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Probability Inequalities of the Tchebycheff Type

I. Richard Savage [1]

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Thirteen basic inequalities relating tail area probabilities to moments are stated. One-tailed and multidimensional inequalities as well as the classical two-tailed, unidimensional inequalities are presented. Sufficient detail is given for each inequality so that the material can be used in handbook style without cross referencing or familiarity with the entire article. Examples of uses of the inequalities, tables comparing the relative strengths of the inequalities, and bibliographic data through 1960 are included.

I. Introduction

1. Background

The author originally prepared this survey and bibliography of probability inequalities of the Tchebycheff type, early in 1952, for convenient reference within the Statistical Engineering Laboratory of the National Bureau of Standards. Copies of the bibliography and of the survey were given limited circulation, within the Bureau and among a few individual specialists in this area, in April and June 1952, respectively, with a view to revision for ultimate publication.

Shortly thereafter the author, in his capacity as Associate Editor of the *Journal of the American Statistical Association*, learned of the preparation by H. J. Godwin, University College of Swansea, Wales, of a much more comprehensive treatment of inequalities of the Tchebycheff type, and gave his full support to this undertaking, which resulted in a definitive publication on the subject [Godwin (1955)].[2]

As the years have passed, however, many persons who have had access to the present author's original survey have found it to be a far more convenient source of directly applicable information for practical application than Godwin's more comprehensive paper. Consequently, in response to the urgings of various professional colleagues, the author has brought together and combined in the present paper his 1952 survey and bibliography of probability inequalities of the Tchebycheff type, with a few additions and revisions necessitated by the passage of time and the author's increased knowledge of the subject.

2. Scope and Organization

Tchebycheff inequalities give bounds for the probability of certain events. In particular they give estimates for deviations from the mean in terms of the moments.

A selected collection of Tchebycheff inequalities is given. They have been selected for their diverse nature and for their usefulness in applied and theoretical work.

In section II the various inequalities are presented with some notes on their uses and the conditions under which they may be used. In several cases more than one form of the inequality is presented in order to make it easier to work with the inequality. With each inequality the nature of the random variable is specified; that is, it is indicated whether the random variable is arbitrary, a sum, or some other function. Other special conditions are stated, always including the dimension of the random variables, and the moments which are assumed to exist.

In section III, tables are given for finding the particular probabilities associated with a specific inequality. Tables are also given for determining the required size of the variable parameters when a particular probability is desired. These tables will be found useful in choosing which of the inequalities to use. Tables have not been prepared for some of the inequalities which involve several parameters.

Several examples are given in section IV showing how some of these inequalities can be used. These examples show various possible uses, but are by no means exhaustive.

The bibliography in section V is as complete as possible. The books by Uspensky (1937) and Fréchet (1950), and the paper by Godwin (1955) are recommended as good surveys of the subject.

Tchebycheff inequalities are useful for working with distributions whose functional form is unknown. In many situations it is possible to avoid the assumption that random variables are (say) normally distributed. All that is needed for use of these inequalities are good estimates of certain population moments. Sometimes something is required of the functional form of the density function of a distribution. This is true for inequalities **9, 10,** and **11a** in the text. However, it is easy to verify

[1] Harvard University, Cambridge, Mass., and the University of Minnesota, Minneapolis, Minn.
[2] Names followed by dates in parentheses refer to the bibliography at the end of this paper.

whether the necessary conditions are true for the distributions that one is discussing.

In statistical work these inequalities have had several uses. In working theoretical problems, it is often necessary to use these inequalities, for instance, in proving the weak law of large numbers for binomial distributions. These inequalities are particularly useful for testing hypotheses and finding confidence intervals for the mean of a distribution, if one has some information about the other moments. In industrial work, these inequalities have been used to form "tolerance" sets.

Usually one does not have the true values for all of the parameters that are needed for using these inequalities. But if one has upper bounds for the parameters, that is for the moments, then one can use these inequalities. If one has run a process many times with the same type of material, then one usually has a good idea of the variance, even if the process mean has been shifted, so that in a sense one often knows some of the moments, and in this way one can test for the others.

Remember that a sample acts as a population; therefore, once the moments have been computed for a sample, all of these inequalities will be true for that sample; that is, these inequalities will provide bounds for the portion of the sample in various parts of its range. To obtain lower bounds for the probabilities in the tails of the distribution it is usually necessary to assume that the random variables are bounded.

Most of the inequalities presented are for the univariate case. There are several papers that discuss the multivariate case in much more detail; in particular, see Camp (1948), Leser (1942), Pearson (1919), and Marshall and Olkin (1960a). Most of the multivariate inequalities have been omitted because they are quite complicated and hard to apply. For each inequality presented here, the dimension of the random variable is specified, and this is a clue to deciding which one of the inequalities is applicable to a specific problem.

Several of the inequalities given require special assumptions on the shape of the involved distributions. All of these special assumptions require that the distribution has an unique mode. Narumi (1923) treated the opposite case, where the distributions have an unique minimum and increase as you go away from it. This case did not seem to be as important as the other and is omitted.

Winsten (1946) found inequalities that involve the ranges for various sample sizes. These inequalities will undoubtedly prove useful in the future; but they are not entirely analogous to the Tchebycheff inequalities and were omitted.

The Markov Inequality **1**[3] contains many of the other ones as special cases, which is a little surprising since this is the simplest of all the inequalities. This results from the fact that **1** is true for any positive random variable, X, that has a finite expected value. In particular, **1a** is derived from **1** by replacing X by a sum of random variables. Inequality **2** is obtained by replacing X by the square of the difference between a random variable and its expected value. One can derive many of the other inequalities in this manner.

In cases where the inequality is given only for the random variable X minus its mean, there are also inequalities for a sample average minus the expected value of that average.

Most of the inequalities are stated in the form of upper bounds for the probability that a random variable is greater than or equal to some number. There are opposite inequalities, lower bounds for the probability that the random variable is less than the same number. These are the same expressions, with the inequality reversed within the probability symbol (the "greater than or equal" symbol being replaced by a "less than" symbol), and with the right-hand side replaced by one minus the original right-hand side.

As given, some of the inequalities are very weak, for the right-hand sides may be greater than unity; but a probability is always less than or equal to unity, so the right-hand sides should be interpreted as the minimum of the given expression and unity.

Most of these inequalities cannot be improved; that is, the right-hand sides cannot be replaced by smaller quantities. That is, usually the left-hand side equals the right-hand side for some distribution that satisfies the conditions under which the inequality holds. Of course this will only occur for certain exceptional cases. If the exceptional case is known to be impossible, there might be a better inequality available [see Godwin (1955)].

In the following inequalities, unless otherwise noted, λ is any positive number. EX equals the expected value of the random variable, and will be denoted by μ; if need be this will be given a subscript. The expectation sign E will be used to denote other expected values depending on the argument that follows it. That is, $E(X-\mu)^2$ will be the variance and will be denoted by σ^2. The symbol $P(A)$ means the probability of the event A.

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[3] Each inequality has been given a name, mostly for convenience. These names do not necessarily reflect priority. It is hoped that the names do not conflict with common usage.

II. Inequalities

1. Markov

$$P\left(\frac{X}{\mu}\geq\lambda\right)\leq 1/\lambda \tag{1}$$

$$P(X\geq\lambda)\leq\mu/\lambda \tag{1'}$$

Random variable: X
Restrictions: X is non-negative, that is, $P(X<0)=0$
Dimension: One
Moments: $\mu=EX$
References: Cramér (1946), Fréchet (1937).

Notes:

1. This is a fundamental inequality from which inequalities **1a**, **2**, **2a**, **2b**, **3**, **5**, **6**, **7**, **8**, **13**, and **13a** can be derived by picking X as a special function of the random variables of interest.

2. In using this inequality, note that one needs to know only one moment, or one is testing an hypothesis about only one moment.

3. By itself this is a rather weak inequality, for the probability is bounded by $1/\lambda$; this is of course to be expected, since only one moment is being used and therefore one has very little knowledge about the involved distribution, or at least is only using very little of this knowledge.

4. Equality holds in **1**, with $\lambda>1$, for the random variable X which takes only the values zero and $\mu\lambda$, with probabilities $(\lambda-1)/\lambda$ and $1/\lambda$, respectively. If $\lambda<1$, the left-hand side of **1** is unity for the degenerate random variable which is equal to μ with probability one. A similar argument shows that **1′** is also a sharp inequality.

1a. Markov

$$P\left(\frac{X}{\mu}\geq\lambda\right)\leq 1/\lambda \tag{1a}$$

$$P\left(\sum_{i=1}^{n}X_i\Big/\sum_{i=1}^{n}\mu_i\geq\lambda\right)\leq 1/\lambda \tag{1a'}$$

Random variable:

$$X=\sum_{i=1}^{n}X_i$$

Restrictions: Each X_i is non-negative.

Dimension: Each X_i is one-dimensional, but actually the X_i may be considered as one observation on an n-dimensional random variable; i.e., the X_i may be dependent.

Moments: $\mu_i=EX_i$

$$\mu=\sum_{i=1}^{n}\mu_i$$

References: Cramér (1946), Fréchet (1937).

Notes:

1. This inequality is formally the same as **1**, but shows how **1** can be used where the random variable of interest is actually the sum of several random variables.

2. It is clear how this inequality is derived from the first one, for it is the same as that one, except that the random variable can be written in two ways, that is, either as X or as a sum of X_i.

2. Bienaymé-Tchebycheff

$$P(|X-\mu|\geq\lambda\sigma)\leq 1/\lambda^2 \tag{2}$$

$$P(X\geq\lambda\sigma+\mu \text{ or } X\leq\mu-\lambda\sigma)\leq 1/\lambda^2 \tag{2'}$$

$$P(|X-\mu|\geq\lambda)\leq\sigma^2/\lambda^2 \tag{2''}$$

Random variable: X

Restrictions: None

Dimension: One

Moments: $\sigma^2=E(X-\mu)^2$

References: Cramér (1946), Fréchet (1937), Uspensky (1937).

Notes:

1. This is the standard Tchebycheff inequality for one random variable.

2. Now the probability is decreasing as $1/\lambda^2$, which means that the probability of large deviations from the mean becomes quite small. It is to be noted that for the normal distribution and for large λ this is actually a very poor estimate of the probability of large deviations, for there the probability of a large deviation is smaller than $e^{-\frac{1}{2}\lambda^2}$, but for intermediate values this is not a bad approximation.

3. If one has a fairly good estimate of σ, then this inequality can be used for testing hypotheses about the mean, and for finding confidence intervals for the mean. In many industrial applications this inequality is used for estimating how much of the production will be near the mean of the process, where one has a good idea of the variance.

2a. Bienaymé-Tchebycheff

$$P(\sqrt{n}|\overline{X}-\mu|\geq\lambda\sigma)\leq 1/\lambda^2 \tag{2a}$$

$$P(|\overline{X}-\mu|\geq\lambda)\leq\sigma^2/n\lambda^2 \tag{2a'}$$

Random variable: $\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$

Restrictions: If $i\neq j$, then X_i and X_j are uncorrelated; that is, $E(X_i-EX_i)(X_j-EX_j)=0$

Dimension: One

Moments: $\mu = EX_i$

$$\sigma^2 = E(X_i - \mu)^2$$

References: Cramér (1946), Fréchet (1937), Uspensky (1937).

Notes:

1. This is one of the most useful of the Tchebycheff inequalities. One can use this whenever he has sample averages of identically and independently distributed random variables.

