w}_{\gamma,\beta}) + (\alpha-1)\overline{w}_{k,k}\delta_{\beta\gamma}] \tag{4.11}$$

satisfies the mean value theorem

$$r^{N+3}\overline{\tau}_{\beta\gamma}(0) = \frac{\mu N}{w_N(N+2\alpha+2)}\left[(N+2)(1-\alpha)\oint_{\Sigma(r)} r^2(\overline{w}_\beta x_\gamma + \overline{w}_\gamma x_\beta)dS + \alpha(N+2)(N+4)\oint_{\Sigma(r)} x_\beta x_\gamma x_k\overline{w}_k dS\right.$$

$$\left. + (2\alpha^2 - 4\alpha - N - 2)\delta_{\beta\gamma}\oint_{\Sigma(r)} r^2\overline{w}_k x_k dS\right]. \tag{4.12}$$

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Upon integrating with respect to r as before from 0 to R we obtain

$$R^{N+4}\overline{\tau}_{\beta\gamma}(0) = \frac{\mu N(N+4)}{w_N(N+2\alpha+2)} \int_{S(R)} [(N+2)(1-\alpha)r^2(x_\beta\delta_{\gamma k} + x_\gamma\delta_{\beta R})$$
$$+ \alpha(N+2)(N+4)x_\beta x_\gamma x_k + (2\alpha^2 - 4\alpha - N - 2)\delta_{\beta\gamma}x_k]\overline{w}_k dV, \quad (4.13)$$

where $S(R)$ again is required to lie interior to D. An application of Schwarz's inequality then yields

$$|\overline{\tau}_{\beta\gamma}(0)|^2 \le K_{\beta\gamma}(N)R^{-(N+2)} \int_{S(R)} \overline{w}_k\overline{w}_k dV, \quad (4.14)$$

where $K_{\beta\gamma}(N)$ is an easily computable function of N.

To obtain a bound for $\tau'_{\beta\gamma}(0)$ we differentiate (4.7) to obtain

$$\tau'_{\beta\gamma}(0) = \oint_C \tau_{\beta\gamma}(z^k)[w'_{kj}n_j + \alpha w'_{j,j}n_k]dS + \int_D \tau_{\beta\gamma}(z^k)L_k(w')dV \quad (4.15)$$

where we have used the notation

$$\mu^{-1}\tau_{\beta\gamma}(z^k) = \frac{\partial z^k_\beta(P,Q)}{\partial x_\gamma} + \frac{\partial z^k_\gamma(P,Q)}{\partial x_\beta} + \alpha \frac{\partial z^k_l(P,Q)}{\partial x_l}\delta_{\beta\gamma} \quad (4.16)$$

the differentiation being taken with respect to the variables of the point P. Thus the $\tau_{\beta\gamma}(z^k)$ occurring in (4.15) is formed by first taking $z^k_i(P,Q)$ (the displacement at Q due to a point load at P) performing the indicated differentiation with respect to the variables of the point P and then evaluating at the origin. As before, from Schwarz's inequality we obtain

$$|\tau'_{\beta\gamma}(0)|^2 \le \left(\left\{ \oint_C t^{-1}[\tau_{\beta\gamma}(z^k)\tau_{\beta\gamma}(z^k) + (\tau_{\beta\gamma}(z^k)n_k)^2]dS \right. \right.$$
$$\left. \left. \cdot \oint_C t(w'_{k,l}w'_{k,l} + \alpha w'_{k,k}w'_{l,l})dS \right\}^{\frac{1}{2}} + \left| \int_D \tau_{\beta\gamma}(z^k)L_k(w)dV \right| \right)^2. \quad (4.17)$$

The insertion of (2.19) into (4.17) then leads to the result

$$|\tau'_{\beta\gamma}(0)|^2 \le \left\{ M_{\beta\gamma}(N)\left[\int_D L_j(w)L_j(w)dV\right]^{\frac{1}{2}} + \left| \int_D \tau_{\beta\gamma}(z^k)L_k(w)dV \right| \right\}^2 \quad (4.18)$$

where $M_{\beta\gamma}(N)$ is an easily computed constant. The desired bound for $\tau_{\beta\gamma}(0)$ now follows since

$$|\tau_{\beta\gamma}(0)|^2 \le 2\{|\overline{\tau}_{\beta\gamma}(0)|^2 + |\tau'_{\beta\gamma}(0)|^2\}. \quad (4.19)$$

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(Paper 65B4–66)

Publications of the National Bureau of Standards*

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Selected Abstracts

The nonsingular embedding of transition processes with a more general framework of coupled variables, J. Heading, *J. Research NBS* **65D** *(Radio Prop.)*, *No. 6*, *(Nov.–Dec. 1961)* 75 cents.

