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Measurement of Wave Fronts Without a Reference Standard:

Part 1. The Wave-Front-Shearing Interferometer

James B. Saunders

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The wave-front-shearing interferometer may be used to test any converging wave front regardless of whether or not it is symmetrical. A mathematical operation is described that permits complete analysis of the data. This operation yields values of the deviations of wave fronts under test from a close fitting sphere. The reference surface may be chosen statistically so that the results are the deviations from a best fitting sphere.

Introduction

The wave-front-shearing interferometer [1, 2, 3] [1] provides a comparison of a converging wave front with a sheared image of itself. Similarly, the wave-front-reversing interferometer [4] provides a comparison of either a converging or plane wave front with a reversed image of itself. A detailed description of a method of analyzing the data from these interferometers, except for the special case of revolution symmetry, has not been published. Consequently, the absolute shape of a completely unknown wave front could not be obtained.

The method of analysis involves a mathematical operation. This operation is similar for the two aforementioned interferometers. Its description is more clearly portrayed in its application to the wave-front-shearing interferometer (≡WSI) than to the wave-front-reversing interferometer (≡WRI), although the method evolved from a study of the latter. There are, however, sufficient differences in the two operations to warrant separate treatments. Accordingly, part 1, which follows, will describe the analysis of data from a WSI and part 2, to be published later, will describe the analysis of data from a WRI. Part 3 of this series will describe the absolute testing of optical flats. This method is not new but has not been described adequately in the literature.

Part 1

1. Analysis of WSI Fringes

The WSI will yield unique solutions of a wave front. The assumption of symmetry (i.e., that it have revolution symmetry) is unnecessary. Any convergent wave front may be tested along any chosen diameter or along the line of its intersection with any plane that passes through the point of convergence. Several equally spaced reference points are chosen along this line. The line of reference points must be parallel to the direction of shear and the shear must be an integral multiple of the separation of the points. This arrangement and spacing of the reference points causes their images, in the region of overlapping wave fronts, to coincide in pairs as is shown in figure 1A.

The separation of the reference points will be defined as unity for this discussion. If the magnitude of the shear is unity also, one adjustment of the system (one set of fringes) is sufficient to evaluate the deviations of the wave front at the chosen reference points from a statistically chosen sphere. For a shear that exceeds unity, more than one fringe pattern (i.e., two or more shear values) must be used to obtain a solution.

In figure 1, $P_\nu (\nu=0, 1, 2 \ldots N)$ represents the $(N+1)$ reference points in one of the images, W, of the sheared wave front and P'_ν the corresponding points in the other image, W'. Let δ_ν (see fig. 1B) equal to the deviation of the wave front at P_ν, from the corresponding point, T_ν, on a reference circle, C, (to be chosen later), ϵ equals the angle between the two images, C and C', of the reference circle at their point of intersection and μ equals the distance from the intersection point to P_0. The distance, μ, is positive if P_0 is below (in fig. 1B) the intersection of the two circles and negative if above. The deviation, δ_ν, is positive if P_ν is on the concave side of the circle, and negative if it lies on the convex side of the reference circle. The distance from P_0 to P_ν (or from T_0 to T_ν), measured along the circle, is always positive and is represented by ν.

The separation, S_ν (distance from T_ν to $T'_{\nu-1}$), of the two images of the reference circle at any pair of points, P_ν and $P'_{\nu-1}$, (assuming unit shear) may be obtained from figure 2. The centers of the two circles are located at C_1 and C'_1. The distance $C_1C'_1$ equals $2h$, R equals the radii of the two circles, ρ equals the distance from E to T_ν, E bisects the line $C_1C'_1$ and ϕ is the angle subtended by $(\nu+\mu)$ at E. Applying the law of cosines (from trigonometry) to triangles $EC'_1T'_{\nu-1}$ and EC_1T_ν we obtain

$$R^2=h^2+(\rho+S_\nu)^2-2h(\rho+S_\nu)\cos(90-\phi) \text{ and}$$

$$R^2=h^2+\rho^2-2h\rho\cos(90+\phi).$$

[1] Figures in brackets indicate the literature references at the end of this paper.