2. This inequality gives the "square root of the sample size" law. That is, $|\bar{X} - \mu|$ is of the order of magnitude $1/\sqrt{n}$ in probability.

3. The uses of this inequality are much like those of **2**, but it can be used when working with sample averages.

2b. Bienaymé-Tchebycheff

$$P(|X - \mu| \geq \lambda\sigma) \leq 1/\lambda^2 \tag{2b}$$

$$P(|X - \mu| \geq \lambda) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}/\lambda^2 \tag{2b$'$}$$

Random variable: $X = \sum_{i=1}^{n} X_i$

Restrictions: None

Dimension: n

Moments: $\mu_i = EX_i$, $\mu = \sum_{i=1}^{n} \mu_i$

$$\sigma_{ij} = E(X_i - \mu_i)(X_j - \mu_j)$$

$$\sigma^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}$$

Reference: Uspensky (1937).

Notes:

1. This is another form of the Tchebycheff inequality. In this case the random variables forming the sum can have different variances and can be correlated.

2. In the future, inequalities are presented only for samples of one observation. The detailed treatments of the versions of **1** and **2** for sample sums and sample averages can be modified to apply to the other inequalities.

3. Another point of interest is that, although the inequalities appear to require exact estimates of certain moments before they can be used, it is possible to get similar inequalities by substituting upper bounds for the moments that are involved. For instance, in this inequality one might not actually know the value of σ, but from previous experience he might know that it can not under any circumstances be greater than, say, σ'. In this case, if one uses σ' instead of σ, then the inequality will not be as good as the given one, but still may prove useful. This technique can, with some care, be used for all the inequalities.

3. Pearson

$$P\left(\frac{|X - \mu|}{\beta_r^{1/r}} \geq \lambda\right) \leq 1/\lambda^r \tag{3}$$

$$P(|X - \mu| \geq \lambda) \leq \beta_r/\lambda^r \tag{3$'$}$$

$$P\left(\frac{|X - \mu|}{\sigma} \geq \lambda\right) \leq \frac{\beta_r}{\sigma^r \lambda^r} \tag{3$''$}$$

Random variable: X

Restrictions: None

Dimension: One

Moments: $\mu = EX$

$$\sigma^2 = E(X - \mu)^2$$

$$\beta_r = E|X - \mu|^r$$

References: Fréchet (1937), Narumi (1923), Pearson (1919).

Notes:

1. In order to use this inequality, an absolute moment of the random variable is required.

2. If several absolute moments are available and one needs an inequality for a particular λ, then use that moment that makes the right-hand side of **3$'$** the smallest for that particular λ. Thus, for instance, **3$''$** should be compared to **2**: for large values of λ it may provide a smaller bound.

4. Birnbaum, Raymond, and Zuckerman

If n is an even integer,

$$P\left(\sum_{i=1}^{n} (X_i - \mu_i)^2 \geq \lambda^2\right) \leq \begin{cases} 1 & \text{if } \lambda^2 \leq n\sigma^2 \\ \dfrac{n\sigma^2}{2\lambda^2 - n\sigma^2} & \text{if } n\sigma^2 \leq \lambda^2 \leq \dfrac{n\sigma^2}{4}(3+\sqrt{5}) \\ \dfrac{n\sigma^2}{\lambda^2}\left(1 - \dfrac{n\sigma^2}{4\lambda^2}\right) & \text{if } \dfrac{n\sigma^2}{4}(3+\sqrt{5}) \leq \lambda^2 \end{cases}$$

If n is an odd integer,

$$P\left(\sum_{i=1}^{n}(X_i-\mu_i)^2\geq\lambda^2\right)\leq\begin{cases}1 & \text{if } \lambda^2\leq n\sigma^2\\ \dfrac{(n+1)\sigma^2}{2\lambda^2-(n-1)\sigma^2} & \text{if } n\sigma^2\leq\lambda^2\leq\dfrac{\sigma^2}{4}(3n+1+\sqrt{5n^2+6n+5})\\ \dfrac{n\sigma^2}{\lambda^2}-\dfrac{(n^2-1)}{4}\dfrac{\sigma^4}{\lambda^4} & \text{if } \dfrac{\sigma^2}{4}(3n+1+\sqrt{5n^2+6n+5})\leq\lambda^2\end{cases}$$

Random variable: $\sum_{i=1}^{n}(X_i-\mu_i)^2$

Restrictions: The X_i's are independently distributed with common variance.

Dimension: n

Moments: $\mu_i=EX_i$

$$\sigma^2=E(X_i-\mu_i)^2$$

Reference: Birnbaum, Raymond, and Zuckerman (1947).

Notes:

1. This inequality is an upper bound on the probability of the sample point falling outside of a hypersphere centered at the population mean. Birnbaum, Raymond, and Zuckerman (1947) also gives bounds for hyper-ellipses.
2. The application of this inequality to bombing and other aiming problems is obvious.
3. This inequality is multidimensional in that the probability of a multidimensional set is bounded. The random variables, however, are assumed to be independent. Inequality **5** is multidimensional in both senses, e.g., the probability of falling in a rectangle is bounded and the random variables are independent.
4. As n becomes large, the results for odd and even integers n approach each other.

5. Berge

$$P\left(\text{either } \frac{|X_1-\mu_1|}{\sigma_1}\geq\lambda \text{ or } \frac{|X_2-\mu_2|}{\sigma_2}\geq\lambda\right)\leq\frac{1+\sqrt{1-\rho^2}}{\lambda^2} \tag{5}$$

$$P\left(\max\left\{\frac{|X_1-\mu_1|}{\sigma_1},\frac{|X_2-\mu_2|}{\sigma_2}\right\}\geq\lambda\right)\leq\frac{1+\sqrt{1-\rho^2}}{\lambda^2} \tag{5'}$$

$$P\ (|X_1-\mu_1|\geq\lambda_1\sigma_1 \text{ or } |X_2-\mu_2|\geq\lambda_2\sigma_2)$$

$$\leq\frac{1}{2\lambda_1^2\lambda_2^2}\left[\lambda_1^2+\lambda_2^2+\sqrt{(\lambda_1^2+\lambda_2^2)^2-4\rho^2\lambda_1^2\lambda_2^2}\right] \tag{5''}$$

Random variable: $X=(X_1, X_2)$

Restrictions: None

Dimension: Two

Moments: $\mu_i=EX_i$

$$\sigma_i^2=E(X_i-\mu_i)^2$$

$$\sigma_{12}=E(X_1-\mu_1)(X_2-\mu_2)$$

$$\rho=\frac{\sigma_{12}}{\sigma_1\sigma_2}$$

References: Berge (1938), Lal (1955), Olkin and Pratt (1958), Whittle (1958b).

Notes:

1. This inequality bounds the probability of falling outside of a rectangle centered at the means for a bivariate sample.
2. This inequality uses the dependence between the random variables, and therefore, in order to apply this inequality one needs actually to have some knowledge about the correlation.
3. The right-hand side of this inequality is a decreasing function of the correlation. Thus, the most impressive results are obtained when the correlation is one. Even when the random variables are independent (zero correlation) the right-hand side is $2/\lambda^2$ which is not quite as strong as could be obtained from **2** but still useful.
4. Marshall and Olkin (1960a) found a "one-sided" version of this inequality and its extension to p dimensions.

6. Guttman

$$P\left[(\bar{X}-\mu)^2\geq\frac{S^2}{n-1}+\sigma^2\sqrt{\frac{2(\lambda^2-1)}{n(n-1)}}\right]\leq1/\lambda^2 \tag{6}$$

$$\lambda\geq1$$

Random variable: $\bar{X}=\dfrac{\sum_{i=1}^{n}X_i}{n}$

Restrictions: X_i are identically and independently distributed

Dimension: One

Moments: $\mu = EX_i$

$$\sigma^2 = E(X_i - \mu)^2$$

and define

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

References: Guttman (1948b), Midzuno (1950).

Note:

Inequality **6** is applicable whenever **2a** can be used, however, **6** takes more computing than **2a** does, and that is its only disadvantage. If we compare these two inequalities, they differ only in the quantity on the right of the inequality sign within the probability statement. For **6** that quantity is a random variable, and it is not for **2a**. The expected size of this random variable is much smaller than that of the quantity that occurs in **2a**, and therefore with probability near one it is a better inequality to use than **2a**.

7. Kolmogorov

$$P(T \geq \sigma\lambda) \leq 1/\lambda^2 \tag{7}$$

Random variable: $S_i = \sum_{j=1}^{i}(X_j - \mu_j);\ i = 1, 2, \ldots, n$

$$\text{Let } T = \max_{1 \leq i \leq n}[|S_1|, \ldots, |S_i|, \ldots |S_n|]$$

Restrictions: X_i and X_j independent

Dimension: One

Moments: $\mu_i = EX_i$

$$\sigma^2 = \sum_{i=1}^{n} E(X_i - \mu_i)^2$$

References: Fréchet (1937), Kolmogoroff (1928), Marshall (1960), Uspensky (1937).

Note:

This inequality can be used whenever **2a** is applicable. A typical use of this inequality is in the extreme-value situations. For instance, if one is putting together an assembly one might ask what is the probability that the cumulative error ever exceeds a certain quantity, and this inequality would give the answer.

8. Cantelli

$$P(X - \mu \leq \lambda) \leq \frac{\sigma^2}{\sigma^2 + \lambda^2} \qquad \text{if } \lambda < 0 \tag{8}$$

$$\geq 1 - \frac{\sigma^2}{\sigma^2 + \lambda^2} \qquad \text{if } \lambda \geq 0$$

Random variable: X

Restrictions: None

Dimension: One

Moments: $\mu = EX$

$$\sigma^2 = E(X - \mu)^2$$

References: Cantelli (1928), Cramér (1946), Uspensky (1937).

Notes:

1. This inequality is applicable whenever inequality **2a** can be used.
2. In this case one is interested in one-sided alternatives; that is, one wishes to detect large positive deviations from the mean. This occurs, for instance, whenever one is using one-sided confidence intervals or one-sided test regions.
3. The derivation of this inequality essentially depends on the Schwartz inequality.
4. Comparisons of "one-sided" and "two-sided" alternatives for certain convex sets are obtained by Marshall and Olkin (1960b).

9. Gauss (Camp-Meidell)

$$P(|X - \mu_0| \geq \lambda\tau) \leq \begin{cases} 1 - \dfrac{\lambda}{\sqrt{3}} & \text{for } \lambda \leq \dfrac{2}{\sqrt{3}} = 1.1547 \\[2ex] \dfrac{4}{9\lambda^2} & \text{for } \lambda \geq \dfrac{2}{\sqrt{3}} \end{cases} \tag{9}$$

$$P(|X - \mu| \geq \lambda\sigma) \leq \frac{4}{9} \cdot \frac{1 + s^2}{(\lambda - s)^2} \text{ if } \lambda > s \tag{9'}$$

Random variable: X

Restrictions: X has a density function with one mode, μ_0

Dimension: One

Moments: $\mu = EX$

$$\sigma^2 = E(X - \mu)^2$$

$$\tau^2 = \sigma^2 + (\mu - \mu_0)^2$$

$$s = \left|\frac{\mu - \mu_0}{\sigma}\right|$$

References: Cramér (1946), Fréchet (1937), Gauss (1880), Narumi (1923).

