JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 65B, No. 3, July–September 1961

Theory of an Accurate Intermediary Orbit for Satellite Astronomy [1]

John P. Vinti

(May 16, 1961)

This paper derives an accurate intermediary orbit of an artificial satellite of an oblate planet. The drag-free motion takes place under the action of a gravitational potential which fits the even zonal harmonics exactly through the second and approximately through the fourth, in the case of the earth. This potential leads to separability of the Hamilton-Jacobi equation.

Two alternative sets of orbital elements are set forth. The first set is related directly to initial conditions, but requires numerical factoring of a certain quartic to evaluate some of the integrals. The second set, on the other hand, permits exact factoring of both quartics that appear, but is not related directly to initial conditions, so that its members can best be obtained by a least-square fit of the solution over many orbital revolutions.

The final solution is given in terms of certain uniformizing variables, whose periodic terms are correct through the second order in the oblateness parameter and whose secular terms are exact, for the intermediary orbit. These exact solutions for the secular terms are expressed by means of certain rapidly converging series, with complete avoidance of elliptic integrals of the third kind. Sections 9 and 10 give a summary and a list of symbols.

1. Introduction

The author has introduced a gravitational potential [1] [2] for an axially symmetric planet which accounts rather accurately for its oblateness and yet still leads to separability of the problem of satellite motion. The resulting solution is expected to yield an "intermediary orbit" somewhat more accurate than those heretofore used, in that it accounts for all of the second zonal harmonic and for more than half of the fourth zonal harmonic. (Previous intermediary orbits [2, 3] have accounted for only part of the second harmonic and have neglected the fourth harmonic.) The residual fourth harmonic, the odd harmonics, the tesseral harmonics, the lunar-solar forces, and aerodynamic and electromagnetic drag are then to be considered as producing perturbations of this intermediary orbit.

Izsak [4] has already given an analytical solution for this intermediary orbit, with both periodic and secular terms correct through the second order in a certain oblateness parameter. His solution makes rather heavy demands on the reader's knowledge of linear fractional transformations and the theory of elliptic functions in the complex plane. The present paper avoids these complications, with elliptic integrals occurring only in the simple forms of the complete first and second kinds. Furthermore the resulting solution not only gives the periodic terms correctly to the second order, but gives the secular terms "exactly"; i.e., to arbitrarily high order. I wish to acknowledge very explicitly, however, that I am greatly indebted to Izsak for the introduction of one of the sets of orbital elements that I use. Knowledge of this set, which permits exact factoring of a certain refractory quartic, has influenced my treatment of the whole problem.

2. Statement of Problem

If ρ, η, ϕ are the oblate spheroidal coordinates introduced in [1] and if r, θ, ϕ and X, Y, Z are the corresponding spherical and rectangular coordinates, then

$$X+iY=r\cos\theta\exp i\phi=[(\rho^2+c^2)(1-\eta^2)]^{\frac{1}{2}}\exp i\phi, \tag{2.1}$$

$$Z=r\sin\theta=\rho\eta, \qquad (-1\leqq\eta\leqq 1). \tag{2.2}$$

[1] This work was supported by the U.S. Air Force, through the Office of Scientific Research of the Air Research and Development Command.
[2] Figures in brackets indicate the literature references at the end of the paper.

Here r is the geocentric distance to the satellite and θ and ϕ are respectively its geocentric declination and right ascension. For sufficiently large r, $\rho \approx r$ and $\eta \approx \sin \theta$.

With the origin taken at the planet's center of mass, the intermediary orbit is then the path of a particle in the approximate potential field

$$V_a = -\mu\rho(\rho^2 + c^2\eta^2)^{-1}, \tag{2.3}$$

where μ is the product of the gravitational constant and the planetary mass and where

$$c^2 = r_e^2 J_2. \tag{2.4}$$

Here r_e is the equatorial radius of the planet and J_2 is the coefficient of the second zonal harmonic in the expansion of the planet's true potential in spherical harmonics. For the earth $J_2 = (1.08)$ 10^{-3}, to three significant figures.

According to [1], the coordinates ρ, η, ϕ satisfy the following equations, involving quadratures:

$$t + \beta_1 = \pm \int_{\rho_1}^{\rho} \rho^2 F^{-\frac{1}{2}}(\rho) d\rho \pm c^2 \int_0^{\eta} \eta^2 G^{-\frac{1}{2}}(\eta) d\eta, \tag{2.5}$$

$$\beta_2 = \mp \alpha_2 \int_{\rho_1}^{\rho} F^{-\frac{1}{2}}(\rho) d\rho \pm \alpha_2 \int_0^{\eta} G^{-\frac{1}{2}}(\eta) d\eta, \tag{2.6}$$

$$\phi - \beta_3 = \mp c^2\alpha_3 \int_{\rho_1}^{\rho} (\rho^2 + c^2)^{-1} F^{-\frac{1}{2}}(\rho) d\rho \pm \alpha_3 \int_0^{\eta} (1 - \eta^2)^{-1} G^{-\frac{1}{2}}(\eta) d\eta. \tag{2.7}$$

Here $F(\rho)$ and $G(\eta)$ are the quartic polynomials

$$F(\rho) = c^2\alpha_3^2 + (\rho^2 + c^2)(-\alpha_2^2 + 2\mu\rho + 2\alpha_1\rho^2), \tag{2.8}$$

$$G(\eta) = -\alpha_3^2 + (1 - \eta^2)(\alpha_2^2 + 2\alpha_1 c^2 \eta^2). \tag{2.9}$$

The α's and β's are the Jacobi constants, with the energy $\alpha_1 < 0$ for satellite motion and with the polar component of angular momentum $\alpha_3 \gtrless 0$ according as the orbit is direct or retrograde. To orient oneself, note that in the limiting case $c \to 0$ of Keplerian motion the separation constant α_2 reduces to the total angular momentum, $-\beta_1$ to the time of passage through perigee, β_2 to the argument ω of perigee, and β_3 to the right ascension Ω of the ascending node. The Jacobi constants may be determined, at least in principle, from the initial conditions; we have more to say about this point later. The constant ρ_1 is the next-to-the-largest real zero of $F(\rho)$ and thus is that zero of $F(\rho)$ which is closest to the smaller zero of

$$f(\rho) \equiv -\alpha_2^2 + 2\mu\rho + 2\alpha_1\rho^2. \tag{2.10}$$

To solve (2.5) through (2.9) for ρ, η, and ϕ as functions of t, we must first solve (2.5) and (2.6) for ρ and η and then substitute the results $\rho(t)$ and $\eta(t)$ into (2.7) to determine $\phi(t)$. To do so we must first evaluate the above six integrals, which we shall obtain in terms of certain uniformizing variables. In turn, evaluating these integrals presupposes knowing how to factor the quartics $F(\rho)$ and $G(\eta)$.

3. Factoring the Quartics: Orbital Elements a_0, e_0, i_0, β_1, β_2, β_3

In the case of elliptic motion ($c=0$) the perigee and apogee radii r_1 and r_2 would be the two zeros of $f(\rho)$, viz,

$$r_1 = a_0(1 - e_0), \tag{3.1}$$

$$r_2 = a_0(1 + e_0), \tag{3.2}$$

where

$$a_0 \equiv -\frac{1}{2}\mu\alpha_1^{-1} \tag{3.3}$$

$$e_0^2 \equiv 1+2\alpha_1\alpha_2^2\mu^{-2}. \tag{3.4}$$

In our present problem, with $c \neq 0$, we may still *define* constants a_0 and e_0 by (3.3) and (3.4), as well as another constant

$$i_0 \equiv \cos^{-1}(\alpha_3/\alpha_2). \tag{3.5}$$

The constants a_0, e_0, i_0, β_1, β_2, and β_3 constitute *one possible set of orbital constants.* We may also introduce the corresponding semi-latus rectum p_0, defined by

$$p_0 \equiv a_0(1-e_0^2), \tag{3.6}$$

so that

$$\alpha_2^2 = \mu p_0. \tag{3.7}$$

A determination of α_1, α_2, and α_3 would then furnish a_0, e_0, and i_0. If the subscript i denotes initial values, then (per unit mass)

$$\alpha_1 = \tfrac{1}{2}u_i^2 - \mu\rho_i(\rho_i^2+c^2\eta_i^2)^{-1}, \tag{3.8}$$

$$\alpha_3 = r_i^2\cos^2\theta_i\dot{\phi}_i = X_i\dot{Y}_i - Y_i\dot{X}_i, \tag{3.9}$$

where u means speed and a superscript dot denotes the time derivative. Also, by equations (50), (59.1), (13.2), (10.2), and (55) of [1],

$$\alpha_2^2 = (1-\eta_i^2)^{-1}[(\rho_i^2+c^2\eta_i^2)^2\dot{\eta}_i^2 + \alpha_3^2 - 2\alpha_1 c^2\eta_i^2(1-\eta_i^2)]. \tag{3.10}$$

Thus a knowledge of the initial coordinates and their initial derivatives would determine the α's and thus the constants a_0, e_0, and i_0.

A knowledge of their numerical values would then permit a numerical solution of the quartic equation $F(\rho)=0$ and thus furnish the numerical values of ρ_1, ρ_2, A, and B necessary to factor $F(\rho)$ into the form

$$F(\rho) \equiv -2\alpha_1(\rho-\rho_1)(\rho_2-\rho)(\rho^2+A\rho+B), \tag{3.11}$$

where ρ_1 and ρ_2 are the zeros of $F(\rho)$ closest to the values r_1 and r_2. Then, in the intermediary orbit, ρ is restricted to the interval $\rho_1 \leqq \rho \leqq \rho_2$ between two spheroids.

By equating the coefficients of corresponding powers of ρ in (3.11) and (2.8) we find

$$\rho^3: \quad \rho_1+\rho_2-A = -\mu\alpha_1^{-1} = 2a_0, \tag{3.12}$$

$$\rho^2: \quad B+\rho_1\rho_2-(\rho_1+\rho_2)A = c^2-\tfrac{1}{2}\alpha_2^2\alpha_1^{-1} = c^2+a_0p_0, \tag{3.13}$$

$$\rho: \quad (\rho_1+\rho_2)B-\rho_1\rho_2A = -\mu c^2\alpha_1^{-1} = 2a_0c^2, \tag{3.14}$$

$$\rho^0: \quad \rho_1\rho_2B = -\tfrac{1}{2}c^2(\alpha_2^2-\alpha_3^2)\alpha_1^{-1} = a_0p_0c^2\sin^2 i_0, \tag{3.15}$$

with use of (3.3) through (3.7).