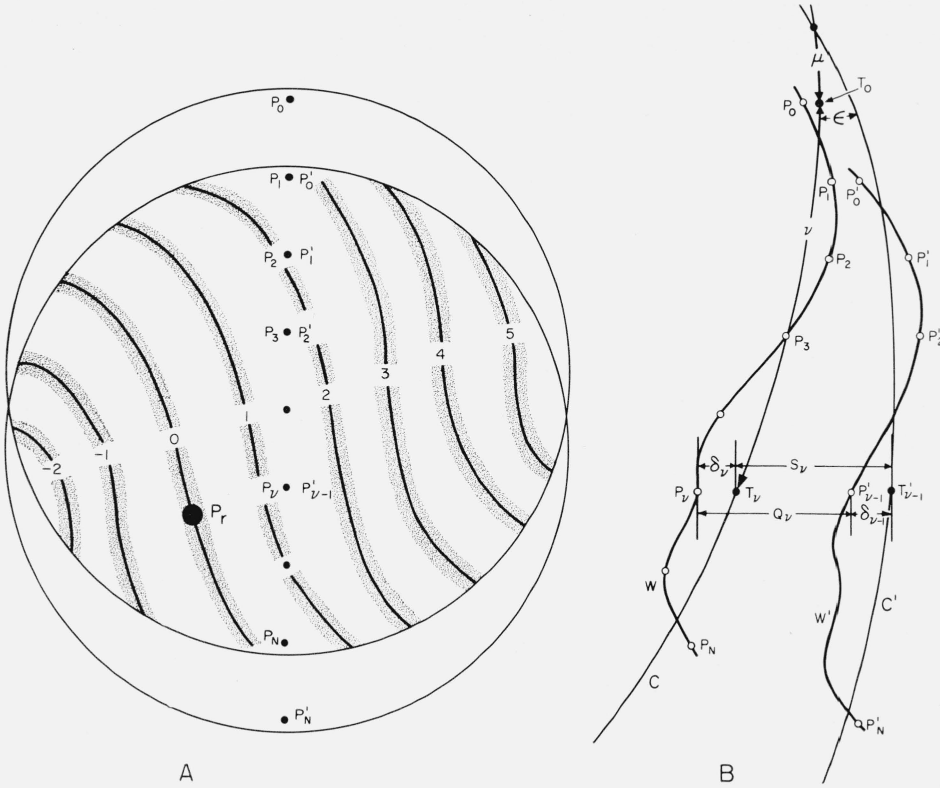

FIGURE 1A *Two images of a wave front sheared laterally relative to itself.*
B *Illustration of the two wave fronts relative to the images of a reference circle.*

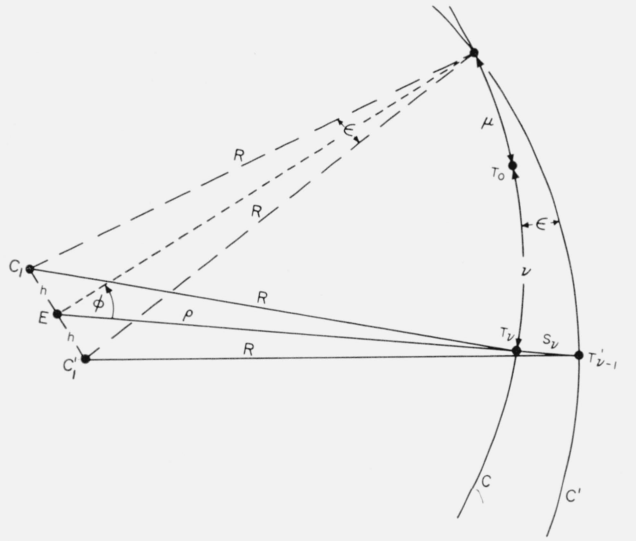

FIGURE 2 *Relationship between parameters and other variables.*

On taking differences and solving for S_ν we obtain,

$$S_\nu = 2h \sin \phi. \tag{1}$$

This equation may be applied rigorously, but if the focal ratio (radius of curvature of the circle divided by the diameter of the wave front) exceeds six, no significant error is introduced by replacing $\sin \phi$ by its approximate equivalent, $(\nu+\mu)/R$. Also, since the angle ϵ will always be quite small, the value of $2h$ is approximately equal to $R\cdot\epsilon$ and the value of S_ν in eq (1) becomes

$$S_\nu = (\nu+\mu)\epsilon. \tag{2}$$

If we let Q_ν represent the separation of the two wave fronts, W and W', at P_ν, it is seen from figure 1B that

$$\left.\begin{array}{l} \delta_\nu = \delta_{\nu-1} + Q_\nu - S_\nu = \delta_{\nu-1} + Q_\nu - \epsilon(\nu+\mu) \\ \nu = 1,\ 2,\ 3\ \ldots\ N. \end{array}\right\} \tag{3}$$