Notes:

1. This inequality requires the same knowedge of moments as does inequality **2**; but it is also necessary to know the mode of the distribution. For a symmetric distribution, of course, the mode is the same as the mean.
2. For a symmetric unimodal distribution, this is a better inequality than **2**, since the bound in **2** is here multiplied by 4/9. The inequality **9′** has a particularly simple form when $s = 0$. Indeed, for any unimodal distribution such that a bound for s is known, this inequality is better than **2** for sufficiently large values of λ.
3. If this decreasing property actually is true, then this is a better inequality to use than **2**, for it essentially multiplies the bound by 4/9.
4. This inequality in the form **9** is a special case of inequality **10′**.

10. Narumi, Gauss (Camp-Meidell)

$$P(|X-\mu_0|\geq\lambda\sqrt{\nu_r})\begin{cases}\leq 1-\dfrac{r+1}{rb}\left(1-\dfrac{1}{b^r}\right)\lambda \text{ if } 0\leq\lambda\leq\dfrac{rb}{r+1}\\[2ex] \leq\dfrac{1}{b^r} \text{ if } \dfrac{rb}{r+1}\leq\lambda\leq b\\[2ex] \leq\dfrac{1}{\lambda^r} \text{ if } b\leq\lambda\end{cases} \tag{10}$$

$$P(|X-\mu_0|\geq\lambda\sqrt{\nu_r})\begin{cases}\leq 1-\dfrac{\lambda}{(1+r)^{1/r}} \text{ if } \lambda\leq\dfrac{r}{(1+r)^{1-1/r}}\\[2ex] \leq\left(\dfrac{r}{r+1}\right)^r(\lambda)^{-r} \text{if} \dfrac{r}{(1+r)^{1-1/r}}\leq\lambda\end{cases} \tag{10'}$$

Random variable: X

Restrictions: X has a density function $f(x)$, and $f(x)$ has an unique maximum in the interval $(\mu_0-b,\ \mu_0+b)$ at μ_0 and $b>0$. Use (10) if b is finite, and (10′) if b is infinite.

Dimension: One

Moments: $(\nu_r)^r=E|X-\mu_0|^r$

References: Fréchet (1937), Narumi (1923).

Notes:

1. In **10** and **10′** the absolute moments about the mode are used rather than the absolute moments about the mean. For unimodal symmetric distributions that is not a restriction.
2. For many of the common distributions b will be infinity and **10′** can be used in preference to **10**. In applications, however, one might only be sure of the behavior of the density function near the mode and thus **10** is required.
3. Actually, in most cases one would probably use inequality **9**, which is a special case of these inequalities; this is why it has been given by itself.

11. Peek

$$P(|X-\mu|\geq\lambda\sigma)\leq\frac{1-\delta^2}{\lambda^2-2\lambda\delta+1},\ \lambda\geq\delta \tag{11}$$

Random variable: X

Restrictions: None

Dimension: One

Moments: $\mu=EX$
$\sigma^2=E(X-\mu)^2$
$\nu=E|X-\mu|$
$\delta=\nu/\sigma$

Reference: Peek (1933).

Notes:

1. This inequality is much like inequality **2**, except that here one needs to know ν, the mean deviation. If one has this additional information, this may be a better inequality to use.
2. This is a special case of inequality **12**.

11a. Peek

$$P(|X-\mu|\geq\lambda\sigma)\leq\frac{4}{9}\,\frac{1-\delta^2}{(\lambda-\delta)^2} \tag{11a}$$

Random variable: X

Restrictions: X has a density function whose only mode is its mean

Dimension: One

Moments: $\mu=EX$
$\sigma^2=E(X-\mu)^2$
$\nu=E|X-\mu|$
$\delta=\nu/\sigma$

Reference: Peek (1933).

Note:

This is an improvement over **9** for relatively large λ, but can only be used if one has an estimate of the mean deviation. For a particular value of λ, this should be compared with **9** and with **12**.

12. Cantelli

$$P(|X-\mu|\geq\lambda)\leq\frac{\beta_r}{\lambda^r}\qquad \text{if } \lambda^r\leq\frac{\beta_{2r}}{\beta_r}$$
$$\leq\frac{\beta_{2r}-\beta_r^2}{(\lambda^r-\beta_r)^2+\beta_{2r}-\beta_r^2}\qquad \text{if } \frac{\beta_{2r}}{\beta_r}\leq\lambda^r \tag{12}$$

Random variable: X

Restrictions: None

Dimension: One

Moments: $\mu=EX$
$\beta_{2r}=E|X-\mu|^{2r}$
$\beta_r=E|X-\mu|^r$

References: Cantelli (1910), Fréchet (1937), Peek (1933).

Notes:

1. Of course in this case one needs information about two moments.
2. The first part of **12** is equivalent to **3′**; for larger values of λ, **12** is better than **3′** when the required two moments are known.

13. Bernstein

$$P(|X-\mu|\geq\lambda)\leq 2e^{-\lambda^2/(2\sigma^2+2C\lambda)} \tag{13}$$

Random variable: $X=\Sigma X_i$

Restrictions: X_i and X_j are independent

Dimension: One

Moments: $\mu_i=EX_i,\qquad \mu=\Sigma\mu_i$

$$\sigma_i^2=E(X_i-\mu_i)^2,\qquad \sigma^2=\Sigma\sigma_i^2$$

$$E|X_i-\mu_i|^s\leq\frac{\sigma_i^2 s!\,C^{s-2}}{2} \text{ for all integers } s \text{ and}$$

some constant $C\,(C>0)$.

References: Curtiss (1950), Craig (1933), Fréchet (1937), Uspensky (1937).

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Notes:

1. This is a very nice inequality in the sense that the bound goes to zero very rapidly as λ increases.
2. The one difficulty in applying this inequality is that essentially one has to know at least an upper bound for every moment of each distribution involved. The next inequality treats a useful particular case.

13a. Bernstein

$$P(|X-\mu|\geq\lambda)\leq 2e^{-\lambda^2/(2\sigma^2+\frac{2}{3}m\lambda)} \tag{13a}$$

Random variable: $X=\Sigma X_i$

Restrictions: $P(|X_i-\mu_i|>m)=0$, X_i and X_j are independent $(i\neq j)$. The maximum deviation of a random variable X_i from its mean μ_i will not exceed m, with probability one.

Dimension: One

Moments; $\mu_i=EX$, $\qquad \mu=\Sigma\mu_i$

$$\sigma_i^2=E(X_i-\mu_i)^2, \qquad \sigma^2=\Sigma\sigma_i^2$$

References: Fréchet (1937), Uspensky (1937).

Note:

Each X_i has a distribution which does not allow deviations greater than m from μ_i. This condition also bounds all of the central moments.

13b. Bernstein

$$P(|X-\mu|\geq\sigma\lambda)\leq e^{-\lambda^2/2} \tag{13b}$$

Random variable: $X=\Sigma X_i$

Restrictions: X_i, X_j are independent $(i\neq j)$; X_i is symmetrical about μ_i; i.e., $E(X_i-\mu_i)^{2r+1}=0$ $(r=0, 1, 2, \ldots)$

Dimension: One

Moments: $\mu_i=EX_i$, $\mu=\Sigma\mu_i$

$$E(X_i-\mu_i)^{2r}\leq\left(\frac{\sigma_i^2}{2}\right)^r\frac{(2r)!}{r!} \qquad (r \text{ an integer})$$

$$\sigma_i^2=E(X_i-\mu_i)^2,\ \sigma^2=\Sigma\sigma_i^2$$

Reference: Uspensky (1937).

Note:

This form of the Bernstein inequality places many restrictions on the underlying distributions through their moments. The normal distribution does satisfy these conditions.

III. Tables

Essentially tables 1 and 2 are inverses of each other, column by column. The first table answers the question: How large is the probability associated with a specific "deviation" λ? The second table gives the "deviation" associated with specific probabilities P.

$$P(\overline{X}-\mu\geq\lambda)\leq 0.10,$$

TABLE 1. *Probability associated with deviation* λ

Probability associated with / λ	1	2	3	4	5	6	7
	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{1}{\lambda^4}$	$\frac{4}{9\lambda^2}$	$\left(\frac{4}{5}\right)^4\frac{1}{\lambda^4}$	$\frac{1}{1+\lambda^2}$	$e^{-\lambda^2/2}$
1.0	1.000	1.000	1.000	0.444	0.4096	0.5000	0.607
1.5	0.667	0.444	0.1975+	.198	.0809	.3077	.325
2.0	.500	.250	.0625	.111	.0256	.2000	.135
2.5	.400	.160	.0256	.071	.0105−	.1379	.044
3.0	.333	.111	.0123	.049	.0050	.1000	.011
3.5	.286	.082	.0067	.036	.0027	.0755−	.002
4.0	.250	.062	.0039	.028	.0016	.0588	.000
4.5	.222	.049	.0024	.022	.0010	.0471	↑
5.0	.200	.040	.0016	.018	.0007	.0385−	
5.5	.182	.033	.0011	.015−	.0005−	.0320	
6.0	.167	.028	.0008	.012	.0003	.0270	↓
6.5	.154	.024	.0006	.011	.0002	.0231	.000

TABLE 2. *Deviation associated with probability* P

Deviation associated / P	1	2	3	4	5	6	7
	$\frac{1}{P}$	$\frac{1}{\sqrt{P}}$	$\frac{1}{P^{1/4}}$	$\frac{2}{3\sqrt{P}}$	$\frac{4}{5P^{1/4}}$	$\sqrt{\frac{1}{P}-1}$	$\sqrt{2\log_e\frac{1}{P}}$
0.99	1.0101	1.0050	1.0025	0.6700	0.802	0.1005−	0.1418
.95	1.0526	1.0260	1.0129	.6840	.810	.2293	.3202
.90	1.1111	1.0541	1.0267	.7027	.821	.3333	.4590
.75	1.3333	1.1547	1.0746	.7698	.860	.5773	.7585
.50	2	1.4142	1.1892	.9428	.951	1	1.1774
.25	4	2	1.4142	1.3333	1.131	1.7321	1.6651
.10	10	3.1623	1.7783	2.1082	1.423	3	2.1460
.05	20	4.4721	2.1147	2.9814	1.692	4.3589	2.4477
.01	100	10	3.1623	6.6667	2.530	9.9499	3.0348
.001	1000	31.6228	5.6234	21.0818	4.500	31.6070	3.7169

An example is now given showing how to use these tables. Suppose one has a sample of nine independent observations from a distribution whose variance is four units squared. (1) What is an upper bound for the probability that the sample average is more than one unit larger than the population mean? (2) How far above the population mean could the ninety percent point of the distribution of sample means be? (3) If one knew the population had an unique mode at its mean, could these results be improved? Answer: For questions (1) and (2) one needs inequality **8**, and for (3) one needs **9′**.

To answer question (1), determine an upper bound for $P(\overline{X}-\mu\geq 1)$ using the second part of **8**, since $\lambda=1>0$. Since **8** is given for a random variable X with variance σ_X^2, and the present question involves the average of $n=9$ observations from a distribution with $\sigma_X^2=4$, we use the variance of $\overline{X}$, $\sigma_{\overline{X}}^2=4/9$, in inequality **8**. Thus,

$$P(\overline{X}-\mu\geq 1)\leq\frac{\sigma_{\overline{X}}^2}{\sigma_{\overline{X}}^2+\lambda^2}=\frac{1}{1+9/4}.$$

The value of this bound is found in column (6) of table 1 for $\lambda=3/2=1.5$, which shows $1/(1+\lambda^2)=0.3077$.