By beginning with the zero-order solution $A=B=0$, $\rho_1+\rho_2=r_1+r_2=2a_0$, $\rho_1\rho_2=r_1r_2=a_0p_0$, one can solve this set of equations for the four unknowns A, B, $\rho_1+\rho_2$, and $\rho_1\rho_2$, by a method of successive approximations. If

$$k_0 \equiv c^2/p_0^2 \equiv (r_e/p_0)^2 J_2, \tag{3.16}$$

$$x \equiv (1-e_0^2)^{\frac{1}{2}}, \tag{3.17}$$

$$y \equiv \alpha_3/\alpha_2 = \cos i_0, \tag{3.18}$$

the second-order solution, through k_0^2, is

$$A = -2k_0 p_0 y^2[1+k_0(2x^2-3x^2y^2-4+8y^2)+\ldots], \tag{3.19}$$

$$B = k_0 p_0^2(1-y^2)[1+k_0(4y^2-x^2y^2)+\ldots], \tag{3.20}$$

$$\rho_1+\rho_2 = 2p_0x^{-2}[1-k_0x^2y^2-k_0^2x^2y^2(2x^2-3x^2y^2-4+8y^2)+\ldots], \tag{3.21}$$

$$\rho_1\rho_2 = p_0^2x^{-2}[1+k_0y^2(x^2-4)-k_0^2y^2(12x^2-x^4-20x^2y^2-16+32y^2+x^4y^2)+\ldots]. \tag{3.22}$$

The constants

$$a \equiv \tfrac{1}{2}(\rho_1+\rho_2), \tag{3.23}$$

$$e = \frac{\rho_2-\rho_1}{\rho_2+\rho_1}, \tag{3.24}$$

$$p \equiv a(1-e^2) \tag{3.25}$$

will occur throughout the evaluation of the ρ-integrals. In terms of a_0, e_0, and i_0, their values, to the second order in k_0, then satisfy

$$p/p_0 = 1+2k_0y^2(x^2-2)+k_0^2y^2(3x^4-2x^4y^2-16x^2+24x^2y^2+16-32y^2)+\ldots, \tag{3.26}$$

$$p_0/p = 1+2k_0y^2(2-x^2)+k_0^2y^2(-3x^4+6x^4y^2+16x^2-40x^2y^2-16+48y^2)+\ldots, \tag{3.27}$$

$$\frac{1-e^2}{1-e_0^2} = 1+k_0y^2(3x^2-4)+k_0^2y^2(5x^4-2x^4y^2-20x^2+28x^2y^2+16-32y^2)+\ldots, \tag{3.28}$$

$$\left(\frac{1-e^2}{1-e_0^2}\right)^{\frac{1}{2}} = 1+\frac{1}{2}\,k_0y^2(3x^2-4)+\frac{1}{8}\,k_0^2y^2(20x^4-17x^4y^2-80x^2+136x^2y^2+64-144y^2)+\ldots, \tag{3.29}$$

$$(1-e^2)^{\frac{1}{2}}p^{-1} = a_0^{-1}(1-e_0^2)^{-\frac{1}{2}}[1+\tfrac{1}{2}\,k_0y^2(4-x^2)+\tfrac{1}{8}\,k_0^2y^2(-4x^4+7x^4y^2+48x^2 -104x^2y^2-64+176y^2)+\ldots]. \tag{3.30}$$

In those places where e occurs alone in the theory, i.e., not in the combination $1-e^2$, it may of course be found by use of

$$e \equiv [1-(1-e^2)]^{\frac{1}{2}}. \tag{3.31}$$

By (3.28) this results in

$$e = [e_0^2+k_0x^2y^2(4-3x^2)+\ldots]^{\frac{1}{2}}, \tag{3.32}$$

so that when e_0 is comparable to k_0 it is not feasible to expand e in a power series in k_0; indeed if $e_0=0$, we should need a power series in $k_0^{\frac{1}{2}}$.

Direct use of this second-order solution in factoring $F(\rho)$ will lead to ρ-integrals that have secular terms correct only to $O(k_0^2)$. Since we are aiming at arbitrarily high accuracy for the secular terms, we include it here for other purposes. The most important of these purposes is to furnish information about the orders of various quantities in k_0; e.g., A and B are both of order k_0. Such information will be necessary in carrying out the solution of (2.5) through (2.7) for the periodic terms. The second purpose is to furnish a convenient starting point for any investigators who may choose to use a_0, e_0, and i_0, along with the β's, as orbital elements, and who will therefore have to solve the equation $F(\rho)=0$ numerically. A third purpose is for use in calculating the mean motions to first order, for comparison with other theories.

The quartic $G(\eta)$, which is quadratic in η^2, may be factored either as

$$G(\eta) \equiv -2\alpha_1c^2(\eta_0^2-\eta^2)(\eta_2^2-\eta^2) \tag{3.33}$$

or as

$$G(\eta)\equiv(\alpha_2^2-\alpha_3^2)\eta^4(\eta^{-2}-\eta_0^{-2})(\eta^{-2}-\eta_2^{-2}). \tag{3.34}$$

If we use the latter form, we find on comparing (3.34) with (2.9) that η_0^{-2} and η_2^{-2} are the roots of the following quadratic equation in η^{-2}:

$$(\alpha_2^2-\alpha_3^2)\eta^{-4}+(2\alpha_1c^2-\alpha_2^2)\eta^{-2}-2\alpha_1c^2=0. \tag{3.35}$$

Thus

$$\eta_0^{-2}=\tfrac{1}{2}(\alpha_2^2-2\alpha_1c^2)(\alpha_2^2-\alpha_3^2)^{-1}(1+W^{\frac{1}{2}}), \tag{3.36}$$

$$\eta_2^{-2}=\tfrac{1}{2}(\alpha_2^2-2\alpha_1c^2)(\alpha_2^2-\alpha_3^2)^{-1}(1-W^{\frac{1}{2}}), \tag{3.37}$$

where

$$W\equiv1+8\alpha_1c^2(\alpha_2^2-\alpha_3^2)(\alpha_2^2-2\alpha_1c^2)^{-2}. \tag{3.38}$$

From (3.36) and (3.38) it follows that for $\alpha_1<0$

$$\eta_0^2\leqq\alpha_2^{-2}(\alpha_2^2-\alpha_3^2)\leqq1 \tag{3.39}$$

and we shall see below that for satellite motion $\eta_2^2\gg1$. Since η^2 cannot exceed 1, it follows that in the actual motion η always lies in the interval $-\eta_0\leqq\eta\leqq\eta_0$ between two hyperboloids.

We readily find that

$$\eta_0=(\sin i_0)\,[1-\tfrac{1}{2}k_0x^2y^2+\tfrac{1}{8}k_0^2x^4y^2(7y^2-4)+\ldots], \tag{3.40}$$

$$(1-\eta_0^2)^{-\frac{1}{2}}=|\sec i_0|[1-\tfrac{1}{2}k_0x^2(1-y^2)+\tfrac{1}{8}k_0^2x^4(1-y^2)(5y^2-1)+\ldots], \tag{3.41}$$

$$\eta_2^{-2}=k_0x^2(1-k_0x^2y^2+\ldots), \tag{3.42}$$

$$(\eta_0/\eta_2)^2=k_0x^2(\sin^2 i_0)(1-2k_0x^2y^2+\ldots). \tag{3.43}$$

Note that $\eta_2^{-2}\leqq k_0\approx10^{-3}$, so that $\eta_2^2\geqq1000$.

4. Factoring the Quartics: Orbital Elements a, e, I, β_1, β_2, β_3

If we equate the coefficients of powers of η^2 in (3.33) with those of corresponding powers in (2.9), we find

$$\eta_0^2+\eta_2^2=1-\frac{\alpha_2^2}{2\alpha_1c^2}=1+a_0p_0/c^2, \tag{4.1}$$

$$\eta_0^2\eta_2^2=-\frac{\alpha_2^2-\alpha_3^2}{2\alpha_1c^2}=\frac{a_0p_0}{c^2}\sin^2 i_0. \tag{4.2}$$

If in (3.12) through (3.15) we use (3.23) through (3.25), we find

$$2a-A=2a_0, \tag{4.3}$$

$$B+ap-2Aa=c^2+a_0p_0, \tag{4.4}$$

$$2aB-Aap=2a_0c^2, \tag{4.5}$$

$$Bap=a_0p_0c^2\sin^2 i_0. \tag{4.6}$$

Suppose we now regard a, e, and

$$\eta_0\equiv\sin I \tag{4.7}$$

as known. (When we adopt a, e, I, and the β's as orbital elements we are certainly assuming so; we discuss later how they may be determined.) Then in treating the ρ-integrals we have five unknowns, viz, a_0, e_0, i_0, A, and B, and in treating the η-integrals one additional unknown, viz, η_2. Altogether then, we have six unknowns and we have six equations with which to determine them, (4.1) through (4.6). With *these* orbital elements, viz, a, e, and η_0, however, it turns out that the system can be solved *exactly* and with considerable ease, for the required unknowns. This property of a, e, and η_0 was first pointed out by Izsak [4].