Reflection and coupling processes exhibited by plane electromagnetic waves propagated in an inhomogeneous horizontally-stratified anisotropic ionosphere are associated with discrete transition points or with continuous coupling regions. These arise when the fourth order differential equations are written in first order coupled form, and many terms in these equations become infinite at the transition points. This procedure is rendered more precise by means of a special linear transformation that reformulates the equations in a new way, thereby exhibiting the manner in which local coupling processes are embedded in the more general background process of otherwise independently propagated characteristic waves. To exhibit the power of the matrix algebra involved, the case of an arbitrary number of characteristic waves is considered; moreover, Försterling-type coupled equations are produced in a more generalized form than hitherto considered, and a discussion of the equations governing continuous coupling completes the paper.

The robustness of life testing procedures derived from the exponential distribution, M. Zelen and M. C. Dannemiller, *Technometrics* **3**, *No. 1, 29–49 (Feb. 1961)*.

Almost all the statistical procedures in current use for evaluating the reliability of components or equipment rest on the assumption that the failure times follow the exponential distribution. However, in practical situations one rarely has enough data to determine whether failure times are actually exponential. This paper studies the behavior of several statistical life testing procedures based on the exponential failure law if the true failure law is the Weibull distribution. It is found that these statistical techniques, which are widely used, are very sensitive to departures from initial assumptions. Applying these techniques to life test data when the exponential failure law is not satisfied may result in substantially increasing the probability of accepting components or equipments having poor mean-time-to-failure.

This paper also develops convenient analytic techniques for approximating (i) the distribution of sums of independent random variables, and (ii) the characteristics of sequential procedures, for non-negative random variables. These techniques are based on a general life distribution which consists of an expansion of generalized Laguerre polynomials. Application of these methods to the Weibull distribution yields excellent results.

Applications of graphs and Boolean matrices to computer programming, R. B. Marimont, *SIAM Rev.* **2**, *No. 4, 259–268 (Oct. 1960)*.

It is shown how the techniques of graph theory and Boolean matrices may be used in the analysis of digital computer programs for the elimination of inconsistencies and redundancies. The inadequacy of the present methods to deal with the more complicated problems of computer networks is illustrated by discussion of some important unsolved problems.

On some partial differential equations of Brownian motion of a free particle, A. Ghaffari, *(Abstract)Proc. Intern. Conf. on Partial Differential Equations and Continuum Mechanics (Univ. of Wisconsin, Madison, Wisc., June 7–15, 1960), 348–350 (1961)*.

The most general one-dimensional solutions of the Chapman-Kolmogoroff functional equation are well known as series of parabolic cylinder functions. The present paper gives a derivation of the corresponding parabolic partial differential equations and their simplification and solution for a special case. These solutions differ from those of A. N. Kolmogoroff and are of class $L^{(2)}$ over $(-\infty, +\infty)$. The same method furnishes the most general one-dimensional solutions, as series of Laguerre polynomials which converge uniformly over $(0, +\infty)$ under certain conditions.

Nonequilibrium processes in isotopically disordered crystals. Dependence on degree of order, R. Zwanzig, *J. Math. Phys.*, **2**, *No. 3, 370–372 (Mar.–June 1961)*.

A previous paper [A. A. Maradudin, G. H. Weiss, and D. H. Jepsen, J. Math. Phys. 2, 349 (1961)] has shown that approximate normal modes of isotopically disordered crystals decay irreversibly when the arrangement of isotopes is completely random. These results are generalized to crystals with an arbitrary degree of order. In particular, it is shown irreversible behavior occurs whenever the spatial correlation between isotopic species extends over a fixed finite range, as the size of the crystal tends to infinity.

A diffraction theory for LF sky-wave propagation, J. R. Wait, *J. Geophys. Research* **66**, *No. 6, 1713–1724 (June 1961)*; **An additional note**, J. R. Wait and A. M. Conda, *J. Geophys. Research* **66**, *No. 6, 1725–1729 (June 1961)*.

The concept that radio waves propagate from transmitter to receiver via discrete ionospheric reflections (i.e., hops) is put on a sound theoretical basis. The earth and the ionosphere are represented by two (smooth) concentric spherical surfaces. The (local) reflecting characteristics of these boundaries are assumed to be known. It is shown that the m'th hop sky wave may be generally represented by a complex integral. The geometrical-optical representations for the field are retrieved as a special case. For low-radio-frequencies, it is shown that these optical-type formulae, which are often used, have severe limitations. Generally, they are not valid near the caustic points. Techniques for evaluating the complex integral for the sky waves near and beyond the caustics are then discussed in some detail. Where possible, a physical interpretation of the results is given.

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