All quantities that represent optical distances are given in units of the wavelength that is used. Accordingly, these quantities should be multiplied by the wavelength to convert to standard units of length. The quantity Q_ν may be observed directly if white light is used to adjust the zero order to a known point, P_r, and then, using monochromatic light of known wavelength, observe the order difference between points P_r and P_ν. It is inconvenient and unnecessary to use white light. Let Q_r represent the unknown order of interference at an arbitrarily chosen point, P_r, in the fringe pattern (fig. 1A) and q_ν the difference in order (number of fringes) between this point and P_ν. The order of interference, Q_ν, at

P_ν is, therefore, equal to (Q_r+q_ν). This introduces another unknown into the equations of observation; but this may be absorbed by replacing the product $\epsilon\mu$ with a new parameter, r, such that

$$\epsilon\mu=(Q_r-r). \tag{4}$$

With these values for Q_ν and $\epsilon\mu$ replaced by their respective equals, eq (3) becomes

$$\left.\begin{aligned}\delta_\nu&=\delta_{\nu-1}+q_\nu+r-\nu\epsilon\\ \nu&=1,2\;.\;.\;.\;N\end{aligned}\right\} \tag{5}$$

Equation (5) represents N equations that contain $(N+3)$ unknowns: namely, $(N+1)$ δ_ν's plus the two parameters ϵ and r. We need three additional equations relating these $(N+3)$ unknowns if a solution is to be possible. It appeared that one could use a second set of fringes formed with a shear of two units, as shown in figure 3A. This would yield $(N-1)$ additional equations of observations with only two new unknowns, i.e., two more parameters, ϵ_2 and r_2, to be added to the above-mentioned $(N+3)$ unknowns. If N is 6 or larger, we would have as many equations as unknowns and a solution would seem to be possible, but it can be shown that these equations are not entirely independent.

The δ_ν's do not have significance until the reference circle, C, shown in figure 1B, is defined. Any circle may be defined by three conditions. In analytical geometry, these might be the coordinates of the center and the radius of the circle. A circle might also be defined as the one that passes through three given points. The circle of reference may be fully defined by three equations of condition. When these are combined with the set of observation equations (eq (5)) data from a single set of fringes is sufficient for a complete solution.

The values of the deviations may be obtained without knowing the value of R. However, when testing a parabolic mirror, for instance, the deviations represent the difference between the parabola and a sphere. If comparison is to be made between the measured deviations of the wave front and the computed deviations of a parabola from a circle, the radius of curvature of the circle must be known. The radius, R, can be measured independently of the interferometer. The ideal image point, E, (fig. 2) or the mean point of convergence of the wave front, can be located and its distance from any chosen point can be measured directly. The center of the reference circle will be located at the ideal image point if the reference circle is chosen as the circle that best fits the observed wave front. This may be done most precisely by the method of least squares. The value of R is the measured distance from the ideal image point (point of convergence) to any chosen reference point plus the computed deviation of this reference point from the reference circle.

In the test of a concave mirror, R is approximately equal to the radius of the mirror. In the test of a lens (simple or compound system), R is the distance from the point of convergence (image point) to the back surface of the lens.

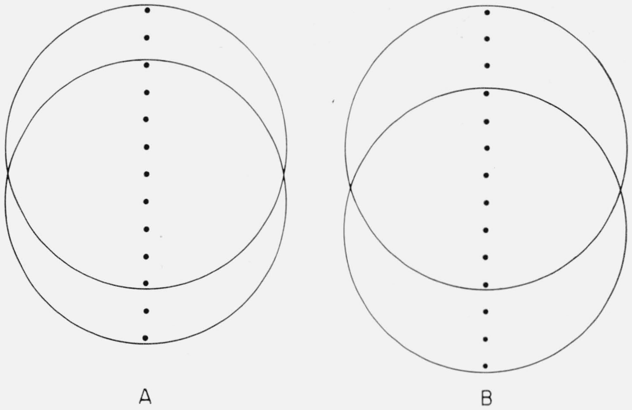

FIGURE 3A *Wave fronts sheared 2 units.*
B *Wave fronts sheared 3 units.*

2. Specifications for the Reference Circle

There are three relatively simple methods of defining the circle of reference. Each method requires three equations of condition. These conditions involve three well-known principles: namely, the method of coincidence, the method of averages, and the method of least squares.