To carry out the solution, first eliminate η_2 from (4.1) and (4.2). The result is

$$\frac{\alpha_2^2-\alpha_3^2}{\alpha_2^2}\equiv\sin^2 i_0=\eta_0^2+\frac{c^2\eta_0^2(1-\eta_0^2)}{a_0p_0}. \tag{4.8}$$

On inserting (4.8) into (4.6), we find

$$Bap=c^2\eta_0^2\,[a_0p_0+c^2(1-\eta_0^2)]. \tag{4.9}$$

If we now use (4.3) to eliminate a_0 from (4.5) and (4.9) to eliminate a_0p_0 from (4.4), we find a pair of simultaneous linear equations for A and B:

$$(ap-c^2)A-2aB=-2ac^2, \tag{4.10}$$

$$2c^2\eta_0^2aA+(ap-c^2\eta_0^2)B=c^2\eta_0^2(ap-c^2\eta_0^2). \tag{4.11}$$

Their solution is

$$A=-\frac{2ac^2(1-\eta_0^2)(ap-c^2\eta_0^2)}{(ap-c^2)(ap-c^2\eta_0^2)+4a^2c^2\eta_0^2}, \tag{4.12}$$

$$B=c^2\eta_0^2\,\frac{(ap-c^2)(ap-c^2\eta_0^2)+4a^2c^2}{(ap-c^2)(ap-c^2\eta_0^2)+4a^2c^2\eta_0^2}. \tag{4.13}$$

Then, from (4.3) and (4.12)

$$-\frac{\mu}{2a\alpha_1}\equiv\frac{a_0}{a}=1-\frac{A}{2a}=1+\frac{c^2(1-\eta_0^2)(ap-c^2\eta_0^2)}{(ap-c^2)(ap-c^2\eta_0^2)+4a^2c^2\eta_0^2} \tag{4.14}$$

and from (4.9) and (4.13)

$$-\frac{\alpha_2^2}{2\alpha_1}\equiv a_0p_0=-c^2(1-\eta_0^2)+ap\,\frac{(ap-c^2)(ap-c^2\eta_0^2)+4a^2c^2}{(ap-c^2)(ap-c^2\eta_0^2)+4a^2c^2\eta_0^2}. \tag{4.15}$$

Equation (4.8) and the relation $\eta_0\equiv\sin I$ give

$$\alpha_3=\alpha_2\left(1-\frac{c^2\eta_0^2}{a_0p_0}\right)^{\frac{1}{2}}\cos I, \tag{4.15a}$$

so that (4.15) and (4.15a) determine α_3. Finally, to obtain η_2, combine (4.2) and (4.6) to obtain $\eta_2^{-2}=c^4\eta_0^2\,(Bap)^{-1}$ and then use (4.13). The result is

$$\eta_2^{-2}=\frac{c^2}{ap}\,\frac{(ap-c^2)(ap-c^2\eta_0^2)+4a^2c^2\eta_0^2}{(ap-c^2)(ap-c^2\eta_0^2)+4a^2c^2}. \tag{4.16}$$

This completes the solution for the required unknowns when the orbital elements are a, e, and I. In terms of these orbital elements we can now factor the two quartics $F(\rho)$ and $G(\eta)$ exactly and thus evaluate all the integrals.

With use of the old oblateness parameter $k_0\equiv c^2/p_0^2$ and a new one, suitable for use with this second set of orbital elements, viz,

$$k\equiv c^2/p^2\equiv(r_e/p)^2J_2, \tag{4.17}$$

we can readily show that, to the first order, the equations of sections 3 and 4 give similar results. Thus we readily obtain

$$A\approx-2k_0p_0\cos^2 i_0\approx-2kp\cos^2 I, \tag{4.18}$$

$$B\approx k_0p_0^2\sin^2 i_0\approx kp^2\sin^2 I, \tag{4.19}$$

$$\frac{a_0}{a}\approx1+k_0(1-e_0^2)\cos^2 i_0\approx1+k(1-e^2)\cos^2 I, \tag{4.20}$$

$$\frac{p_0}{p} \approx 1+2k_0(1+e_0^2) \cos^2 i_0 \approx 1+2k(1+e^2) \cos^2 I, \tag{4.21}$$

$$\frac{1-e_0^2}{1-e^2} \approx 1+k_0(1+3e_0^2) \cos^2 i_0 \approx 1+k(1+3e^2) \cos^2 I, \tag{4.22}$$

$$\frac{\sin i_0}{\sin I} \approx 1+\frac{1}{2}k_0(1-e_0^2) \cos^2 i_0 \approx 1+\frac{1}{2}k(1-e^2) \cos^2 I, \tag{4.23}$$

$$\eta_2^{-2} \approx k_0(1-e_0^2)+ \ . \ . \ . \approx k(1-e^2)+ \ . \ . \ ., \tag{4.24}$$

$$(\eta_0/\eta_2)^2 \approx k_0(1-e_0^2) \sin^2 i_0 \approx k(1-e^2) \sin^2 I+ \ . \ . \ . \ . \tag{4.25}$$

With either set of orbital elements, the final solution will be given in terms of a, e, I, β_1, β_2, and β_3, and certain angles E, v, and ψ, analogous respectively to the eccentric anomaly, the true anomaly, and the argument of latitude in elliptic motion. Once one knows a, e, and I, one can then determine the β's by observing, at various times, whatever quantities will best serve to determine E, v, and ψ.

If one is using a_0, e_0, and i_0 as elements, one has to determine them from initial values or from some procedure equivalent to determining initial values. One then has to factor the quartic $F(\rho)$ numerically to determine a, e, and $\eta_0 \equiv \sin I$.

If one is using a, e, and I as elements, one has to determine them by following the orbit for many revolutions and then applying some sort of iterated least-square process. In this case one can then find the Jacobi constants α_1, α_2, and α_3 and thus a_0, e_0, and i_0 by means of the equations of this section.

In any event, the determination of orbital elements, by comparison of theory and observation, is ordinarily considered a completely separate problem in celestial mechanics from the calculation of the motion for given orbital elements. We have included the above remarks only to aid in the possible application of the present theoretical solution. Indeed the problem is further complicated by the small perturbations that occur in practice.

In the rest of the paper we simply assume a, e, $\eta_0 \equiv \sin I$, β_1, β_2, and β_3 to be known and then complete the solution for the intermediary orbit. There will be one restriction, however. The method of evaluating the ρ-integrals will be found to depend on the orbital inclination i_0 or I. For values of I less than 1°54′ or greater than 178°6′, a different approach would be needed. We shall therefore restrict considerations in the present paper to orbits that have inclinations between these two values, thereby ruling out equatorial or almost equatorial orbits.

5. The ρ-Integrals

In (2.5) through (2.7) the ρ-integrals are

$$R_1 \equiv \pm \int_{\rho_1}^{\rho} \rho^2 F^{-\frac{1}{2}}(\rho) d\rho, \tag{5.1}$$

$$R_2 \equiv \pm \int_{\rho_1}^{\rho} F^{-\frac{1}{2}}(\rho) d\rho, \tag{5.2}$$

$$R_3 \equiv \pm \int_{\rho_1}^{\rho} (\rho^2+c^2)^{-1} F^{-\frac{1}{2}}(\rho) d\rho, \tag{5.3}$$

where $F(\rho)$ is given by (3.11), A and B by (4.12) and (4.13), p by (3.25), and ρ_1 and ρ_2 by

$$\rho_1 = a(1-e), \qquad \rho_2 = a(1+e). \tag{5.4}$$

Equations (5.4) follow from (3.23) and (3.24). Then

$$F^{-\frac{1}{2}}(\rho) = (-2\alpha_1)^{-\frac{1}{2}} [(\rho-\rho_1)(\rho_2-\rho)]^{-\frac{1}{2}} \rho^{-1} (1+A\rho^{-1}+B\rho^{-2})^{-\frac{1}{2}}. \tag{5.5}$$

If, for convenience, we put

$$A \equiv -2b_1, \qquad B \equiv b_2^2, \qquad \lambda \equiv b_1/b_2 \tag{5.6}$$

and

$$h \equiv b_2/\rho, \tag{5.7}$$

then

$$(1+A\rho^{-1}+B\rho^{-2})^{-\frac{1}{2}} \equiv (1-2\lambda h+h^2)^{-\frac{1}{2}} \tag{5.8}$$

$$= \sum_{n=0}^{\infty} h^n P_n(\lambda), \tag{5.9}$$

if $|h|<1$ and $|\lambda|<1$.

The expansion (5.9) will be a convenient tool for evaluating all the ρ-integrals, whenever it can be used. To see when, use (5.6), (5.7), (4.18), and (4.19). Then to $O(k)$

$$b_1 = kp\cos^2 I, \qquad b_2 = k^{\frac{1}{2}} p \sin I,$$

so that $b_2/\rho \ll 1$ and

$$\lambda = k^{\frac{1}{2}} \cos^2 I \csc I > 0.$$

Thus to $O(k)$, $|\lambda|<1$ whenever

$$k^{\frac{1}{2}} \cos^2 I \csc I < 1,$$

or whenever

$$\tan^2 I + \tan^4 I > k$$

or

$$2\tan^2 I > (1+4k)^{\frac{1}{2}} - 1.$$

To $O(k)$, this becomes

$$\tan^2 I > k.$$

But $k=(r_e/p)^2 J_2$ and $J_2=(1.08)10^{-3}$ for the earth, so that $\lambda<1$ whenever

$$|\tan I| > 0.033\, r_e/p.$$

For close orbits $r_e/p \approx 1$, so that the necessary condition becomes

$$I_c < I < 180° - I_c, \qquad I_c \approx 1°54'.$$

The expansion will thus work for all satellite orbits that are inclined more than 1°54′ to the equator. For those orbits that lie closer to the equator one must use some other method to evaluate the ρ-integrals. On inserting (5.7) and (5.9) into (5.5), we find

$$F(\rho)^{-\frac{1}{2}} = (-2\alpha_1)^{-\frac{1}{2}} \sum_{n=0}^{\infty} b_2^n P_n(\lambda) \rho^{-1-n} [(\rho-\rho_1)(\rho_2-\rho)]^{-\frac{1}{2}}. \tag{5.10}$$

With use of (5.1), (5.10), and (5.6), we find

$$(-2\alpha_1)^{\frac{1}{2}} R_1 = \int_{\rho_1}^{\rho} (\rho+b_1)[(\rho-\rho_1)(\rho_2-\rho)]^{-\frac{1}{2}}(\pm d\rho) + \sum_{n=2}^{\infty} b_2^n P_n(\lambda) \int_{\rho_1}^{\rho} \rho^{1-n}[(\rho-\rho_1)(\rho_2-\rho)]^{-\frac{1}{2}}(\pm d\rho). \tag{5.11}$$