To answer question (2), determine the smallest value of λ for which again using the second part of **8.** The required value of λ satisfies

$$\frac{\sigma_{\overline{X}}^2}{\sigma_{\overline{X}}^2+\lambda^2}=0.10$$

or

$$\frac{1}{1+\lambda^2/\sigma_{\overline{X}}^2}=0.10.$$

The smallest value of $\lambda/\sigma_{\overline{X}}$ which satisfies inequality **8** is found in column (6) of table 2 for $P=0.10$, which shows $\lambda/\sigma_{\overline{X}}=3$, whence $\lambda=3\sigma_{\overline{X}}=2$.

To answer question (3), the two previous questions are treated again using inequality **9′**. This makes use of the additional information that the distribution is unimodal with its mode μ_0 at the mean ($s=0$ in **9′**). On the other hand, **9′** is a two-sided inequality. The first step is to evaluate an upper bound for

$$P(\overline{X}-\mu\geq 1)\leq P(|\overline{X}\leq\mu|\geq 1)$$

using **9′** with $\lambda\sigma_{\overline{X}}=1$, that is with $\lambda=3/2$. This is found in column (4) of table 1, which shows $4/9\lambda^2=0.198$. This is a smaller upper bound than the one obtained from **8**.

The second part of (3) is to find the smallest value of λ for which

$$P(|\overline{X}\leq\mu|\geq\lambda)\leq 0.10.$$

This value is given by

$$\frac{4}{9}\frac{1}{(\lambda/\sigma_{\overline{X}})^2}=0.10.$$

and from column (4) of table 2 for $P=0.10$ is found to be $\lambda/\sigma_{\overline{X}}=2.1082$ or $\lambda=1.41$. This is better (smaller) than the value obtained by use of **8**.

Thus for this problem it seems better to use inequality **9′** than **8**.

These tables will facilitate choosing which inequality to use when several are available, by comparing the associated probabilities (deviations) with the deviation (probability) of interest, thus making it possible to choose the inequality that gives the smallest probability (deviation) for the problem at hand.

The columns of the tables are associated with the inequalities as follows:

Column	*Inequalities*
1	(**1**), (**1a**), (**1a′**)
2	(**2**), (**2′**), (**2a**), (**2b**), (**6**), (**7**)
3	(**3** with $r=4$)
4	(**9**)
5	(**10′** for $r=4$)
6	(**8** for $\sigma^2=1$)
7	(**13** for $2\sigma^2+2c\lambda=2$; **13a** for $2\sigma^2+\frac{2}{3}m\lambda=2$; **13b**)

IV. Examples

Example (1). Assuming that all soldiers are between 60 and 78 inches tall, what is the probability that the average height of 500 soldiers is more than 1 inch away from the average height of all soldiers?

Solution: Although the population is finite, it is safe to assume that the measurements in the sample are independent. The largest possible variance occurs if half the soldiers have height 60 inches, and half have height 78 inches, in which case the variance is 81 inches squared. First apply inequality **2a′**. Here $\lambda=1$ inch, $\sigma^2=81$ inches squared, and $n=500$. Thus the answer is $\frac{\sigma^2}{n\lambda^2}=\frac{81}{500}=0.162$. One can also apply inequality **13a**. Here $\lambda=500$ inches, $\sigma^2=500\times.81$ inches, and $m=18$ inches. The probability is 0.11, and thus for this example inequality **13** gives more precise results than **2**.

Example (2). In the course of deposit and withdrawal transactions, such as money in a bank, or radioactive material in a hospital, one often wishes to control the absolute error. That is, in a sequence of, say, 100 transactions (a day's activity) one does not want one's books to differ from one's assets, at any time, by more than some fixed amount, say 1,000 units. Assume that the variance due to errors of measuring and of counting for each transaction is 400 units squared (this value being obtained by previous experience). The question then naturally arises: what is an upper bound for the probability of having an accumulated error of more than 1,000 units at any time during the day?

Solution: Inequality **7** is suited for this problem. Here $n=100$, $\sigma=20$, and $\sqrt{n}\sigma\lambda=1{,}000$. Thus $\lambda=\frac{1{,}000}{10\cdot 20}=\frac{1{,}000}{200}=5$, and $1/\lambda^2=1/25=0.04$. If instead of 100 transactions there had been 400, then $n=400$, $\sigma=20$, $\sqrt{n}\sigma\lambda=1{,}000$, $\lambda=\frac{1{,}000}{20\cdot 20}=\frac{1{,}000}{400}=2.5$; and the resulting probabilty is at most $\left(\frac{1}{2.5}\right)^2=\frac{1}{6.25}=0.161$.

Example (3). From previous experience assume that the correlation between two variables (height and weight, rainfall and crop yield) is at least 0.8. If a sample of 25 is made on this bivariate distribution, what is an upper bound for the deviations from the population means that will not be exceeded more than 10 percent of the time, where deviations are measured in standard units?

Solution: Here one can use inequality **5**. First one must solve the following equation:

$$0.10=\frac{1+\sqrt{1-0.64}}{\lambda^2}\text{ or }0.10=\frac{1+\sqrt{0.36}}{\lambda^2},$$

$$\lambda^2=\frac{1+0.6}{0.1}=\frac{1.6}{0.1}=16;\ \lambda=4.$$

Thus

$$P\left(\frac{|\overline{x}-\mu|}{\sigma_{\overline{x}}}\geq 4\text{ or }\frac{|\overline{y}-\mu|}{\sigma_{\overline{y}}}\geq 4\right)\leq 0.1,$$

or

$$P\left(\frac{|\overline{x}-\mu|}{\sigma_x}\geq\frac{4}{5}\text{ or }\frac{|\overline{y}-\mu|}{\sigma_y}\leq\frac{4}{5}\right)\leq 0.1.$$

Thus both sample means will be within 0.80 standard units of their respective population means with probability at least 0.9.

V. Bibliography

Anderson, O. (1935). *Einführung in die mathematische Statistik*, 187–193 (Julius Springer, Wien).

Anderson, T. W. (1955). The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. Proc. Amer. Math. Soc. **6,** 170–176.

Aoyama, Hirojiro (1952). On Midzuno's inequality, Ann. Inst. Stat. Math. **3,** 65–67.

Barnard, G. A. (1944). *An Analogue of Tchebycheff's Inequality in Terms of Range*, British Ministry of Supply Advisory Service on Statistical Method and Quality Control. Technical Report No. Q.C./R/11. (London), 1 sheet.

Beretta, L. (1941). A proposita del lemma do Bienaymé, Giorn. Ist. Ital. Attuari. (Rome) **12,** 122–126.

Berge, P. O. (1932). Über das Theorem von Tchebycheff und andere Grenzen einer Wahrscheinlichkeitsfunktion. Skand. Aktuarietidskr. **15,** 65–77.

Berge, P. O. (1938). A note on a form of Tchebycheff's theorem for two variables, Biometrika **29,**, 405–406.

Bergström, H. (1949). On the central limit theorem in the case of not equally distributed random variables, Skand. Aktuarietidskr. **32,** 37–62.

Bernstein, S. (1924). Sur une modification de l'inégalité de Tchebichef, Ann. Sc. Instit. Sav. Ukraine, Sect. Math. I. 38–49.

Bernstein, S. (1937). Sur quelques modifications de l'inégalité de Tchebycheff, C. R. (Doklady) Acad. Sc. URSS. **17,** 279–282.

Bernstein, S. (1946). *Theory of Probability*, 4th ed. (Gostehizdat, Moscow) 1–412.

Berry, A. C. (1941). The accuracy of the Gaussian approximation to the sum of independent variates, Trans. Amer. Math. Soc. **49,** 122–136.

Bienaymé, J. (1853). Considérations à l'appui de la découverte de Laplace sur la loi de probabilité dans la méthode des moindres carrés, C. R. Acad. Sci. (Paris) **37,** 309–326.

Bienaymé, J. (1867). Considérations à l'appui de la découverte de Laplace sur la loi de probabilité dans la méthode des moindres carrés, J. Math. Pures et Appl. **12,** 158–176.

Bignardi, F. (1947a). Sur les généralisations de l'inégalité de Bienaymé-Tchebycheff dans l'étude des distributions de fréquence, Ann. Fac. di Econ. e Commer., Universita di Palermo. **1,** 47–49.

Bignardi, F. (1947b). De quelques critères pour l'application des inégalités de Bienaymé: Tchebycheff et de Vinci dans l'étude des distributions de fréquence, Ann. Fac. di Econ. e Commer., Universita di Palermo. **1,** 84–108.

Birnbaum, Z. W. (1948). On random variables with comparable peakedness, Ann. Math. Stat. **19,** 76–81.

Birnbaum, Z. W., and H. S. Zuckerman (1944). An inequality due to H. Hornich, Ann. Math. Stat. **15,** 328–329.

Birnbaum, Z. W., J. Raymond, and H. S. Zuckerman (1947). A generalization of Tshebyshev's inequality to two dimensions, Ann. Math. Stat. **18,** 70–79.

Birnbaum, Z. W., and O. M. Klose (1955). Bounds for the variance of the U-statistics in terms of probability $Y<X$, Bull. Amer. Math. Soc. (Abstract), p. 329.

Birnbaum, Z. W., and A. W. Marshall (1960). *Some Multivariate Chebyshev Inequalities with Extensions to Continuous Parameter Processes*, Appl. Math. and Stat. Lab. Technical Report No. **32**. (Stanford University, Stanford, Calif.)

Blackwell, David (1954). On optimal systems, Ann. Math. Stat. **25,** 394–397.

Borel, Émile (1958). *Principes et Formules Classiques du Calcul des Probabilités*, (Gauthier-Villars, Paris), 106–148.

Brunk, H. D., G. M. Ewing, and W. R. Utz (1957). Minimizing integrals in certain classes of monotone functions Pacific J. Math. **7,** 833–847.

Camp, Burton H. (1922). A new generalization of Tchebycheff's statistical inequality. Bull. Amer. Math. Soc. **28,** 427–432.

Camp, Burton H. (1923). Note on Professor Narumi's paper, Biometrika. **15,** 421–423.

Camp, Burton H. (1948). Generalization to N dimensions of inequalities of the Tchebycheff type, Ann. Math. Stat. **19,** 568–574.

Cantelli, F. P. (1910). Intorno ad un teorema fundamentale della teoria del rischio, Bol. Assoc. degli Attuari Italiani (Milan) 1–23.

Cantelli, F. P. (1911). Intorno ad un teorema di calcolo della probabilità, Giorn. mat. Battaglini (Napoli) **49,** 341–352.

Cantelli, F. P. (1928). Sui confini della probabilità. Atti. Congr. Internaz. Matemat. **6,** 47–59.

Cantelli, F. P. (1958). *Alcune Memorie Matematiche*, xxx+448. (Publicatione Facolta Economia e Commercio, Universita di Roma, Milano: A. Guiffre).

Chapelon, Jacques M. (1937). Sur l'inégalité fondamentale du calcul des probabilités, Bull. Soc. Math. France (Paris) **65,** 100–108.

Chernoff, H., and S. Reiter (1954). *Selection of a Distribution Function to Minimize an Expectation Subject to Side Conditions*, Appl. Math. and Stat. Lab. Technical Report No. **23**. (Stanford University, Stanford, Calif.)