2.1. The Method of Coincidence

We may define the reference circle, shown in figures 1B and 2, by requiring it to pass through any three of the chosen reference points. Thus, we might require that it pass through the two end points, P_0 and P_N, plus some other point near the axis of the mirror (or lens). If the points, P_0 and P_N, are too close to a dubbed edge, a better fitting circle is obtained by requiring it to pass through the two points adjacent to P_0 and P_N, respectively. These two choices for reference circle are made by equating the corresponding δ_ν's to zero. Any three of the points can be chosen to define the circle and the results would be the deviations of the wave front at all chosen reference points from the chosen circle.

For illustration we will choose the fringe pattern shown in figure 1A. There are 8 reference points. We will arbitrarily require the reference circle to pass through points P_0, P_3, and P_7. This is done by requiring that

$$\delta_0=\delta_3=\delta_7=0. \tag{6}$$

For this illustration, eq (5) represents 7 equations of observation. These 7 equations, when combined with the 3 condition equations, (6), form a set of 10 linear equations with 10 unknowns; and all δ_ν's that are not evaluated in eq (6) can be computed in terms of known quantities (the q_ν's and ν's). Table 1 and figure 4 show the results obtained by this method.

2.2. The Method of Averages

The method of averages will usually yield a better fitting circle than the method of coincidence, because of the averaging of the δ_ν's. This method requires an assignment of weights. A reasonable correlation

TABLE 1. *Deviations of a wave front from reference circles that are chosen as follows: Column 3, by the method of coincidence; column 4, by the method of averages; and column 5, by the method of least squares*

ν	q_ν	δ_ν		
		MC	MA	LS
0	--------	0.00	−0.20	−0.10
1	2.74	.23	.13	.14
2	2.10	.15	.11	.07
3	1.70	.00	.00	−.07
4	1.45	−.06	−.05	−.13
5	1.30	.06	.04	−.01
6	1.00	.21	.14	.15
7	0.30	.00	−.16	−.06

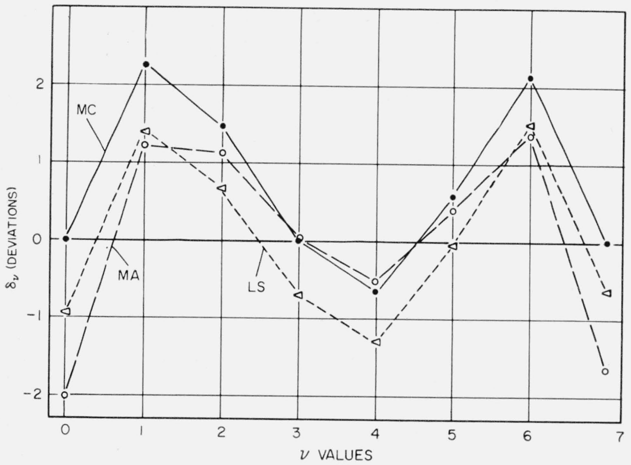

FIGURE 4 *Graphical representation of the deviations of a wave front from reference circles that are chosen by the method of coincidence (MC), the method of averages (MA), and the method of least squares (LS).*

of image quality, formed by the measured wave front from an optical, image-forming system, with wave-front shape would normally require that the weights should be proportional to some function of the distance from the reference points to the center of the circular aperture of the optical system. This would allow the assignment of equal weights to equal areas of the wave front. Equal weights will, however, be used in this discussion.

In previous applications of the method of averages [5], the deviations of the observations from a curve, which is to be fitted to the observations by adjustment of parameters, are equated to functions of the parameters. These are then grouped into as many groups as there are parameters to be adjusted. The algebraic sum of the deviations in each group is equated to zero. When the functions of the parameters (eq (5)) are substituted for the deviations, there are as many equations as parameters and the parameters can be evaluated. Any manner of grouping the deviations will yield a solution. A very poor and sometimes useless solution will be obtained if the points that are allocated to each group are scattered among those of the other two groups. This author prefers to avoid any intermingling of the points of one group with those of other groups.