Each of the separate integrals in (5.11) is a multiple-valued function of ρ. It is appropriate to change variables in each to a uniformizing variable E or v, analogous respectively to the eccentric and true anomalies in elliptic motion. We define E and v by requiring them to satisfy

$$\rho = a(1-e\cos E) = (1+e\cos v)^{-1} p \tag{5.12}$$

and always to increase with time. Then, from (5.12), (5.4), and (3.23) through (3.25), we find

$$[(\rho-\rho_1)(\rho_2-\rho)]^{-\frac{1}{2}}(\pm d\rho) = dE = (1-e^2)^{\frac{1}{2}}(1+e\cos v)^{-1} dv \tag{5.13}$$

5.1. The Integral R_1

On introducing E into the first integral in (5.11) and v into each of the integrals in the series, we find

$$(-2\alpha_1)^{\frac{1}{2}}R_1=b_1E+a(E-e\sin E)+(1-e^2)^{\frac{1}{2}}p\sum_{n=2}^{\infty}(b_2/p)^nP_n(\lambda)\int_0^v(1+e\cos v)^{n-2}dv. \quad (5.14)$$

To investigate the convergence of the series we write

$$S_1\equiv\sum_{n=2}^{\infty}(b_2/p)^nP_n(\lambda)\int_0^v(1+e\cos v)^{n-2}dv \quad (5.15)$$

$$=(b_2/p)^2\sum_{m=0}^{\infty}(b_2/p)^mP_{m+2}(\lambda)\int_0^v(1+e\cos v)^m dv. \quad (1.16)$$

Then

$$|S_1|\leqq(b_2/p)^2\sum_{m=0}^{\infty}(b_2/p)^m|P_{m+2}(\lambda)|(1+e)^m v \quad (5.17)$$

and since $|P_n(\lambda)|\leqq 1$ for all $|\lambda|\leqq 1$, we have

$$|S_1|\leqq(b_2/p)^2v\sum_{m=0}^{\infty}\left[\frac{b_2(1+e)}{p}\right]^m \quad (5.18)$$

$$\leqq\frac{(b_2/p)^2v}{1-b_2(1+e)/p}. \quad (5.19)$$

By (4.19) and (5.6), however, $b_2/p\approx k^{\frac{1}{2}}\sin I$ and of course $1+e\leqq 2$, so that

$$b_2(1+e)/p\leqq 2k^{\frac{1}{2}}\sin I\leqq 2k^{\frac{1}{2}}. \quad (5.20)$$

Since $k\approx 10^{-3}$, we have $b_2(1+e)/p\leqq 0.063$. Thus the series S_1 converges, and converges more rapidly than a geometric series of common ratio $\approx 1/16$. Actually, since we have shown that the series S_1 converges absolutely, we can regroup it into the sum of a series S_{1e} containing only the even values of n and a series S_{10} containing only the odd values of n. It is then a simple matter to show that $S_1\equiv S_{1e}+S_{10}$ converges more rapidly than the geometric series $[1+b_2(1+e)/p]\cdot\sum_{n=0}^{\infty}[b_2(1+e)/p]^{2n}$. That is, we can actually expect the convergence of S_1 to be as rapid as that of a geometric series of common ratio $[b_2(1+e)/p]\leqq 4k\approx 1/250$.

To decompose the series S_1 into a part proportional to v and a periodic part, note first that if

$$f_m(v)\equiv\int_0^v(1+e\cos v)^m dv, \quad (5.21)$$

then $f_n(v)-(2\pi)^{-1}f_m(2\pi)v$ is an odd function of v, of period 2π. But $f_m(2\pi)=2f_m(\pi)$, so that we obtain

$$f_m(v)\equiv\int_0^v(1+e\cos v)^m dv=\pi^{-1}v\int_0^{\pi}(1+e\cos v)^m dv+\sum_{j=1}^{\infty}c_{mj}\sin jv. \quad (5.22)$$

To obtain, to any order in k, the parts of the ρ-integrals proportional to v, we shall need to consider all integral values of m in $\pi^{-1}f_m(\pi)$. To obtain the periodic parts correct to order k^2, we shall need values of m only up to 4.

To obtain a convenient expression for the v-term of $f_m(v)$, note that [5]

$$\int_0^{\pi}(z+\sqrt{z^2-1}\cos v)^m dv=\pi P_m(z), \quad (5.23)$$

for all values of z, including real values greater than unity. Here $P_m(z)$ is the Legendre polynomial

$$P_m(z)\equiv\frac{2^{-m}}{m!}\frac{d^m}{dz^m}(z^2-1)^m, \quad (5.24)$$

of the same polynomial form in z that holds when $|z|<1$, when it can be defined by the usual generating function. If we put

$$z=(1-e^2)^{-\frac{1}{2}} \tag{5.25}$$

in (5.23), we find

$$\int_0^\pi (1+e\cos v)^m dv=\pi(1-e^2)^{m/2}P_m[(1-e^2)^{-\frac{1}{2}}] \tag{5.26}$$

$$=\pi R_m(\sqrt{1-e^2}), \tag{5.27}$$

where

$$R_m(x)\equiv x^m P_m(1/x) \qquad (0\leqq x\leqq 1). \tag{5.28}$$

Thus $R_m(x)$ is a polynomial of degree $[m/2]$ in x^2. The first few of these polynomials are given in table 1.

TABLE 1

m	$R_m(x)$
0	1
1	1
2	$\frac{1}{2}(3-x^2)$
3	$\frac{1}{2}(5-3x^2)$
4	$(\frac{1}{8})(35-30x^2+3x^4)$
5	$(\frac{1}{8})(63-70x^2+15x^4)$
6	$(\frac{1}{16})(231-315x^2+105x^4-5x^6)$

From (5.22) and (5.27) we then obtain

$$\int_0^v (1+e\cos v)^m dv=R_m(\sqrt{1-e^2})v+\sum_{j=1}^{\infty} c_{mj}\cdot\sin jv. \tag{5.29}$$

Through $m=4$, the coefficients c_{mj} are easy to find, simply by binomial expansion of the integrand and conversion to a trigonometric polynomial.
The results are given in the following table:

TABLE 2. *Coefficients c_{mj} in (5.29)*

$m \backslash j$	1	2	3	4
0				
1	1			
2	$2e$	$e^2/4$		
3	$3e+3e^3/4$	$3e^2/4$	$e^3/12$	
4	$4e+3e^3$	$3e^2/2+e^4/4$	$e^3/3$	$e^4/32$

In (5.14), $b_2^n=O(k^{n/2})$ and $P_n(\lambda)=O(\lambda^0)=O(k^0)$ if n is even or $P_n(\lambda)=O(\lambda)=O(k^{\frac{1}{2}})$ if n is odd. On inserting (5.29) into (5.14), using table 2, and keeping periodic terms through $O(k^2)$ only, we find

$$(-2\alpha_1)^{\frac{1}{2}}R_1=b_1E+a(E-e\sin E)+A_1v+\sum_{j=1}^{2}A_{1j}\cdot\sin jv, \tag{5.30}$$

where

$$A_1=(1-e^2)^{\frac{1}{2}}p\sum_{n=2}^{\infty}(b_2/p)^n P_n(\lambda)R_{n-2}(\sqrt{1-e^2}) \tag{5.31}$$

and

$$A_{11}=\frac{3(1-e^2)^{\frac{1}{2}}}{4p^3}(-2b_1b_2^2p+b_2^4)e, \tag{5.32}$$

$$A_{12}=\frac{3(1-e^2)^{\frac{1}{2}}}{32p^3}b_2^4e^2. \tag{5.33}$$

The above proof of convergence of the series for R_1 also shows the rapid convergence of the series (5.31) for the coefficient A_1 of the v-term.

5.2. The Integral R_2

On inserting (5.10) into (5.2) and using (5.12) and (5.13), we find

$$(-2\alpha_1)^{\frac{1}{2}} R_2 = (1-e^2)^{\frac{1}{2}} p^{-1} \sum_{n=0}^{\infty} (b_2/p)^n P_n(\lambda) \int_0^v (1+e\cos v)^n dv. \tag{5.34}$$

As above, one can show at once that this series converges more rapidly than the geometric series $v\sum_{n=0}^{\infty}[b_2(1+e)/p]^n$, with common ratio $b_2(1+e)/p \leqq 2k^{\frac{1}{2}} \approx 0.066$. The same proof then applies to the coefficient A_2 in

$$(-2\alpha_1)^{\frac{1}{2}} R_2 = A_2 v + \sum_{j=1}^{4} A_{2j} \sin jv. \tag{5.35}$$

Using the same methods as for R_1, we then find

$$A_2 = (1-e^2)^{\frac{1}{2}} p^{-1} \sum_{n=0}^{\infty} (b_2/p)^n P_n(\lambda) R_n(\sqrt{1-e^2}) \tag{5.36}$$

and, through periodic terms of $O(k^2)$:

$$A_{21} = (1-e^2)^{\frac{1}{2}} p^{-1} e[b_1 p^{-1} + (3b_1^2 - b_2^2) p^{-2} - \tfrac{9}{2} b_1 b_2^2 (1+e^2/4) p^{-3} + \tfrac{3}{8} b_2^4 (4+3e^2) p^{-4}], \tag{5.37}$$

$$A_{22} = (1-e^2)^{\frac{1}{2}} p^{-1} \left[\frac{e^2}{8}(3b_1^2 - b_2^2) p^{-2} - \tfrac{9}{8} e^2 b_1 b_2^2 p^{-3} + \frac{3b_2^4}{32}(6e^2+e^4) p^{-4}\right], \tag{5.38}$$

$$A_{23} = (1-e^2)^{\frac{1}{2}} p^{-1} \frac{e^3}{8} (-b_1 b_2^2 p^{-3} + b_2^4 p^{-4}), \tag{5.39}$$

$$A_{24} = (1-e^2)^{\frac{1}{2}} p^{-5} \frac{3}{256} b_2^4 e^4. \tag{5.40}$$

5.3. The Integral R_3

If in (5.3) we now use the binomial expansion

$$(\rho^2 + c^2)^{-1} = \rho^{-2} \sum_{j=0}^{\infty} (-1)^j c^{2j} \rho^{-2j} \tag{5.41}$$

and insert the expressions (5.10), (5.12), and (5.13), we find

$$(-2\alpha_1)^{\frac{1}{2}} R_3 = (1-e^2)^{\frac{1}{2}} p^{-3} \int_0^v \sum_{j=0}^{\infty} (-1)^j (c/p)^{2j} (1+e\cos v)^{2j} \sum_{n=0}^{\infty} (b_2/p)^n P_n(\lambda) (1+e\cos v)^{n+2} dv, \tag{5.42}$$

where the integrand is the product of two series, each of which converges absolutely for any value of v. Then [6] it is equal to the series formed by summing the products of the individual terms, taken in any order, and this resulting series is itself absolutely convergent, for any value of v. It is therefore uniformly convergent, by the Weierstress M-test [7], so that it may be integrated term by term.

