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Note on the "Baffled Piston" Problem

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King's integral expression governing the problem of the time-harmonic motion of a piston in a rigid wall is simplified.

In a previous paper [4][2] the case of an arbitrarily (not necessarily time-harmonic) moving disk in a rigid wall was discussed. The starting point was the eq (12) in the above quoted paper,

$$U=a\int_0^\infty (\tau^2+\gamma^2)^{-\frac{1}{2}} \exp[-z(\tau^2+\gamma^2)^{-\frac{1}{2}}] J_0(\tau\rho)J_1(\tau a)d\tau, \tag{1}$$

which, when γ is replaced by ik $(k=2\pi/\lambda)$ represents the velocity potential for the time harmonic case (King's integral [2]). The investigation of transient phenomena as carried out in [4] did not involve (1) directly, but its inverse Laplace transform with respect to γ, and therefore not much attention was paid to (1) itself. But the computation of the acoustic pressure distribution for the time-harmonic case makes an investigation of (1) itself imperative. Numerous approximate evaluations of (1) (with γ replaced by ik, $k>0$) have been carried out. (for instance, [1], [2], [3], [5], [6]). It may be noted in this connection that (1) can be considerably simplified in as much as that it can be transformed into an integral expression with an elementary integrand which is extended over a finite interval of integration. However, the expressions (22) and (23) in [4] for the transient field can be obtained as easily from this simplified expression for U.

We proceed now to replace the factor $(\tau^2+\gamma^2)^{-\frac{1}{2}}\exp[-z(\tau^2+\gamma^2)^{\frac{1}{2}}]$ by the integral formula in [4(9)]:

$$(\tau^2+\gamma^2)^{-\frac{1}{2}} \exp[-z(\tau^2+\gamma^2)^{\frac{1}{2}}] = \int_0^\infty J_0(\tau t)t(t^2+z^2)^{-\frac{1}{2}} \exp[-\gamma(t^2+z^2)^{\frac{1}{2}}]\,dt.$$

Upon interchanging the order of integration, one obtains instead of (1)

$$U=a\int_0^\infty t(t^2+z^2)^{-\frac{1}{2}} \exp[-\gamma(t^2+z^2)^{\frac{1}{2}}] \left\{\int_0^\infty J_0(\tau\rho)J_0(\tau t)J_1(\tau a)d\tau\right\}dt.$$

The inner integral is known [4(20)] and therefore, instead of (1),

$$U=a\int_0^\infty t(t^2+z^2)^{-\frac{1}{2}} \exp[-\gamma(t^2+z^2)^{\frac{1}{2}}]f(t)dt, \tag{2}$$

where

$$f(t)=\begin{cases} 0, & 0<a<|t-\rho| \\ (\pi a)^{-1} \arccos \dfrac{t^2+\rho^2-a^2}{2\rho t}, & |t-\rho|<a<t+\rho \\ a^{-1}, & a>t+\rho \end{cases}$$

Again, as before, two cases A and B have to be distinguished. The results are

Case A, $a>\rho$:

$$f(t)=\begin{cases} a^{-1} & 0<t<a-\rho \\ (\pi a)^{-1} \arccos \dfrac{t^2+\rho^2-a^2}{2\rho t} & a-\rho<t<a+\rho \\ 0 & t>a+\rho \end{cases}$$

Case B, $a<\rho$:

$$f(t)=(\pi a)^{-1} \arccos \frac{t^2+\rho^2-a^2}{2\rho t} \qquad \rho-a<t<\rho+a$$

$$=0 \qquad \text{otherwise.}$$

Therefore, inserting $f(t)$ into (2),

Case A, $\rho<a$:

$$U=\int_0^{a-\rho} t(t^2+z^2)^{-1/2} \exp[-\gamma(t^2+z^2)^{1/2}]dt + \pi^{-1}\int_{a-\rho}^{a+\rho} t(t^2+z^2)^{-1/2} \exp[-\gamma(t^2+z^2)^{1/2}] \arccos\left(\frac{t^2+\rho^2-a^2}{2\rho t}\right)dt.$$

or, upon evaluating the first integral,

$$U=\frac{1}{\gamma}\exp(-\gamma z)-\frac{1}{\gamma}\exp\{-\gamma[z^2+(a-\rho)^2]^{1/2}\} + \pi^{-1}\int_{a-\rho}^{a+\rho} t(t^2+z^2)^{-1/2} \exp[-\gamma(t^2+z^2)^{1/2}] \arccos\left(\frac{t^2+\rho^2-a^2}{2\rho t}\right)dt.$$

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[1] Figures in brackets indicate the literature references at the end of this paper.

Case B, $\rho > a$:

$$U = \pi^{-1} \int_{\rho-a}^{\rho+a} t(t^2+z^2)^{-1/2}$$

$$\exp[-\gamma(t^2+z^2)^{1/2}] \arccos\left(\frac{t^2+\rho^2-a^2}{2\rho t}\right) dt.$$

Substitute $t^2+z^2=\alpha^2$ and get

$$U = \frac{1}{\gamma}\exp(-\gamma z) - \frac{1}{\gamma}\exp\{-\gamma[z^2+(a-\rho)^2]^{1/2}\} + \pi^{-1}\int_{R'}^{R}$$

$$\exp(-\gamma\alpha) \arccos\left[\frac{\alpha^2-z^2+\rho^2-a^2}{2\rho(\alpha^2-z^2)^{1/2}}\right] d\alpha, \qquad \rho < a, \quad (3)$$

$$U = \pi^{-1}\int_{R'}^{R} \exp(-\gamma\alpha)$$

$$\arccos\left[\frac{\alpha^2-z^2+\rho^2-a^2}{2\rho(\alpha^2-z^2)^{1/2}}\right] d\alpha, \qquad \rho > a. \quad (4)$$

As in [4(21)], R and R' denote the smallest and the largest distance of a point of observation from the circumference of the disk respectively. These properties are given by

$$R = [z^2+(a+\rho)^2]^{1/2}, \qquad R' = [z^2+(a-\rho)^2]^{1/2}.$$

The expressions (3) and (4) for U are considerably simpler than [4(12)]. For the time-harmonic case, γ has to be replaced by ik. The expressions [4(22)] and [4(23)] for a "Dirac pulse" are immediately obtained from (3) and (4) by taking their Laplace inversions with respect to γ.

References

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