The parameters of the reference circle shown in figure 1A are ϵ, r and any one of the deviations, δ_ν.

The parameter ϵ (see figs. 1B and 2), together with R, determines the distance, h, from the center, C_1, of the reference circle to the point of convergence, E, of the wave front. The center of this circle is displaced from the ideal image point, E (mean point of convergence) for two reasons. One is that the axis of the mirror or lens cannot be adjusted to pass absolutely through the image of the source, and, secondly, the interference fringes would be too broad for reading when the wave front is nearly spherical. Consequently, the angle ϵ is adjusted to be large enough to provide fringes of a most convenient width for accurate readings.

The parameter, μ, together with R, determines the direction of the point C_1 from E, relative to the axis of the mirror. The parameter r is related to and replaces μ through eq (4). Consequently, the parameters r and ϵ define the center of the circle of reference relative to the wave front.

Any one of the δ_ν's may be chosen as a parameter. By adding δ_ν to the directly measurable distance from E to P_ν we have the radius R of the circle of reference. We will choose δ_0 for the third parameter because of simplification in the formulas that follow.

This relationship of the three parameters to the coordinates of the center of the circle and its radius is somewhat vague, involved, and perhaps not clearly explained, but it is sufficient to understand that they correspond to the three parameters that define the circle of reference.

If equal weights are to be assigned, there must be an equal number of points allocated to each of the three equations of condition. In the example chosen above (fig. 1A), the number of points are not integrally divisible by three. However, we may consider that each of the observation equations represents three identical observations. The total number of observation equations is then $3N$ and division by 3 is now possible. The three equations of condition are:

$$3\delta_0+3\delta_1+2\delta_2=\delta_2+3\delta_3+3\delta_4+\delta_5=2\delta_5+3\delta_6+3\delta_7=0. \quad (7)$$

The set of equations represented by (5) may be replaced by the equivalent set

$$\delta_\nu=\delta_0+\sum_{\sigma=1}^{\nu} q_\sigma+\nu\cdot r-\epsilon\sum_{\sigma=0}^{\nu-1}\sigma=f_\nu, \quad (8)$$

where f_ν is a function of the three parameters, δ_0, ϵ, and r. Any equation in set (8) is obtained by adding the first ν equations in set (5). The functions, f_ν, may now be substituted for the corresponding δ_ν's in (7). It will be noted that there is no f_0 corresponding to δ_0. This, however, does not present any difficulty because δ_0 is itself a parameter and may be left in eq (7) as is.

The three equations of (7), after replacing δ_ν by its equal, f_ν, contain the three parameters as unknowns which may be evaluated. The computed values for the parameters are then substituted in eq (8) for evaluating the remaining δ_ν's. The results for the chosen example, shown in figure 1A, are given in table 1, column MA, and are also represented graphically in figure 4.

The number of reference points, in the unit-shear method, is determined by the shear angle and diameter of the element being tested. For testing elements (mirrors or lenses) during the polishing and figuring operations, the optician is only interested in departures from a desired figure. The easiest method of computing this is, therefore, the most practical. The shear may be adjusted to such a value that the resultant number of points is some multiple of three. The computation is a minimum when the number of points is the same in each group.

2.3. The Method of Least Squares

The method of evaluating a wave front by least squares is more elaborate than the others but would usually be used when highest precision is desired. In applying this method, the sum of the squared deviations $\left(\sum_{\sigma=0}^{N}\delta_\sigma^2\right)$ is to be minimized with respect to the three parameters required to define the reference sphere.

This application of least squares differs from the usual method of least squares in that the observations are assumed to be free from error and the number of observations needs not exceed the number of unknowns whose values are sought (namely the δ's). In fact, a unique solution may be obtained even if the number of observations is one less than the number of chosen reference points.