Let us now rewrite (5.42) in the form

$$(-2\alpha_1)^{\frac{1}{2}} R_3 = (1-e^2)^{\frac{1}{2}} p^{-3} \int_0^v \sum_{m=0}^{\infty} D_m (1+e\cos v)^{m+2} dv. \tag{5.43}$$

Here

$$D_m = \sum d_j \delta_{n'}, \tag{5.44}$$

summed over all those non-negative integral values of j and n' for which

$$2j + n' = m, \tag{5.45}$$

and where

$$d_j=(-1)^j(c/p)^{2j}, \tag{5.46}$$

$$\delta_{n'}=(b_2/p)^{n'}P_{n'}(\lambda). \tag{5.47}$$

Then, because of the uniform convergence,

$$(-2\alpha_1)^{\frac{1}{2}}R_3=(1-e^2)^{\frac{1}{2}}p^{-3}\sum_{m=0}^{\infty}D_m\int_0^v(1+e\cos v)^{m+2}dv. \tag{5.48}$$

It is interesting to investigate here the rapidity of convergence of the series

$$S\equiv\sum_{n=0}^{\infty}D_m\int_0^v(1+e\cos v)^{m+2}dv. \tag{5.49}$$

If m is even, we have $m=2i$ and $n'=2n$, so that $2j+2n=2i$ and $j=i-n$. Then

$$D_m=D_{2i}=\sum_{n=0}^{i}d_{i-n}\delta_{2n}=\sum_{n=0}^{i}(-1)^{i-n}(c/p)^{2i-2n}(b_2/p)^{2n}P_{2n}(\lambda), \tag{5.50}$$

so that

$$|D_{2i}|\leqq(c/p)^{2i}\sum_{n=0}^{i}(b_2/c)^{2n}. \tag{5.51}$$

But $c^2=kp^2$ and $b_2^2\approx kp^2\sin^2 I$, by (4.19), so that

$$(b_2/c)^2\approx\sin^2 I\leqq 1.$$

Then

$$|D_{2i}|\leqq(i+1)k^i. \tag{5.52}$$

If m is odd, we have $m=2i+1$ and $n'=2n+1$, so that

$$D_{2i+1}=\sum_{n=0}^{i}d_{i-n}\delta_{2n+1}=\sum_{n=0}^{i}(-1)^{i-n}(c/p)^{2i-2n}(b_2/p)^{2n+1}P_{2n+1}(\lambda). \tag{5.53}$$

Then

$$|D_{2i+1}|\leqq(c/p)^{2i}(b_2/p)\sum_{n=0}^{i}(b_2/c)^{2n}\leqq k^ik^{\frac{1}{2}}(i+1)\sin I. \tag{5.54}$$

Then, breaking up S into an even series S_e and an odd series S_o, we find

$$|S_e|\leqq\sum_{i=0}^{\infty}(i+1)k^i(1+e)^{2i+2}v\leqq(1+e)^2v\sum_{i=0}^{\infty}(i+1)[k(1+e)^2]^i. \tag{5.55}$$

Using $\sum_0^\infty x^i=(1-x)^{-1}$ and $\sum_1^\infty ix^i=x(1-x)^{-2}$, we find $\sum_0^\infty(i+1)x^i=(1-x)^{-2}$, so that

$$|S_e|\leqq\frac{(1+e)^2v}{[1-k(1+e^2)]^2}. \tag{5.56}$$

Similarly

$$|S_o|\leqq k^{\frac{1}{2}}(1+e)^3v\sin I\sum_{i=0}^{\infty}(i+1)[k(1+e)^2]^i \tag{5.57}$$

$$\leqq\frac{k^{\frac{1}{2}}(1+e)^3v\sin I}{[1-k(1+e)^2]^2} \tag{5.58}$$

Thus,

$$|S|\leqq\frac{(1+e)^2v[1+k^{\frac{1}{2}}(1+e)\sin I]}{[1-k(1+e)^2]^2}. \tag{5.59}$$

The series (5.49) for R_3 thus converges more rapidly than the series expansion of the function given in (5.59).

Through $O(k^2)$ the values of D_m are given by

TABLE 3

m	Order	D_m
0	k^0	$d_0\delta_0=1$
1	k	$d_0\delta_1=\delta_1=b_1/p\approx k\cos^2 I$
2	k	$d_0\delta_2+d_1\delta_0=(b_2/p)^2P_2(b_1/b_2)-(c/p)^2$
3	k^2	$d_0\delta_3+d_1\delta_1=(b_2/p)^2P_3(b_1/b_2)-(c/p)^2(b_1/p)$
4	k^2	$d_0\delta_4+d_1\delta_2+d_2\delta_0=(b_2/p)^4P_4(b_1/b_2)-(c/p)^2(b_2/p)^2P_2(b_1/b_2)+(c/p)^4$

As with R_1 and R_2, we find

$$(-2\alpha_1)^{\frac{1}{2}}R_3=A_3v+\sum_{j=1}^{\infty}A_{3j}\sin jv, \tag{5.60}$$

where

$$A_3=(1-e^2)^{\frac{1}{2}}p^{-3}\sum_{m=0}^{\infty}D_mR_{m+2}(\sqrt{1-e^2}). \tag{5.61}$$

The remarks about the convergence of the series (5.49) apply also to (5.61).

Since R_3 is multiplied by $c^2=kp^2$ in (2.7), we need periodic terms only through $O(k)$, in order to have periodic terms in the final solution correct through $O(k^2)$. By (5.43) and tables 2 and 3, their coefficients are

$$A_{31}=(1-e^2)^{\frac{1}{2}}p^{-3}e\left[2+b_1p^{-1}\left(3+\frac{3}{4}e^2\right)-p^{-2}\left(\frac{b_2^2}{2}+c^2\right)(4+3e^2)\right], \tag{5.62}$$

$$A_{32}=(1-e^2)^{\frac{1}{2}}p^{-3}\left[\frac{e^2}{4}+\frac{3}{4}\frac{b_1}{p}e^2-p^2\frac{e^4}{4}+\frac{3}{2}e^2\left(\frac{b_2^2}{2}+c^2\right)\right], \tag{5.63}$$

$$A_{33}=(1-e^2)^{\frac{1}{2}}p^{-3}\left[\frac{e^3b^1}{12p}-\frac{e^3}{3p^2}\left(\frac{b_2^2}{2}+c^2\right)\right], \tag{5.64}$$

$$A_{34}=(1-e^2)^{\frac{1}{2}}p^{-3}\left[-\frac{e^4}{32p^2}\left(\frac{b_2^2}{2}+c^2\right)\right]. \tag{5.65}$$

6. The η-Integrals

In (2.5) through (2.7) the η-integrals are

$$N_1=\pm\int_0^{\eta}\eta^2G^{-\frac{1}{2}}(\eta)d\eta, \tag{6.1}$$

$$N_2=\pm\int_0^{\eta}G^{-\frac{1}{2}}(\eta)d\eta, \tag{6.2}$$

$$N_3=\pm\int_0^{\eta}(1-\eta^2)^{-1}G(\eta)^{-\frac{1}{2}}d\eta, \tag{6.3}$$

where $G(\eta)$ is given by (3.34), η_2 by (4.16), α_2^2 by (4.14) and (4.15), and $\alpha_3^2-\alpha_2^2$ finally by (4.8).

In evaluating N_1 and N_2 it is convenient to put

$$\eta=\eta_0\sin\psi, \tag{6.4}$$

where ψ is to be an angle that always increases with time. In the limiting case $c=0$ we should have $\sin\psi=\sin\theta/\sin I$ and ψ would thus reduce to the argument of latitude, i.e., to the angle between the line of nodes and the radius vector to the satellite.

Then

$$\pm G^{-\frac{1}{2}}(\eta)d\eta=(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0(1-q^2\sin^2\psi)^{-\frac{1}{2}}d\psi, \tag{6.5}$$

where

$$q \equiv \eta_0/\eta_2 = O(k^{\frac{1}{2}}) \ll 1, \tag{6.6}$$

η_2 being given by (4.16).

6.1. The Integrals N_1 and N_2

Insertion of (6.4) and (6.5) into (6.1) gives

$$N_1 = (\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} \eta_0^3 \int_0^{\psi} (1 - q^2 \sin^2 \psi)^{-\frac{1}{2}} \sin^2 \psi d\psi. \tag{6.7}$$

Because of the identity

$$(1 - q^2 \sin^2 \psi)^{-\frac{1}{2}} q^2 \sin^2 \psi \equiv (1 - q^2 \sin^2 \psi)^{-\frac{1}{2}} - (1 - q^2 \sin^2 \psi)^{\frac{1}{2}}, \tag{6.8}$$

$$N_1 = (\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} \eta_0^3 q^{-2} \left[\int_0^{\psi} (1 - q^2 \sin^2 \psi)^{-\frac{1}{2}} d\psi - \int_0^{\psi} (1 - q^2 \sin^2 \psi)^{\frac{1}{2}} d\psi \right] \tag{6.9}$$

$$= (\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} \eta_0^3 q^{-2} \left[F(\psi, q) - E(\psi, q) \right], \tag{6.10}$$

where $F(\psi,q)$ and $E(\psi,q)$ are respectively the elliptic integrals of the first and second kinds, with modulus q.

Insertion of (6.4) and (6.5) into (6.2) gives

$$N_2 = (\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} \eta_0 \int_0^{\psi} (1 - q^2 \sin^2 \psi)^{-\frac{1}{2}} d\psi \tag{6.11}$$

$$= (\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} \eta_0 F(\psi, q). \tag{6.12}$$

Thus N_1 and N_2 can both be expressed exactly in terms of elliptic integrals of the first and second kinds. Our ultimate purpose, however, is to express each uniformising variable as the sum of an exact secular term and periodic terms correct through $O(k^2)$. For this purpose it is convenient to express each elliptic integral as a linear combination of ψ and a Fourier series $\sum_1^{\infty} B_n \sin 2n\psi$.