Chow, Y. S. (1960). A martingale inequality and the law of large numbers, Proc. Amer. Math. Soc. **11,** 107–111.

Combes, B. (1953). Le problème de la généralisation de l'inégalité de Bienaymé, Bull. Trimest. Inst. Actuaires Franç. (Paris) **52,** 61–69.

Craig, C. C. (1933). On the Tchebycheff inequality of Bernstein, Ann. Math. Stat. **4,** 94–102.

Cramér, H. (1946). *Mathematical Methods of Statistics*, xvi+575. (Princeton University Press, Princeton, N.J.)

Curtiss, J. H. (1950). Lot quality measured by average or variability. *Acceptance Sampling*, 79–116. (The American Statistical Association, Washington, D.C.)

van Dantzig, D. (1951). Une nouvelle généralisation de l'inégalité de Bienaymé, Ann. Inst. Poincaré. **12(1),** 31–43.

Deltheil, R. (1930). *Erreurs et Moindres Carrés*, (Gauthier-Villars, Paris), 15–22.

Doob, J. L. (1951). Continuous parameter martingales, Proc. 2d. Berkeley Symp. Math. Stat. and Prob. 269–277. (University of California Press, Berkeley).

Dor, L. (1943). Sur la fonction caractéristique des schémas de Poisson et de Lexis, Mem. Soc. Roy. Sci. Liège. **1,** 211–249.

Dresher, M. (1953). Moment spaces and inequalities, Duke Math. **20,** 261–271.

Edmundson, H. P. (1957). *Bounds on the Expectation of a Convex Function of a Random Variable*, (Rand Corporation, Santa Monica, Calif., April 9, 1957).

Faber, G. (1922). Bemerkungen zu Satzen der Gausschen theoria combinationis observationum, S.-B. Math.-Nat. K1. Bayer. Akad. Wiss. (München) 7–21.

Fréchet, M. (1931). La generalizzazione delle ineguaglianza di Bienaymé, Giorn. Ist. Ital. Attuari. **2,** 22–36.

Fréchet, M. (1937). *Généralités sur les probabilités. Variables aléatoires.* (Borel Series, Traité du calcul des probabilités et de ses applications, Div. I, Pt. III, Vol. 1, 123–126, Gauthier-Villars, Paris).

Fréchet, M. (1950). *Généralités sur les probabilités. Éléments aléatoires.* 2d edition of the above reference. (Gauthier-Villars, Paris), 130–168.

Fréchet, M. (1956). Sur les tableaux de corrélation dont les marges sont données, C.R. Acad. Sci. (Paris) **242,** 2426–2428.

Gauss, C. F. (1880). Theoria combinationis observationum, (1821), *Werke* **4,** 10–11 (Goettingen).

Godwin, H. J. (1944). *Inequalities Related to Tchebycheff's Inequality.* British Ministry of Supply Advisory Service on Statistical Method and Quality Control, Technical Report No. QC/R/13. (London).

Godwin, H. J. (1955). On generalizations of Tchebycheff's inequality, J. Amer. Stat. Assoc. **50,** 923–945.

Guldberg, Alf (1922a). Sur le théorème de M. Tchebycheff, C.R. Acad. Sci. (Paris) **175,** 418–420.

Guldberg, Alf (1922b). Sur un théorème de M. Markoff, C.R. Acad. Sci. (Paris) **175,** 679–680.

Guldberg, Alf (1922c). Sur les valeurs moyennes, C.R. Acad. Sci. (Paris) **175,** 1035–1037.

Guldberg, Alf (1922d). Sur quelques inégalités dans le calcul des probabilités, C.R. Acad. Sci. (Paris) **175,** 1382–1384.

Gumbel, E. J. (1954). The maxima of the mean largest value and of the range, Ann. Math. Stat. **25,** 76–84.

Guttman, L. (1948a). An inequality for kurtosis, Ann. Math. Stat. **19,** 277–278.

Guttman, L. (1948b). A distribution-free confidence interval for the mean, Ann. Math. Stat. **19,** 410–413.

Hájek, J., and A. Rényi (1955). Generalization of an inequality of Kolmogorov, Acta Math. Acad. Sci. Hungar. **6,** 281–283.

Halmos, Paul (1950). *Measure Theory,* xi+304. (D. van Nostrand, New York).

Hartley, H. O., and H. A. David (1954). Universal bounds for mean range and extreme observations, Ann. Math. Stat. **25,** 85–99.

Hoeffding, Wassily (1955). The extreme of the expected value of a function of independent random variables, Ann. Math. Stat. **26,** 268–275.

Hoeffding, Wassily (1956). On the distribution of the number of successes in independent trials, Ann. Math. Stat. **27,** 713–721.

Hoeffding, Wassily (1960). An upper bound for the variance of Kendall's 'tau' and of related statistics, *Contributions to Probability and Statistics: Essays in Honor of Harold Hotelling,* 258–264. (Stanford University Press, Stanford, Calif.).

Hoeffding, Wassily, and S. S. Shrikhande (1955). Bounds for the distribution function of a sum of independent, identically distributed random variables, Ann. Math. Stat. **26,** 439–449.

Hotelling, Harold, and Leonard M. Solomons (1932). The limits of a measure of skewness, Ann. Math. Stat. **3,** 141–142.

Hsu, P. L. (1945). The approximate distributions of the mean and variance of a sample of independent variables, Ann. Math. Stat. **16,** 1–29.

Isaacson, S., and Herman Rubin (1954). *On Minimizing an Expectation Subject to Certain Side Conditions,* Appl. Math. and Stat. Lab. Technical Report No. **25.** (Stanford University, Stanford, Calif.).

Isii, Keiiti (1957). Some investigations of the relation between distribution functions and their moments, Ann. Inst. Stat. Math. **9,** 1–11.

Isii, Keiiti (1959a). On a method for generalizations of Tchebycheff's inequality, Ann. Inst. Stat. Math. **10,** 65–88.

Isii, Keiiti (1959b). On Tchebycheff type inequalities, Proc. Inst. Stat. Math. **7,** 123–143.

Isii, Keiiti (1959c). Bounds on probability for nonnegative random variables, Ann. Inst. Stat. Math. **11,** 89–99.

Johnson, N. L., and C. A. Rogers (1951). The moment problem for unimodal distributions, Ann. Math. Stat. **22,** 433–439.

de Jongh, B. H. (1941). General minimum-probability theorem, Nederl. Akad. Wetensch, Proc. **44,** 738–743.

Karlin, S., and L. S. Shapley (1953). *Geometry of Moment Spaces,* Memoirs of American Mathematical Society, **12,** 1–93. (American Mathematical Society, Providence, R.I.).

Karlin, S., Frank Proschan, and Richard E. Barlow (1960). *Moment Inequalities of Pólya Frequency Functions,* Appl. Math. and Stat. Lab. Technical Report No. **2.** (Stanford University, Stanford, Calif.).

Khamis, Salem H. (1950). A note on the general Tchebycheff inequality, Proc. Intern. Congr. Math. **1,** 569–570. (American Mathematical Society, Providence, R.I.).

Khamis, Salem H. (1954). On the reduced moment problems, Ann. Math. Stat. **25,** 113–122.

Kolmogoroff, A. N. (1928). Ueber die Summen durch den Zufall bestimmter unabhaengiger Groessen, Math. Ann. **99,** 309–319.

Kolmogoroff, A. N. (1929). Bemerkungen zu meiner Arbeit 'Ueber Summen zufaelliger Groessen', Math. Ann. **102,** 484–488.

Kolmogoroff, A. N. (1933). Ueber die Grenzwertsaetze der Wahrscheinlichkeitsrechnung, Bull. Acad. Sci. URSS **VII,** Nr. 3, 363–372.

Kolmogoroff, A. N. (1950). *Foundations of the Theory of Probability.* 42–43. (Translated by Nathan Morrison, Chelsea, New York.)

Lal, D. N. (1955). A note on a form of Tchebycheff's inequality for two or more variables, Sankhyā. **15,** 317–320.

Leser, C. E. V. (1942). Inequalities for multivariate frequency distributions. Biometrika. **32,** 284–293.

Loève, Michel (1955). *Probability Theory: Foundations, Random Sequences,* xv+515. (D. van Nostrand, New York.)

Lurquin, C. (1922). Sur le critérium de Tchebycheff, C. R. Acad. Sci. (Paris) **175,** 681–683.

Lurquin, C. (1924). Sur une proposition fondamentale de probabilité, C.R. Acad. Sci. (Paris) **178,** 306–308.

Lurquin, C. (1928a). Sur une inégalité fondamentale de probabilité, C.R. Acad. Sci. (Paris) **187,** 868–870.

Lurquin, C. (1928b). Sur une limite de probabilité du sens de Bienaymé-Tchebycheff, C.R. Acad. Sci. (Paris) **187,** 1016–1018.

Lurquin, C. (1928c). Sur des formes d'extension de critérium de Bienaymé-Tchebycheff, C.R. Acad. Sci. (Paris) **188,** 1214–1216.

Lurquin, C. (1928d). Sur un théorème de limite pour les probabilité au sens de Bienaymé-Tchebycheff, Bull. Acad. Bruxelles. **14,** 641–658.

Lurquin, C. (1929a). Sur un critérium élémentaire pour la probabilité continue, Mathesis. **43,** 165–168.

Lurquin, C. (1929b). Sur l'inégalité de probabilité de Markoff, Bull. Sci. Math. **53,** 333–340.

Lurquin, C. (1929c). Sur les critères de probabilité au sens de Bienaymé-Tchebycheff, Bull. Acad. Bruxelles. **15,** 659–669.

Madansky, A. (1959). Bounds on the expectation of a convex function of a multivariate random variable, Ann. Math. Stat. **30,** 743–746.

Mallows, C. L. (1956). Note on the moment-problem for unimodal distributions when one or both terminals are known, Biometrika. **43,** 224–227.

Markoff, A. (1884a). *On Certain Applications of Algebraic Continued Fractions.* (Thesis, St. Petersburg, Russia.)

Markoff, A. (1884b). Demonstration de certaines inégalités de M. Tchebycheff, Math. Ann. **24,** 172–180.

Markoff, A. (1924). *Calculus of probability,* 4th ed. (Soviet Press, Moscow), 1–588.

Marshall, Albert W. (1958). *On the Growth of Stochastic Processes.* Technical Report No. **32,** 1–77. (Department of Mathematics, University of Washington, Seattle.)

Marshall, Albert W. (1960). A one-sided analog of Kolmogoroff's inequality, Ann. Math. Stat. **31,** 483–487.

Marshall, Albert W., and Ingram Olkin (1960a). A one-sided inequality of the Chebyshev type, Ann. Math. Stat. **31,** 488–491.

Marshall, Albert W., and Ingram Olkin (1960b). Multivariate Chebyshev inequalities, Ann. Math. Stat. **31,** 1001–1014.

Marshall, Albert W., and Ingram Olkin (1960c). A bivariate Chebyschev inequality for symmetric convex polygons, *Contributions to Probability and Statistics: Essays in Honor of Harold Hotelling,* 299–308. (Stanford University Press, Stanford, Calif.)

Marshall, Albert W., and Ingram Olkin (1960d). *Game Theoretic Proof that Chebyshev Inequalities are Sharp,* Appl. Math. and Stat. Lab. Technical Report No. **36,** 1–15. (Stanford University, Stanford, Calif.)

Masuyama, Motosaburo (1942). The Bienaymé-Tchebycheff inequality for Hermitic tensors, Proc. Phys. Math. Soc. Japan (3), **24,** 409–411.