To minimize the sum of the squared deviations of the wave front from the reference sphere we require that

$$\sum_{\nu=0}^{N}\delta_\nu^2=\delta_0^2+\sum_{\nu=1}^{N}f_\nu^2=\text{minimum}. \tag{9}$$

This is effected by equating to zero the differentials of eq (9) with respect to r, ϵ, and δ_0, respectively. On performing the differentiation and dividing through by 2, we obtain

$$\delta_0+\sum_{\nu=1}^{N}f_\nu\frac{\partial f_\nu}{\partial\delta_0}=\sum_{\nu=1}^{N}f_\nu\frac{\partial f_\nu}{\partial r}=\sum_{\nu=1}^{N}f_\nu\frac{\partial f_\nu}{\partial\epsilon}=0. \tag{10}$$

The partial differentials, as obtained from eq (8), are

$$\left.\begin{array}{l}\dfrac{\partial f_\nu}{\partial\delta_0}=1,\ (\nu=0,1,2\ldots N)\\[2ex] \dfrac{\partial f_\nu}{\partial r}=\nu,\ \text{and}\ \dfrac{\partial f_\nu}{\partial\epsilon}=-\sum_{\sigma=0}^{\nu-1}\sigma,\ (\nu=1,2,3\ldots N)\end{array}\right\} \tag{11}$$

On substituting these differentials into (10), we have for the example of 8 reference points (fig. 1A),

$$\left.\begin{array}{l}\delta_0+\delta_1+\delta_2+\delta_3+\delta_4+\delta_5+\delta_6+\delta_7=0\\ \delta_1+2\delta_2+3\delta_3+4\delta_4+5\delta_5+6\delta_6+7\delta_7=0\\ \delta_2+3\delta_3+6\delta_4+10\delta_5+15\delta_6+21\delta_7=0\end{array}\right\} \tag{12}$$

On combining sets of eqs (8) and (12), we again have 10 equations with 10 unknowns. Using the same set of fringes (fig. 1A) as was used above for illustration, we obtain for the δ_ν 's the values given in column LS of table 1. These values are also plotted, along with those obtained by the other two methods, in figure 4.

3. Sensitivity

The sensitivity of the WSI varies with the absolute magnitude of the shear. If the shear is maintained at unity the sensitivity of the observations increases with a decrease in the number of points because the unit, as defined above, increases. The reduction of the number of points is, of course, limited.

In testing surfaces that depart very greatly from a sphere, as for a large aperture and large aperture-to-focus-ratio parabolic mirror, it is desirable to reduce the sensitivity. Otherwise, such a mirror cannot be tested interferometrically with the WSI. As the number of points is increased, the absolute spacing of the points decreases. The smaller the shear relative to aperture, the smaller can the optical path difference be made. By reducing the absolute magnitude of the shear the maximum optical path difference may be reduced at will and, consequently, the fringes may be adjusted to any desired width.

The sensitivity may also be increased, without reducing the number of reference points, by using two or more shear values that are greater than unity (fig. 3). This may yield more equations than there are unknowns, but averaging of computed points is always acceptable. Each additional set of fringes, obtained with a different shear value, introduces two additional unknowns—new values for ϵ and r.

4. Symmetrical Surfaces

Optical surfaces that are generated mechanically generally have revolution symmetry about the optical axis. If this is known and the axis is known to be at the center of the mirror (assuming circular apertures) then the deviations are known to be symmetrical because the center of the reference circle can be assumed to lie on the axis. Two other conditions are then sufficient to define the reference circle. The condition for revolution symmetry is defined by the specification

$$\delta_\nu=\delta_{N-\nu} \tag{13}$$

and the condition equations, (6), (7) and (12), become, respectively,

$$\delta_0=\delta_3=0, \tag{14}$$

$$3\delta_0+3\delta_1+2\delta_2=6\delta_3+2\delta_2=0, \tag{15}$$

and

$$\left.\begin{array}{l}\delta_0+\delta_1+\delta_2+\delta_3=0\\ 21\delta_0+15\delta_1+11\delta_2+9\delta_3=0\end{array}\right\}. \tag{16}$$

The third equation in each of the sets (6), (7), and (11) becomes identical, respectively, to the first equation of the set.

The assumption of symmetry (eq (13)) reduces the number of δ_ν's by approximately $\frac{1}{2}N$ (exactly $\frac{1}{2}N$ if N is an even number) and reduces the number of condition equations by one. If N is larger than 2, the number of equations exceeds the number of unknowns, for a unit shear arrangement. Consequently, larger shears (2 or more units) may be used with increased sensitivity, as explained above.

5. References

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