To do so, consider

$$F(\psi, q) \equiv \int_0^{\psi} (1 - q^2 \sin^2 x)^{-\frac{1}{2}} dx. \tag{6.13}$$

Some simple transformations show that

$$F(\psi + \pi, q) = F(\psi, q) + 2K(q), \tag{6.14}$$

where

$$K(q) \equiv \int_0^{\pi/2} (1 - q^2 \sin^2 x)^{-\frac{1}{2}} dx \tag{6.15}$$

is the complete elliptic integral of the first kind. It follows that the function $F(\psi,q) - (2/\pi) K(q) \psi$ is periodic in ψ with period π.
Furthermore it is an odd function of ψ, so that it can be expanded in a Fourier series containing only the sines of even multiples of ψ. Thus,

$$F(\psi, q) = (2/\pi) K(q) \psi + \sum_{m=1}^{\infty} F_{qm} \sin 2m\psi. \tag{6.16}$$

To calculate the Fourier coefficients F_{qm}, differentiate (6.16) with respect to ψ and use (6.13). Then

$$(1 - q^2 \sin^2 \psi)^{-\frac{1}{2}} = (2/\pi) K(q) + 2 \sum_1^{\infty} m F_{qm} \cos 2m\psi. \tag{6.17}$$

The Fourier coefficients F_{qm} are then given by

$$F_{qm}=(2/\pi m)\int_0^{\pi/2}(1-q^2\sin^2 x)^{-\frac{1}{2}}\cos 2mx\,dx. \tag{6.18}$$

Since $\cos 2mx$ is a polynomial in $\cos^2 x$, each F_{qm} can ultimately be expressed as a linear combination of $K(q)$ and the complete elliptic integral

$$E(q)\equiv\int_0^{\pi/2}(1-q^2\sin^2 x)^{\frac{1}{2}}dx. \tag{6.19}$$

Such a procedure, however, would not readily reveal the order of each coefficient in q^2, which itself is of order k. (Actually we shall show that F_{qm} is of order k^m, so that we shall need only F_{q1} and F_{q2}.) Instead we expand the radical, obtaining

$$(1-q^2\sin^2 x)^{-\frac{1}{2}}=1+\sum_{n=1}^{\infty}\frac{(2n)!q^{2n}\sin^{2n}x}{2^{2n}(n!)^2}. \tag{6.20}$$

Then

$$F_{qm}=\frac{2}{\pi m}\sum_{n=1}^{\infty}\frac{(2n)!q^{2n}}{2^n(n!)^2}\int_0^{\pi/2}\sin^{2n}x\cos 2mx\,dx. \tag{6.21}$$

Also

$$\sin^{2n}x=\frac{(2n)!}{(n!)^2 2^n}+(-1)^n 2^{1-2n}\sum_{j=0}^{n-1}(-1)^j\frac{(2n)!}{(2n-j)!j!}\cos(2n-2j)x,\qquad (n\geqq 1) \tag{6.22}$$

and

$$\cos[(2n-2j)x]\cos 2mx=\tfrac{1}{2}\cos[(2n+2m-2j)x]+\tfrac{1}{2}\cos[(2n-2m-2j)x]. \tag{6.23}$$

The integral of this product from 0 to $\pi/2$ fails to vanish only if $j=n+m$ or if $j=n-m$. The value $j=n+m$ is absent from (6.22), so that only the term $j=n-m$ in (6.22) contributes to the integral in (6.21). Thus

$$\int_0^{\pi/2}\sin^{2n}x\cos 2mx\,dx=(-1)^m 2^{-2n}\frac{(2n)!}{(n+m)!(n-m)!}\frac{\pi}{2},\qquad (n\geqq m). \tag{6.24}$$

On inserting (6.24) into (6.21), we find

$$F_{qm}=(-1)^m m^{-1}\sum_{n=m}^{\infty}\frac{(2n)!^2q^{2n}}{(n+m)!(n-m)!2^{4n}(n!)^2}. \tag{6.25}$$

Inspection of (6.25) now shows that $F_{qm}=O(q^{2m})=O(k^m)$. The first two values are, to order q^4 or order k^2:

$$F_{q1}=-\frac{q^2}{8}\left(1+\frac{3}{4}q^2\right)+\ldots, \tag{6.26}$$

$$F_{q2}=\frac{3q^4}{256}+\ldots. \tag{6.27}$$

For our purposes, therefore,

$$F(\psi,q)=(2/\pi)K(q)\psi-\frac{q^2}{8}\left(1+\frac{3}{4}q^2\right)\sin 2\psi+\frac{3q^4}{256}\sin 4\psi+\ldots. \tag{6.28}$$

Similar considerations about periodicity and oddness show that

$$E(\psi,q)\equiv\int_0^{\psi}(1-q^2\sin^2 x)^{\frac{1}{2}}dx=(2/\pi)E(q)\psi+\sum_1^{\infty}E_{qm}\sin 2m\psi. \tag{6.29}$$

Then, as before,

$$(1-q^2\sin^2\psi)^{\frac{1}{2}}=(2/\pi)E(q)+2\sum_1^{\infty}mE_{qm}\cos 2m\psi \tag{6.30}$$

and

$$E_{qm}=(2/\pi m)\int_0^{\pi/2}(1-q^2\sin^2 x)^{\frac{1}{2}}\cos 2mx dx. \tag{6.31}$$

Also

$$(1-q^2\sin^2 x)^{\frac{1}{2}}=1-\sum_{n=1}^{\infty}\frac{(2n-2)!\,q^{2n}\sin^{2n}}{2^{2n-1}n!(n-1)!}x. \tag{6.32}$$

Inserting (6.32) into (6.31), we find

$$E_{qm}=-\frac{1}{\pi m}\sum_{n=1}^{\infty}\frac{(2n-2)!\,q^{2n}}{2^{2n-2}n!(n-1)!}\int_0^{\pi/2}\sin^{2nx}\cos 2mx dx. \tag{6.33}$$

On inserting (6.24) into this, we obtain

$$E_{qm}=\frac{(-1)^{m+1}}{m}\sum_{n=m}^{\infty}\frac{(2n)!\,(2n-2)!\,q^{2n}}{2^{4n-1}n!\,(n-1)!\,(n+m)!\,(n-m)!}. \tag{6.34}$$

Thus E_{qm} is of order q^{2m} or of order k^m and

$$E_{q1}=\frac{q^2}{8}+\frac{q^4}{32}+\ldots, \tag{6.35}$$

$$E_{q2}=-\frac{q^4}{256}+\ldots. \tag{6.36}$$

Then, for our purposes

$$E(\psi,q)=(2/\pi)\,E(q)\,\psi+\left(\frac{q^2}{8}+\frac{q^4}{32}\right)\sin 2\psi-\frac{q^4}{256}\sin 4\psi+\ldots. \tag{6.37}$$

Finally, inserting (6.28) and (6.37) into (6.10) and (6.12), we have for the η-integrals N_1 and N_2:

$$N_1=(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\left[B_1\psi-\frac{1}{4}\left(1+\frac{1}{2}q^2\right)\sin 2\psi+\frac{q^2}{64}\sin 4\psi+\ldots\right], \tag{6.38}$$

$$N_2=(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0\left[B_2\psi-\frac{q^2}{32}(4+3q^2)\sin 2\psi+\frac{3q^4}{256}\sin 4\psi+\ldots\right], \tag{6.39}$$

with

$$B_1\equiv\frac{2q^{-2}}{\pi}[K(q)-E(q)]=\frac{1}{2}+\frac{3}{8}q^2+\frac{15}{128}q^4+\ldots, \tag{6.40}$$

$$B_2\equiv\frac{2}{\pi}K(q)=1+\frac{1}{4}q^2+\frac{9}{64}q^4+\ldots. \tag{6.41}$$

Here the terms in ψ are exact. In N_2 the sines are correct to $O(k^2)$, while in N_1 they are correct only to $O(k)$; this is as much accuracy as we need for N_1, however, since it is multiplied by $c^2=kp^2$ in the first kinetic equation (2.5).

6.2. The Integral N_3

From (6.3) and (3.34) we have

$$(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}N_3=\pm\int_0^{\eta}(1-\eta^2)^{-1}(1-\eta^2/\eta_0^2)^{-\frac{1}{2}}(1-\eta^2/\eta_2^2)^{-\frac{1}{2}}d\eta. \tag{6.42}$$

Then since

$$(1-\eta^2/\eta_2^2)^{-\frac{1}{2}}=\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}(m!)^2}(\eta/\eta_2)^{2m}, \tag{6.43}$$

we obtain

$$(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}N_3=\sum_{m=0}^{\infty}\frac{(2m)!\,\eta_2^{-2m}}{2^{2m}(m!)^2}L_m, \tag{6.44}$$

where

$$L_m\equiv\pm\int_0^{\eta}(1-\eta^2)^{-1}(1-\eta^2/\eta_0^2)^{-\frac{1}{2}}\eta^{2m}d\eta. \tag{6.45}$$

With

$$L_0 \equiv \pm \int_0^\eta (1-\eta^2)^{-1}(1-\eta^2/\eta_0^2)^{-\frac{1}{2}} d\eta \tag{6.46}$$

and

$$(1-\eta^2)^{-1}\eta^{2m} \equiv (1-\eta^2)^{-1} - \sum_{n=0}^{m-1} \eta^{2n}, \qquad (m \geqq 1), \tag{6.47}$$

we then find

$$L_m = L_0 - \sum_{n=0}^{m-1} L_{1n}, \qquad (m \geqq 1), \tag{6.48}$$

where

$$L_{1n} = \pm \int_0^\eta \eta^{2n}(1-\eta^2/\eta_0^2)^{-\frac{1}{2}} d\eta. \tag{6.49}$$