Medolaghi, P. (1909). La teoria del rischio e le sue applicazioni, Trans. 6th. Intern. Actuar. Congr. (Vienna) **6,** 723.

Meidell, Birger (1921). Quelques inégalités sur les fonctions monotones, Skand. Aktuarietidskr. **4,** 230–238.

Meidell, Birger (1922). Sur une problème du calcul des probabilitiés et les statistiques mathematiques, C.R. Acad. Sci. (Paris) **175,** 806–808.

Meidell, Birger (1923). Sur la probabilité des erreurs, C.R. Acad. Sci. (Paris) **176,** 280–282.

Midzuno, Hiroshi (1950). On certain groups of inequalities. Confidence intervals for the mean, Ann. Inst. Stat. Math. **2**, 21–33.

von Mises, R. (1931). Ueber einige Abschaetzungen von Erwartungswerten, Crelles J. Reine Angew. Math. **165**, 184–193.

von Mises, R. (1938). Sur une inégalité pour les moments d'une distribution quasiconvexe, Bull. Sci. Math. (2) **62**, 68–71.

von Mises, R. (1939). The limits of a distribution function if two expected values are given, Ann. Math. Stat. **10**, 99–104.

Moriguti, S. (1951). Extremal properties of extreme value distributions, Ann. Math. Stat. **22**, 523–536.

Moriguti, S. (1953). A modification of Schwarz's inequality with applications to distributions, Ann. Math. Stat. **24**, 107–113.

Münzer, H. (1935). Ueber Verschärfungen der Tchebycheffschen Ungleichung, Skand. Aktuarietidskr. **18**, 279.

Münzer, H. (1950) Zur Abschaetzung von Wahrscheinlichkeiten, Mitteilungsblatt Math. Stat. **2**, 130–137.

Narumi, S. (1923). On further inequalities with possible application to problems in the theory of probability, Biometrika. **15**, 245–253.

Offord, A. C. (1945). An inequality for sums of independent random variables, Proc. London Math. Soc. (2) **48**, 467–477.

Olkin, Ingram, and John W. Pratt (1958). A multivariate Tchebycheff inequality, Ann. Math. Stat. **29**, 226–234.

Pearson, K. (1919). On generalized Tchebycheff theorems in the mathematical theory of statistics, Biometrika. **12**, 284–296.

Peek, R. L. (1933). Some new theorems on limits of variation, Bull. Amer. Math. Soc. **39**, 953–959.

Perks, Wilfred (1947). A simple proof of Gauss's inequality, J. Inst. Actuar. Students' Soc. **7**, 38–41.

Plackett, R. L. (1947). Limits of the ratio of mean range to standard deviation, Biometrika. **34**, 120–122.

Posse, C. (1886). *Sur quelques applications des fractions continuées algebriques,* 1-175. (St. Petersburg, Russia).

Richter, Hans (1957). Parameterfreie Abschaetzung und Realisierung von Erwartungswerten, B1. Deutsch. Ges. Versicherungsmath. **3**, 147–162.

Robbins, H. (1948). Some remarks on the inequality of Tchebycheff, *Courant Anniversary Volume.* 345–350. (Interscience Publishers, Inc., New York).

Romanovski, V. I. (1940). On inductive conclusions in statistics, C. R. (Doklady) Acad. Sci. URSS **27**, 419–421.

Rosenblatt, Alfred (1945). Sobre la ley fuerte de los grandes numeros, Actas Acad. Ci. Lima. **8**, 7–26.

Royden, H. L. (1953). Bounds on a distribution function when its first n moments are given, Ann. Math. Stat. **24**, 361–376.

Rustagi, Jagdish Sharan (1957). On minimizing and maximizing a certain integral with statistical applications, Ann. Math. Stat. **28**, 309–328.

Savage, I. R. (1953). Bibliography of non-parametric statistics and related topics, J. Amer. Stat. Assoc. **48**, 844–906.

Sealy, E. H. (1943). *Specification of tolerances on components and final assembly. Methods of tightening tolerances on final assembly.* British Ministry of Supply Advisory Service on Quality Control, Technical Report No. QC/R/3. (London).

Selberg, Henrik L. (1940a). Ueber eine Ungleichung der mathematischen Statistik, Skand. Aktuarietidskr. **23**, 114–120.

Selberg, Henrik L. (1940b). Zwei Ungleichungen zur Ergaenzung des Tchebycheffschen Lemmas, Skand. Aktuarietidskr. **23**, 121–125.

Selberg, Henrik L. (1940c). Ueber eine Verschaerfung der Tchebycheffschen Ungleichung, Arch. Math. Naturvid. **43**, 30–32.

Selberg, Henrik, L. (1942). On an inequality in mathematical statistics, Norsk Mat. Tidsskr. **24**, 1–12.

Shohat, J. A., and J. D. Tamarkin (1943). *The Problem of Moments,* Mathematical Surveys No. **1**, xiv+140 pp. (American Mathematical Society, Providence, R.I.)

Slutsky, Eugen (1925). Ueber stochastische Asymptoten und Grenzwerte, Metron. **5**, 3–90.

Smith, C. D. (1930). On generalized Tchebycheff inequalities in mathematical statistics, Amer. J. Math. **52**, 109–126.

Smith C. D. (1939). On Tchebycheff approximation for decreasing functions, Ann. Math. Stat. **10**, 190–192.

Smith, C. D. (1955). Tchebycheff inequalities as a basis for statistical tests, Math. Mag. **28**, 185–195.

Stanoyevitch, Tchaslav (1954). Sur une généralisation d'une inégalité de M. Kolmogoroff, C.R. Acad. Sci. (Paris) **239**, 854–856.

Stieltjes, T. J. (1884). Sur l'évaluation approchée des intégrales, C.R. Acad. Sci. (Paris) **97**, 798–799.

Stieltjes, T. J. (1894–1895). Recherches sur les fractions continués, Ann. Fac. Sci. Toulouse, Serie I, **8–J**, 1–122; **9–A**, 5–47.

Tchebycheff, P. L. (1867). Des valeurs moyennes, J. Math. (Liouville, 2d. Series) **12**, 177–184.

Tchebycheff, P. L. (1874). Sur les valeurs limités des intégrales, J. Math. Pures et Appl. (2) **19**, 157–160.

Tchebycheff, P. L. (1890). Sur deux théorèmes relatifs aux probabilités, Acta Math. **14**, 305–315.

Theil, H. (1949). A note on the inequality of Camp and Meidell, Statistica (Rijswijk) **3**, 201–208.

Trybula, S. (1959). On lower bounds for probability moments, Bull. Acad. Polonaise Sci., Ser. Math., Astr., Phys. **7**, 267–269.

Tukey, John W. (1946). An inequality for deviations from medians, Ann. Math. Stat. **17**, 75–78.

Ulin, Bengt (1953). An extremal problem in mathematical statistics, Skand. Aktuarietidskr. **36**, 158–167.

Uspensky, J. V. (1937). *Introduction to Mathematical Probability* ix+411. (McGraw-Hill, New York).

Wald, A. (1938). Generalization of the inequality of Markoff, Ann. Math. Stat. **9**, 244–255.

Wald, A. (1939). Limits of a distribution function determined by absolute moments and inequalities satisfied by absolute moments, Trans. Amer. Math. Soc. **46**, 280–306.

Whittle, P. (1958a). Continuous generalizations of Tchebichev's inequality, Teor. Veroyatnost i Primenen **3**, 386–394.

Whittle, P. (1958b). A multivariate generalization of Tchebichev's inequality, Quart. J. Math. Oxford Ser. (2) **9**, 232–240.

Whittle, P. (1960a). Quadratic forms in Poisson and multinomial variables, J. Australian Math. Soc. **1**, 233–240.

Whittle, P. (1960b). Bounds for the moments of linear and quadratic forms in independent variables, Teor. Veroyatnost. i Primenen. **5**, 331–334.

Winckler, A. (1866). Allgemeine Saetze zur Theorie der unregelmaessigen Beobachtungsfehler, S.-B. Kaiserlichen Akad. Wiss. Math.-Nat. Kl. **53**, 6–41.

Winsten, C. B. (1946). Inequalities in terms of mean range, Biometrika. **33**, 283–295.

Zelen, Marvin (1954). Bounds on a distribution function that are functions of moments to order four, J. Research NBS **53**, 377–381.

(Paper 65B3–59)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
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Publications of the National Bureau of Standards*

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Selected Abstracts

Torsional resonance vibrations of uniform bars of square cross section, W. E. Tefft and S. Spinner, *J. Research NBS* **65A** *(Phys. and Chem.) No. 3, 167 (May–June 1961) 75 cents.*

Relations by which the shear modulus may be computed from the fundamental and overtones of the torsional resonance frequencies of square bars have been established empirically.

The results are analyzed in terms of a proportionality factor, R, defined by the equation

$$G=\left(\frac{2lf_n}{n}\right)^2\rho R$$

R is found to increase with increasing cross section to length ratio. Also, the overtones are less than integral multiples of the fundamental by an amount which increases with increasing cross section to length ratio.

Crystallization of bulk polymers with chain folding: theory of growth of lamellar spherulites, J. D. Hoffman and J. I. Lauritizen, Jr. *J. Research NBS* **65A** *(Phys. and Chem.) No. 4, 297 (July–August 1961) 75 cents.*

A systematic study of the problem of spherulitic growth in linear polymers in bulk has been carried out. A calculation of the radial growth of polymer spherulites is given for four models. These concern growth where the surface nuclei that control the rate are (1) bundlelike and coherent; (2) chain folded and coherent; (3) chain folded and noncoherent; and (4) bundlelike and noncoherent. The required modifications of nucleation theory are given. Then the radial growth rate laws are derived for each model, and the type of "spherulite" that would be formed discussed.

The model with chain folded and coherent growth nuclei leads to a typical lamellar spherulite. The properties of the individual chain folded lamellae that form the spherulite are predicted, including the change of step height with growth temperature, melting behavior, and the behavior on recrystallization. (Chain folded lamellae may also occur in specimens that are not obviously spherulitic.) Under certain conditions, the noncoherent model with chain folds can lead to a modified lamellar spherulite. None of the bundlelike models will lead to a typical lamellar spherulite, though a spherical microcrystalline object might be formed. It is concluded that lamellar spherulites consist largely of chain folded structures.

The factors that could cause chain folded crystals to appear in profusion in bulk polymers are discussed. The case of homogeneous initiation is considered first. Homogeneous initiation of chain folded nuclei in bulk will prevail if the end surface free energy of the bundlelike nucleus exceeds that of the folded. It is shown that the end surface free energy of the bundlelike nucleus, as calculated with a density gradient model, will be larger than had been supposed previously. It is therefore considered to be theoretically possible that the end surface free energy of the bundlelike nucleus may in some cases exceed that of the folded nucleus. Attention is given to the possibility that folded structures appear in large numbers because cumulative strain or large chain ends prevent the growth of bundlelike nuclei to large size, even when the latter type of nucleus is energetically favored when small. Heterogeneous initiation of folded structures is then considered.

Other topics include: (1) conditions that might lead to nonlamellar or nonspherulitic crystallization in bulk; (2) the origin of the twist that is frequently exhibited by the lamellae in spherulites; (3) the transitions that may sometimes occur in the radial growth rate law; and (4) interlamellar links.

Comparison between mode theory and ray theory of VLF propagation, H. Volland, *J. Research NBS* **65D** *(Radio Prop.) No. 4, 357 (July–August 1961) 75 cents.*

It is shown that the field strength according to mode theory and ray theory in the VLF band are derivable from the same expression of the original vector potential, and the result of one theory is the analytic continuation of the other one in another range of convergence. In fact, both ranges of convergence overlap. Estimates of these ranges are made and an example shows that within this overlapping region (between distances of 300 and 2000 km) both theories give the same result. Using this fact calculations of frequency spectra are possible which in the case of a white noise show some similar features to measured frequency spectra of lightning discharges.

On the validity of some approximations to the Appleton-Hartree formula, K. Davies and G. A. M. King, *J. Research NBS* **65D** *(Radio Prop.) No. 4, 323 (July–August 1961) 75 cents.*

The validity of some commonly used quasi-transverse and quasi-longitudinal approximations to the Appleton magnetoionic formula is considered. Using the dipole approximation for the earth's magnetic field the various approximations for refractive index are compared with the values computed from the complete formula for various geomagnetic latitudes and a frequency of 2.0 megacycles per second. It is found that certain approximations become very poor only a short distance from where they are exact and so care must be taken in their use. It is shown that a choice of two suitable approximations yields refractive indices of sufficient accuracy for all geomagnetic latitudes. Certain approximations to the group refractive indices are also considered.

The minima of cyclic sums, K. Goldberg, *J. London Math. Soc.* **35,** *262–264 (1960).*

Given a complex valued function in m variables, defined on a set S, the (average) cyclic sum of this function is defined for $n \geq m$ variables. Letting M_n be the minimum absolute value of this cyclic sum over S, it is proved that $\lim_{n\to\infty} M_n=$ g.l.b. M_n.

An analysis of the accumulated error in a hierarchy of calibrations, E. L. Crow, *IRE Trans. Instrumentation* **I–9** *No. 2, 105–114 (Sept. 1960).*

Calibrations of many types are performed in a hierarchy of calibration laboratories fanning out from a national standard. Often the statement is made that the accuracy of each echelon of the hierarchy should be 10 times the accuracy of the immediately following echelon. The validity of such statements is examined by deriving formulas for the total error accumulated over the entire sequence when systematic and random errors may occur in each echelon, and by determining how a given total error may be achieved at minimum total cost under reasonable assumptions for the form of the cost-error functions.

Generating functions for formal power series in non-commuting variables, K. Goldberg, *Proc. Am. Math. Soc.* **11,** *No. 6, 988–991 (Dec. 1960).*

Generating functions in commuting variables are defined for formal power series in non-commuting variables. The effect on these generating functions of transformations on the non-commuting variables is determined. Application is made to the case of $\log f(x)g(y)$.

Tests for regression coefficients when errors are correlated, M. M. Siddiqui, *Ann. Math. Stat.* **31,** *No. 4, 929–938 (Dec. 1960).*

In a previous paper the covariances of least-squares estimates of regression coefficients and the expected value of the estimate of residual variance were investigated when the errors are assumed to be correlated. In this paper we will investigate the distribution of the usual test statistics for regression coefficients under the same assumptions.

On the nature of the crystal field approximation, C. M. Herzfeld and H. Goldberg, *J. Chem. Phys.* **34,** *No. 2, 643–651 (Feb. 1961).*

A new method is developed for the treatment of molecular interactions, and is applied to a system consisting of a hydrogen atom in a 2p state and a hydrogen molecule in the ground state. The interaction of these two species is calculated using ordinary crystal field theory and also the new method. A comparison of the results shows some of the shortcomings of the conventional crystal field theory, and provides corrections to it. The new method consists of (1) expanding all electron terms of the total Hamiltonian for the system which involve interactions between the atom and the ion, thus transforming the interaction Hamiltonian into sums of products of one-electron operators, and (2) of using properly antisymmetrized wave functions made up of products of atom and molecule eigenfunctions. The calculations show the effect of the neglect of overlap and exchange in ordinary crystal field theory.

List of Titles

Journal of Research 65A (Phys. and Chem.) No. 3 (May–June 1961) 75 cents.

International practical temperature scale of 1948. Text revision of 1960. H. F. Stimson.

Evaluation of the NBS unit of resistance based on a computable capacitor. R. D. Cutkosky.

Wavelengths and intensities in the first spectrum of bromine, 2000 to 13000 A. J. L. Tech and C. H. Corliss.

Torsional resonance vibrations of uniform bars of square cross section. W. E. Tefft and S. Spinner. (See above abstract.)

Infrared studies of aragonite, calcite, and vaterite type structures in the borates, carbonates, and nitrates. C. E. Weir and E. R. Lippincott.

Dielectric properties of polyamides: polyhexamethylene adipamide and polyhexamethylene sebacamide. A. J. Curtis.

Heat of formation of calcium aluminate monocarbonate at 25 °C. H. A. Berman and E. S. Newman.

Thermodynamic constants for association of isomeric chlorobenzoic and toluic acids with 1,3-diphenylguanidine in benzene. M. M. Davis and H. B. Hetzer.

Heats of combustion and formation of trimethylborane, triethylborane, and tri-*n*-butylborane. W. H. Johnson, M. V. Kilday, and E. J. Prosen.

Pyrolysis of linear copolymers of ethylene and propylene. S. Straus and L. A. Wall.

Pyrolysis of fluorocarbon polymers. L. A. Wall and S. Straus.

Preparation of fluoro- and bromofluoroaryl compounds by copyrolysis of bromofluoroalkanes. L. A. Wall, J. E. Fearn, W. J. Pummer, and R. E. Lowry.

Thermal stability of polydivinylbenzene and of copolymers of styrene with divinylbenzene and with tribinylbenzene. S. Straus and S. L. Madorsky.

Conformations of the pyranoid sugars. IV. Infrared absorption spectra of some fully acetylated pyranoses. R. Stuart Tipson and H. S. Isbell.

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Electrical properties and kinetics of electrode reactions. R. J. Brodd.

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Vapor pressures of platinum, iridium, and rhodium. R. F. Hampson, Jr., and R. F. Walker.

Crystallization of bulk polymers with chain folding: theory of growth of lamellar spherulites. J. D. Hoffman and J. I. Lauritzen, Jr. (See above abstract.)

Phase equilibrium relations in the binary system barium oxide-niobium pentoxide. R. S. Roth and J. L. Waring.

Solid state reactions involving oxides of trivalent cations. S. J. Schneider, R. S. Roth, and J. L. Waring.

Gamma irradiation of fluorocarbon polymers. R. E. Florin and L. A. Wall.

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On the validity of some approximations to the Appleton-Hartree formula. K. Davies and G. A. M. King. (See above abstract.)

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Comparison between mode theory and ray theory of VLF propagation. H. Volland. (See above abstract.)

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Bibliography of temperature measurement—January 1953 to June 1960, C. Halpern and R. J. Moffat, NBS Mono. 27 (1961) 15 cents.

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Units of weight and measure (United States customary and metric) definitions and tables of equivalents, L. V. Judson, NBS Misc. Publ. 233 (1960) 40 cents.

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Microtechnique for the infrared study of solids, diamonds, and sapphires as cell materials, E. R. Lippincott, F. E. Welsh, and C. E. Weir, Anal. Chem. 33, 137–143 (Jan. 1961).

Definitions relating to metals and metalworking, M. R. Meyerson and S. J. Rosenberg, Am. Soc. Metals Handb. **1,** 1–41 (1961).

Stress-rupture tests at 1350° F on type 304 stainless steel, W. D. Jenkins, W. A. Willard, and W. J. Youden, ASTM Bull. **1,** No. 2, 104–108 (Feb. 1961)

Techniques in calorimetry. I. A noble-metal thermocouple for differential use, E. D. West, Rev. Sci. Inst. **31,** No. 8, 896–897 (Aug. 1960).

Paratellurite, a new mineral from Mexico, G. Switzer and H. E. Swanson, Am. Mineralogist **45,** 1272 (Nov.–Dec. 1960).

Evaluation of micrometer and microscopical methods for for measuring thickness of floor coverings, E. Horowitz, J. Mandel, R. J. Capott, and T. H. Boone, ASTM Bull. **1,** No. 2, 99–102 (Feb. 1961).

Electrodeposition of metals from nonaqueous media, A. Brenner, Chapter on Electrodeposition, Encyclopedia Chem. Tech. 2d Suppl., 315–324 (Jan. 1961).

The nature, cause, and effect of porosity of electrodeposits VI. Note on a magnetic method of detecting corrosion currents, F. Ogburn and W. H. Roberts, Plating **48,** No. 2, 168–169 (Feb. 1961).

The height of maximum luminosity in an auroral arc, F. E. Roach, J. G. Moore, E. C. Bruner, Jr., H. Cronin, and S. M. Silverman, J. Geophys. Research **64,** No. 11, 3575–3580 (Nov. 1960).

Physical metallurgy and mechanical properties of materials: Ductility and the strength of metallic structures, J. M. Frankland, J. Eng. Mech. Div. Proc. Am. Soc. Civil Eng. **86,** No. EM 6, 45–52 (Dec. 1960).

Cryogenic impurity adsorption from hydrogen, M. J. Hiza, Chem. Engr. Progress **55,** No. 10, 68–71 (Oct. 1960).

Nomenclature for standards of radioactivity, A. G. McNish, Intern. J. Appl. Radiation and Isotopes **8,** 145–146 (Jan. 1960).

Lithium, M. R. Meyerson, Am. Soc. for Metals Handbook **1,** 1213 (1961).

Tensile cryostat for the temperature range 4° to 300° Kelvin, R. M. McClintock and K. A. Warren, ASTM Bull. **1.** No. 2, 95–98 (Feb. 1961).

Standards and testing, the key to quality, B. E. Foster, Eng. Bull., Purdue Univ. **XLIV,** No. 6, 54–62 (Nov. 1960).

A radio-frequency permittimeter, R. C. Powell and A. L. Rasmussen, IRE Trans. Instrumentation **I–9,** No. 2, 179–184 (Sept. 1960).

Subgroups of the modular group and sums of squares, M. Newman, Am, J. Math. **82,** No. 4, 761–778 (Oct. 1960).

Phase equilibria in the system cadmium oxide-niobium oxide, R. S. Roth, J. Am. Ceram. Soc. **44,** No. 1, 49–50 (Jan. 1961).

Redetermination of the choromium and nickel solvuses in the chromium-nickel system, C. J. Bechtoldt and H. C. Vacher, Trans. Metallurgical Soc., Am. Inst. Mining Engrs. **221,** 14–18 (Feb. 1961).

A modulated subcarrier technique of measuring microwave phase shifts, G. E. Schafer, IRE Trans. Instrumentation **I–9,** No. 2, 217–219 (Sept. 1960).

Hydrogen-bonding in calcium-deficient hydroxyapatites, A. S. Posner, J. M. Stutman, and E. R. Lippincott, Nature **188,** No. 4749, 486–487 (Nov. 1960).

Hydrogen sulfide precipitation of the elements from 0.2–0.5 normal hydrochloric acid, J. I. Hoffman, Chemist Analyst **50,** No. 1, 30 (Mar. 1961).

Dielectric constant and diaelectric loss of TiO_2 (Rutile) at low frequencies, R. A. Parker and J. H. Wasilik, Phys. Rev. **120,** No. 5, 1631–1637 (Dec. 1960).