To evaluate L_0, rewrite (6.46) as

$$L_0 = \pm \int_0^\eta \eta^{-3}(\eta^{-2}-1)^{-1}(\eta^{-2}-\eta_0^{-2})^{-\frac{1}{2}} d\eta \tag{6.50}$$

and introduce the new variable χ, defined by the equation

$$\tan \chi = (1-\eta_0^2)^{\frac{1}{2}} \tan \psi = |\cos I| \tan \psi \tag{6.51}$$

and the requirement that χ and ψ shall keep in step. I.e., whenever ψ equals a multiple of $\pi/2$, χ shall be equal to ψ. Then

$$\csc^2 \psi = 1 + \cos^2 I \cot^2 \chi \tag{6.52}$$

and

$$\eta^{-2} = \eta_0^{-2} \csc^2 \psi = 1 + \cot^2 I \csc^2 \chi, \tag{6.53}$$

so that

$$\eta^{-3} d\eta = \cot^2 I \csc^2 \chi \cot \chi \, d\chi \tag{6.54}$$

and $\cot \chi d\chi \gtreqless 0$ according as $d\eta \gtreqless 0$ in (6.50). With $\eta = \eta_0 \sin \psi$ its integrand becomes

$$\pm |\tan I| \cot \chi \, | \tan \chi | \, d\chi = |\tan I| d\chi, \tag{6.55}$$

so that

$$L_0 = |\tan I| \, \chi = \eta_0 (1-\eta_0^2)^{-\frac{1}{2}} \chi. \tag{6.56}$$

To fit the angle χ into one's knowledge of the corresponding elliptic motion, note that as c approaches zero, η approaches $\sin \theta$. Then, by (6.53),

$$\chi \rightarrow \sin^{-1}\left(\frac{\tan \theta}{|\tan I|}\right) = \phi - \Omega, \tag{6.57}$$

where Ω is the right ascension of the node.

To evaluate the integrals L_{1n}, put $\eta = \eta_0 \sin \psi$ in (6.49). Then

$$L_{1n} = \eta_0^{2n+1} \int_0^\psi \sin^{2n} x \, dx, \tag{6.58}$$

so that

$$L_{10} = \eta_0 \psi. \tag{6.59}$$

To handle the cases $n \geqq 1$, we rewrite (6.22) as

$$\sin^{2n} x = \frac{(2n)!}{2^{2n}(n!)^2} + 2^{1-2n} \sum_{j=1}^{n} \frac{(-1)^j (2n)!}{(n+j)!(n-j)!} \cos 2jx, \tag{6.60}$$

thus finding

$$L_{1n} = \frac{\eta_0^{2n+1}(2n)!\psi}{2^{2n}(n!)^2} + \eta_0^{2n+1} 2^{-2n} \sum_{j=1}^{n} \frac{(-1)^j (2n)! \sin 2j\psi}{(n+j)!(n-j)!j}, \qquad (n \geqq 1). \tag{6.61}$$

Insertion of (6.48), (6.59), and (6.61) into (6.44) then yields

$$(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}N_3=L_0\sum_{m=0}^{\infty}\frac{(2m)!\,\eta_2^{-2m}}{2^{2m}(m!)^2}-\eta_0\psi\sum_{m=1}^{\infty}\frac{(2m)!\,\eta_2^{-2m}}{2^{2m}(m!)^2}-\psi\sum_{m=2}^{\infty}\frac{(2m)!\,\eta_2^{-2m}}{2^{2m}(m!)^2}\sum_{n=1}^{m-1}\frac{\eta_0^{2n+1}(2n)!}{2^{2n}(n!)^2}$$

$$-\sum_{m=2}^{\infty}\frac{(2m)!\,\eta_2^{-2m}}{2^{2m}(m!)^2}\sum_{n=1}^{m-1}\eta_0^{2n+1}2^{-2n}\sum_{j=1}^{n}\frac{(-1)^j(2n)!\sin 2j\psi}{(n+j)!(n-j)!j}. \qquad (6.62)$$

If in (6.62) we now use (6.56) and the relation

$$(1-\eta_2^{-2})^{-\frac{1}{2}}=\sum_{m=0}^{\infty}\frac{(2m)!\,\eta_2^{-2m}}{2^{2m}(m!)^2}, \qquad (6.63)$$

we find

$$(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}N_3=\eta_0[(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}}\chi+B_3\psi+\sum_{s=1}^{\infty}B_{3s}\sin 2s\psi], \qquad (6.64)$$

where

$$B_3=1-(1-\eta_2^{-2})^{-\frac{1}{2}}-\sum_{m=2}^{\infty}\gamma_m\eta_2^{-2m}, \qquad (6.65)$$

$$\gamma_m\equiv\frac{(2m)!}{2^{2m}(m!)^2}\sum_{n=1}^{m-1}\frac{(2n)!\,\eta_0^{2n}}{2^{2n}(n!)^2}, \qquad (6.66)$$

and

$$\sum_{s'=1}^{\infty}B_{3s'}\sin 2s'\psi=-\sum_{m=2}^{\infty}\frac{(2m)!\,\eta_2^{-2m}}{2^{2m}(m!)^2}\sum_{n=1}^{m-1}\eta_0^{2n}2^{-2n}\sum_{j=1}^{n}\frac{(-1)^j(2n)!\sin 2j\psi}{(n+j)!(n-j)!j}. \qquad (6.67)$$

The easiest way to isolate the coefficients B_{3s} is to use the orthogonality of the functions $\sin 2s\psi$. Then

$$B_{3s}=-\sum_{m=s+1}^{\infty}\frac{(2m)!\,\eta_2^{-2m}}{2^{2m}(m!)^2}\sum_{n=s}^{m-1}\frac{\eta_0^{2n}2^{-2n}(-1)^s(2n)!}{(n+s)!(n-s)!s} \qquad (6.68)$$

$$=O(\eta_2^{-2s-2}), \qquad (6.69)$$

so that $B_{31}=O(\eta_2^{-4})=O(k^2)$, $B_{32}=O(\eta_2^{-6})=O(k^3)$, etc. Thus, to obtain the sine terms of N_3 to $O(k^2)$, we need only

$$B_{31}=\frac{3}{32}\,\eta_0^2\eta_2^{-4}+\ \ldots\ . \qquad (6.70)$$

To test the convergence of the series $\sum_2^{\infty}\gamma_m\eta_2^{-2m}$, note first that

$$\frac{(2n)!}{2^{2n}(n!)^2}=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}\leqq\frac{1}{2}, \qquad (n\geqq 1), \qquad (6.71)$$

and hence that

$$\sum_{n=1}^{m-1}\frac{(2n)!\,\eta_0^{2n+1}}{2^{2n}(n!)^2}\leqq\frac{m-1}{2} \qquad (6.72)$$

for any orbit, polar or nonpolar. Then, by (6.66), (6.71), and (6.72)

$$\gamma_m\leqq\frac{3}{8}\left(\frac{m-1}{2}\right)\leqq\frac{3}{16}\,(m-1), \qquad (m\geqq 2). \qquad (6.73)$$

Thus $\gamma_m<m$ and

$$\sum_{m=2}^{\infty}\gamma_m\eta_2^{-2m}<\sum_{m=2}^{\infty}m\eta_2^{-2m}. \qquad (6.74)$$

But

$$\sum_2^{\infty}m\eta_2^{-2m}=\frac{\eta_2^{-2}}{(1-\eta_2^{-2})^2}-\eta_2^{-2}=\frac{\eta_2^{-4}(2-\eta_2^{-2})}{(1-\eta_2^{-2})^2}, \qquad (6.75)$$

where $\eta_2^{-2}=O(k)$. Thus the series $\sum_2^{\infty}\gamma_m\eta_2^{-2m}$ converges rapidly.

To test the convergence of the Fourier series $\sum_{s=1}^{\infty} B_{3s}\sin 2s\psi$, note that

$$\left|\sum_{s=1}^{\infty} B_{3s}\sin 2s\psi\right| \leqq \sum_{s=1}^{\infty}|B_{3s}| \leqq \sum_{m=s+1}^{\infty}\frac{(2m)!\eta_2^{-2m}}{2^{2m}(m!)^2}\sum_{n=s}^{m-1}\frac{(2n)!}{2^{2n}(n+s)!(n-s)!}. \tag{6.76}$$

But

$$\frac{(2n)!}{2^{2n}(n+s)!(n-s)!}=\frac{(2n)!n!n!}{2^{2n}(n!)^2(n+s)!(n-s)!}=\frac{(2n)!}{2^{2n}(n!)^2}\,\frac{(n)(n-1)(n-2)\ldots(n-s+1)}{(n+s)(n+s-1)(n+s-2)\ldots(n+1)}\leqq\frac{(2n)!}{2^{2n}(n!)^2}<1,$$

by (6.71). Then

$$\sum_{n=s}^{m-1}\frac{(2n)!}{2^{2n}(n+s)!(n-s)!}<m$$

and

$$\left|\sum_{s=1}^{\infty} B_{3s}\sin 2s\psi\right|<\sum_{m=s+1}^{\infty}\frac{m(2m)!\eta_2^{-2m}}{2^{2m}(m!)^2}<\sum_{m=2}^{\infty} m\eta_2^{-2m}$$

by (6.71) again. Then, by (6.75)

$$\left|\sum_{s=1}^{\infty} B_{3s}\sin 2_s\psi\right|<\frac{\eta_2^{-4}(2-\eta_2^{-2})}{(1-\eta_2^{-2})^2}. \tag{6.77}$$

Since $\eta_2^{-2}=O(k)$, the Fourier series converges rapidly.

Thus all terms in (6.64) remain finite for all $\eta_0^2\leqq 1$, except the term involving χ, which apparently may become infinite for polar orbits. Note, however, that N_3 occurs only in eq (2.7) for the right ascension, when it has a factor α_3, which vanishes for a polar orbit. To see what happens in this case we must investigate the limit of α_3N_3 as $\eta_0^2\rightarrow 1$. For a polar orbit we have

$$\alpha_3N_3=\alpha_3(\alpha_2^2-\tfrac{2}{3}\alpha)^{-\frac{1}{2}}\eta_0(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}}\chi+O(\alpha_3). \tag{6.78}$$

But $|\alpha_3|(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}=|\alpha_3|\alpha_2^{-1}$ to $O(\alpha_3)$ and by (4.8) and (4.2)

$$|\alpha_3|\alpha_2^{-1}=(1-\eta_0^2)^{\frac{1}{2}}(1-\eta_2^{-2}\sin^2 \dot{i}_0)^{\frac{1}{2}}=(1-\eta_0^2)^{\frac{1}{2}}(1-\eta_2^{-2})^{\frac{1}{2}} \text{ to } O(\alpha_3). \tag{6.79}$$

Thus to $O(\alpha_3)$

$$\alpha_3N_3=(\text{sgn }\alpha_3)\chi \qquad \text{(polar orbit)}. \tag{6.80}$$

If we now use (6.51) to plot χ versus ψ for various values of η_0^2, we obtain figure 1.