Application of the method of polarized orbitals to the scattering of electrons from hydrogen, A. Temkin and J. C. Lamkin, Phys, Rev. **121,** No. 3, 788–794 (Feb. 1961).

A recommended standard resistor-noise test system, G. T. Conrad, Jr., N. Newman and A. P. Stansbury, IRE Trans. Component Parts **CP–7,** No. 3, 71–88 (Sept. 1960).

On the absorption spectrum of CF_2 and its vibrational analysis, D. E. Mann and B. A. Thrush, J. Chem. Phys. **33,** No. 6, 1732–1734 (Dec. 1960).

A transfer instrument for the intercomparison of microwave power meters, G. F. Engen, IRE Trans. Instrumentation **I–9,** No. 2, 202–208 (Sept. 1960).

Standards and measurements of microwave surface impedance, skin depth, conductivity and Q, H. E. Bussey, IRE Trans. Instrumentation **I–9,** No. 2, 171–175 (Sept. 1960).

Plating gun bores, V. A. Lamb and J. P. Young, Ordnance **XLV,** No. 245, 725–727 (Mar.–Apr. 1961).

Integrated starlight over the sky, F. E. Roach and L. R. Megill, Astrophys. J. **133,** No. 1, 228–242 (Jan. 1961).

Low temperature static seals using elastomers and plastics, D. H. Weitzel. R. F. Robbins, G. R. Bopp, and W. R. Bjorklund, Rev. Sci. Inst. **31,** No. 12, 1350–1351 (Dec. 1960).

Electrodeposition of metals from nonaqueous media, A. Brenner, Encyclopedia of Chemical Technology, 2d Suppl., 315–324 (Interscience Encyclopedia 1960).

FM and SSB radiotelephone tests on a VHF ionospheric scatter link during multipath conditions, J. W. Koch, W. B. Harding and R. J. Jansen, IRE Trans. Commun. Systems **CS–8,** No. 3, 183–186 (Sept. 1960).

Propagation of error in a chain of standards, A. G. McNish and J. M. Cameron, IRE Trans. Instrumentation **I–9,** No. 2, 101–104 (Sept. 1960).

Absolute measurement of temperatures of microwave noise sources, A. J. Estin, C. L. Trembath, J. S. Wells, and W. C. Daywitt, IRE Trans. Instrumentation **1–9,** No. 2, 209–213 (Sept. 1960).

Electron scattering in high magnetic field, A. H. Kahn, Phys. Rev. **119,** No. 4, 1189–1192 (Aug. 1960).

Measurement of reflections and losses of waveguide joints and connectors using microwave reflectometer techniques, R. W. Beatty, G. F. Engen, and W. J. Anson, IRE Trans. Instrumentation **I–9,** No. 2, 219–226 (Sept. 1960).

Microwave spectrum of cis–difluoroethylene, V. W. Laurie, J. Chem. Phys. **34,** No. 1, (Jan. 1961).

Surface roughness of gold castings, J. J. Barone, R. L. Huff, and G. Dickson, Dental Progr. **1,** No. 2, 78–84 (Jan. 1961).

The sensitivity of photographic film to 3-mev neutrons and to thermal neutrons, M. Ehrlich, Health Phys. **4,** 113–128 (1960).

A prototype rubidium vapor frequency standard, R. J. Carpenter, E. C. Beaty, P. L. Bender, S. Saito, and R. O. Stone, IRE Trans. Instrumentation **I–9,** No. 2 (Sept. 1960).

Low even configurations in the first spectrum of thorium (Th I), R. E. Trees, Physica **26,** 353–360 (1960).

Comparison measurements with intensity standards for high energy bremsstrahlung, J. S. Pruitt and W. Pohlit, Z. Naturforschung **15b,** No. 9, 617–619 (1960).

High-dispersion spectra of Jupiter, C. C. Kiess, C. H. Corliss, and H. K. Kiess, Astrophys. J. **132,** No. 1, 221–231 (July 1960).

Propagation of electromagnetic waves along a thin plasma sheet, J. R. Wait, Can. J. Phys. **38,** 1586–1594 (1960).

Preparation and properties of aromatic fluorocarbons, W. J. Pummer and L. A. Wall, J. Chem. Engrs. Data **6,** No. 1, 76–78 (Jan. 1961).

Stability of thermoset plastics at high temperatures, S. L. Madorsky and S. Straus, Modern Plastics **38,** 134–140 (Feb. 1961).

Immiscibility and the system lanthanum oxide boric-oxide, E. M. Levin, C. R. Robbins, and J. L. Waring, J. Am. Ceram. Soc. **44,** No. 2, 87–91 (Feb. 1961).

On the properties of the vapor pressure curve, E. H. Brown, Cryogenics **1,** No. 1, 37–40 (Sept. 1960).

Absorption and scattering of photons by holmium and erbium, E. G. Fuller and E. Hayward, Proc. Intern. Conf. Nuclear Structure, 761–766 (Kingston, Canada, 1960).

Spectroscopy of fluorine flames. I. Hydrogen-fluorine flame and the vibration-rotation emission spectrum of HF, D. E. Mann, B. A. Thrush, D. R. Lide, J. J. Ball, and N. Acquista, J. Chem. Phys. **34,** No. 2, 420–431 (Feb. 1961).

The absorption spectra of magnesium and manganese atoms in solid rare gas matrices, O. Schnepp, J. Phys. Chem. Solids **17**, Nos. 3/4, 188–195 (1961).

The Franck-Condon factor ($q_{v'v''}$) array to high vibrational quantum numbers for the $O_2(B^3\Sigma_u^- - X^3\Sigma_g^+)$ Schumann-Runge band system, R. W. Nicholls, Can. J. Phys. **38,** 1705–1711 (1960).

NBS—source of American Standards, W. A. Wildhack, ISA J. **8,**, No. 2, 45–50 (Feb. 1961).

Ionospheric mapping by numerical methods, W. B. Jones and R. M. Gallet, Telecommun. J. **12,**, 260–264 (Dec. 1960).

The activation energy for hydrogen atom addition to propylene, M. D. Scheer and R. Klein, J. Phys. Chem. **65,** 375–377 (1961).

Intramolecular rearrangements. I. *sec*-butyl acetate and *sec*-butyl formate, R. Borkowski and P. Ausloos, J. Am. Chem. Soc. **83,** No. 5, 1053–1056 (Mar. 1961).

Titanium (III) chloride and titanium (III) bromide (titanium trichloride and titanium tribromide), J. M. Sherfey, Chapter IVB, Sec. 17, p. 57–61, Book, Inorganic Syntheses, by E. G. Rochow (McGraw-Hill Book Co., New York, N. Y., 1960).

Science news writing, D. M. Gates, J. M. Parker, Science **133,** No. 3447, 211–214 (Jan. 1961).

Elastomers for static seals at cryogenic temperatures, D. H. Weitzel, R. F. Robbins, G. R. Bopp, and W. R. Bjorklund, Rev. Sci. Insts. **31,** No. 12, 1350–1351 (Dec. 1960).

The electromagnetic fields of a dipole in the presence of a thin plasma sheet, J. R. Wait, Appl. Sci. Research **8,** Sec. B, 397–417 (1960).

New standards for the space age, A. T. McPherson, Astronautics **6,** No. 1, 24–25; 50–54 (Jan. 1961).

Near infrared atmospheric transmission to solar radiation, D. M. Gates, J. Opt. Soc. Am. **50,** No. 12, 1299–1304 (Dec. 1960).

Determination of crystallite size distributions from X-ray line broadening, A. Bienenstock, J. Appl. Phys. **32,** No. 2, 187–189 (Feb. 1961).

Correlation of visual and subvisual auroras with changes in the outer Van Allen radiation zone, B. J. O'Brien, J. A. Van Allen, F. E. Roach, and C. W. Gartlein, IGY Bull., No. 45, 1–16 (Mar. 1961).

Arc source for high temperature gas studies, J. B. Shumaker, Jr., Rev. Sci. Instr. **32,** No. 1, 65–67 (Jan. 1961).

The dependency of the melting temperature of bulk homopolymers on the crystallization temperature, L. Mandelkern, J. Polymer Sci. **XLVII** issue 149, 494–496 (July 1960).

Spiral patterns in geophysics. V. Agy, J. Atmospheric and Terrest. Phys. **19,** 136–140 (1960).

Iron (99.9+), G. A. Moore and T. R. Shives, Metals Handb. **1,** 1206–1212 (1961).

The interfacial properties of polyesters at glass and water surfaces, R. R. Stromberg, J. Plastic Engr. **15,** 882–886 (Oct. 1959).

Ionospheric absorption at times of auroral and magnetic pulsations, W. H. Campbell and H. Leinback, J. Geophys. Research **66,** 1, 25–34 (Jan. 1961).

Some properties of new or modified excitation sources, M. Margoshes, Am. Soc. Testing Materials, Spec. Tech. Publ. 259, 46–58 (1959).

Seasonal and day-to-day changes of the central position of the S_q overhead current system, S. Matsushita, J. Geophys. Research **65,** No. 11, 3835–3839 (Nov. 1960).

The mechanism of electrolytic deposition of titanium from fused salt media, W. E. Reid, J. Electrochem. Soc. **108,** No. 4, 393–394 (Apr. 1961).

Studies of elevated temperature corrosion of type 310 stainless steel by vanadium compounds, H. L. Logan, Corrosion **17,** 109–111 (Apr. 1961).

Deep penetration of radiation, U. Fano and M. J. Berger, Proc. Symp. Appl. Math. **XI**, 43–59 (1961).

Deposition of iron from salts of fluoro-acids, J. H. Connor and V. A. Lamb, Plating **48,** No. 4, 388–389 (Apr. 1961).

Calculation of properties of magnetic deflection systems, S. Penner, Rev. Sci. Insts. **32,** No. 2, 150–160 (Feb. 1961).

A simple low-temperature specimen holder for an X-ray diffractometer, D. K. Smith, Norelco Reporter **VII,** No. 1, 11–12 (Jan.–Feb. 1961).

Stepless variable resistor for high currents, C. R. Yokley and J. B. Shumaker, Jr., Rev. Sci. Instr. **32,** No. 1, 6–8 (Jan. 1961).

Microbalance techniques for high temperature application, R. F. Walker, Book, Vacuum microbalance techniques, edited by M. J. Katz. **I,** 87–110, (Plenum Press Inc., New York, N.Y., 1961).

Investigation of bond in beam and pull-out specimens with high-yield-strength deformed bars, R. G. Mathey and D. Watstein, J. Am. Concrete Inst. **32,** No. 9, 1071–1090 (Mar. 1961).

Vacuum ultraviolet photolysis of ethane: Molecular detachment of hydrogen, H. Okabe and J. R. McNesby, J. Chem. Phys. **34,** No. 2, 668–669 (Feb. 1961).

Shielding calculations for civil defense, C. Eisenhauer, Health Phys. **4,** No. 2, 129–132 (1960).

Hydrogen atom reactions with propene at 77° K. Disproportionation and recombination, R. Klein and M. D. Scheer, J. Phys. Chem. **65,** 324–325 (1961).

The effect of solvents of the γ-ray radiolysis of methyle acetate and acetone, P. Ausloos, J. Am. Chem. Soc. **83,** No. 5, 1056–1060 (Mar. 1961).

A study of auroral coruscations, W. H. Campbell and M. H. Rees, J. Geophys. Research **66,** No. 1, 41–55 (Jan. 1961).

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