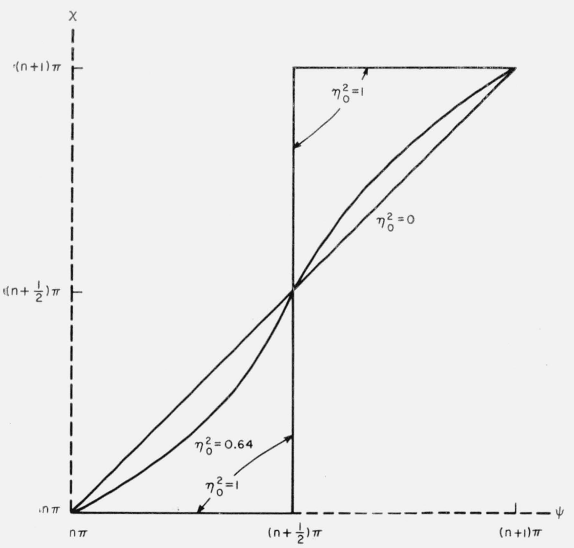

FIGURE 1. *Plot of χ versus ψ for various values of η_0^2.*

The general form of these curves can be checked, as follows. From (6.51)

$$\tan \chi = \epsilon \tan \psi, \qquad \epsilon \equiv (1-\eta_0^2)^{\frac{1}{2}}. \tag{6.81}$$

To find what happens to a polar orbit we must let ϵ approach zero.
If we put

$$\psi = (n+\tfrac{1}{2})\pi + \Delta\psi, \qquad \chi = (n+\tfrac{1}{2})\pi + \Delta\chi \qquad (n=0,1,2.3, \ldots),$$

then

$$\tan \Delta\chi = \epsilon^{-1} \tan \Delta\psi.$$

If we keep $\Delta\psi$ fixed and let $\epsilon \to 0$, then if $0 < \Delta\psi < \pi/2$, it follows that $\Delta\chi \to \pi/2$. If $-\frac{\pi}{2} < \Delta\psi < 0$, then $\Delta\chi \to -\pi/2$. In a polar orbit, whenever the satellite passes over a pole, $\psi = (n+\frac{1}{2})\pi$, with $\dot{\psi} > 0$, so that χ thus jumps by $+\pi$. By (6.80) and (2.7) the right ascension ϕ then jumps by $+\pi$ in a direct orbit or by $-\pi$ in a retrograde orbit. These are expected results, which had to be obtained as a partial check of eq (6.64) for N_3.

7. Mean Motions

The purposes of this section are to obtain expressions for the mean motions correct through the first order, to find if we are on the right track, and to obtain exact expressions for the mean frequencies for later use in section 8 in checking the secular terms in the final solution.

If p_ρ, p_η, and p_ϕ are the generalized momenta conjugate to ρ, η, and ϕ, then the action variables

$$J_1 \equiv \oint p_\rho d\rho = 2\int_{\rho_1}^{\rho_2} p_\rho d\rho, \tag{7.1}$$

$$J_2 \equiv \oint p_\eta d\eta = 4\int_0^{\eta_0} p_\eta d\eta, \tag{7.2}$$

$$J_3 \equiv \oint p_\phi d\phi = \int_0^{2\pi} p_\phi d\phi = 2\pi\alpha_3, \tag{7.3}$$

are functions of the Jacobi constants α_1, α_2, α_3. (Since these J's occur only in this section, there is no danger of confusion with the coefficients of the zonal harmonics in the expansion of the potential.) The α's are then functions of these J's and the mean frequencies [8] are given by

$$\text{mean } \rho\text{-frequency} = \nu_1 \equiv \partial\alpha_1/\partial J_1, \tag{7.4}$$

$$\text{mean } \eta\text{-frequency} = \nu_2 \equiv \partial\alpha_1/\partial J_2, \tag{7.5}$$

$$\text{mean } \phi\text{-frequency} = \nu_3 \equiv \partial\alpha_1/\partial J_3. \tag{7.6}$$

Note that ν_2 and ν_3 are identical with the usual nodal and sidereal frequencies, but that ν_1 is somewhat different from the usual anomalistic frequency.

To compute these frequencies, one may use the system of equations

$$\sum_{m=1}^{3} \frac{\partial\alpha_1}{\partial J_m}\frac{\partial J_m}{\partial\alpha_n} = \frac{\partial\alpha_1}{\partial\alpha_n} = \delta_{1n}, \qquad (n=1,2,3). \tag{7.7}$$

With the use of (7.4) through (7.6) and the abbreviation

$$J_{mn} \equiv \partial J_m/\partial\alpha_n, \tag{7.8}$$

these equations become

$$\nu_1 J_{11}+\nu_2 J_{21} \qquad =1, \tag{7.9}$$

$$\nu_1 J_{12}+\nu_2 J_{22} \qquad =0, \tag{7.10}$$

$$\nu_1 J_{13}+\nu_2 J_{23}+2\pi\nu_3=0, \tag{7.11}$$

with the solution

$$\nu_1=J_{22}/\Delta \tag{7.12}$$

$$\nu_2=-J_{12}/\Delta \tag{7.13}$$

$$2\pi\nu_3=-\nu_1 J_{13}-\nu_2 J_{23} \tag{7.14}$$

$$\Delta\equiv J_{11}J_{22}-J_{12}J_{21} \tag{7.15}$$

With the use of eqs (13), (53), and (55) of [1] and sections (5) and (6) of the present paper we find

$$J_{11}=2R_1(\rho_2)=2\pi(-2\alpha_1)^{-\frac{1}{2}}(a+b_1+A_1), \tag{7.16}$$

$$J_{12}=-2\alpha_2 R_2(\rho_2)=-2\pi\alpha_2(-2\alpha_1)^{-\frac{1}{2}}A_2, \tag{7.17}$$

$$J_{13}=2c^2\alpha_3 R_3(\rho_2)=2\pi c^2\alpha_3(-2\alpha_1)^{-\frac{1}{2}}A_3, \tag{7.18}$$

$$J_{21}=4c^2 N_1(\eta_0)=2\pi c^2(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3 B_1 \tag{7.19}$$

$$J_{22}=4\alpha_2 N_2(\eta_0)=2\pi\alpha_2(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0 B_2, \tag{7.20}$$

$$J_{23}=-4\alpha_3 N_3(\eta_0)=-2\pi\alpha_3(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0[B_3+(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}}]. \tag{7.21}$$

Then

$$2\pi\nu_1=(-2\alpha_1)^{\frac{1}{2}}[a+b_1+A_1+c^2\eta_0^2 A_2 B_1 B_2^{-1}]^{-1} \tag{7.22}$$

$$2\pi\nu_2=(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}A_2 B_2^{-1}[a+b_1+A_1+c^2\eta_0^2 A_2 B_1 B_2^{-1}]^{-1}. \tag{7.23}$$

The above results are all exact, for the potential (2.3), and we shall refer to them again in section 8.

For comparison with other theories it is desirable to express these mean frequencies as power series in the oblateness parameter. For this purpose it is convenient to use the orbital elements a_0, e_0, and i_0, and since we shall carry the series only through the first power we may replace $k_0\equiv c^2/p_0^2$ by $k\equiv c^2/p^2$.

With use of the relations in sections 3 and 4, the above equations then become

$$2\pi\nu_1=n_0+O(k^2) \tag{7.24}$$

$$2\pi\nu_2=n_0[1+\tfrac{3}{4}k(5\cos^2 i_0-1)]+O(k^2), \tag{7.25}$$

where n_0 is given by

$$\mu=n_0^2 a_0^3. \tag{7.26}$$

Similarly

$$J_{13}=\pi k(2+e_0^2)\cos i_0+O(k^2), \tag{7.27}$$

$$J_{23}=-2\pi\operatorname{sgn}\alpha_3+\pi k(1-e_0^2)\cos i_0+O(k^2), \tag{7.28}$$

which together with (7.14), (7.24), and (7.25) lead to

$$2\pi\nu_3/n_0=(\operatorname{sgn}\alpha_3)[1+\tfrac{3}{4}k(5\cos^2 i_0-1)]-\tfrac{3}{2}\,k\cos i_0+O(k^2). \tag{7.29}$$

Here $\operatorname{sgn}\alpha_3=\pm 1$ accordingly as the orbit is direct or retrograde, respectively.

To avoid any use of the concept of an osculating ellipse, we may define the mean motions as follows. We say that the ascending node exists only when the satellite is over the equator, travelling north. Let Ω be its right ascension at such a time. We then define the mean motion $\dot{\bar{\Omega}}$ of the node relative to OX by

$$\dot{\bar{\Omega}}\equiv\lim_{t_i\to\infty}\frac{\Omega_i-\Omega_0}{t_i-t_0}, \tag{7.30}$$

where Ω_0 and t_0 are the values of Ω and t at some (ascending) node and Ω_i and t_i are their values i nodes later. Since the present system is of the conditionally periodic Staeckel type, it follows, after some fairly close reasoning that I shall here omit, that

$$\dot{\bar{\Omega}}=2\pi(|\nu_3|-\nu_2)\ \mathrm{sgn}\ \alpha_3. \tag{7.31}$$

We also say that ρ-perigee exists when, and only when, $\rho=\rho_1$ and we let Φ be its right ascension at such a time. Then the mean motion of ρ-perigee relative to OX must be equal to the mean motion $\dot{\bar{\Phi}}$ of its equatorial projection relative to OX and

$$\dot{\bar{\Phi}}\equiv\lim_{t_i\to\infty}\frac{\Phi_i-\Phi_0}{t_i-t_0} \tag{7.32}$$

Here Φ_0 and t_0 are the values of Φ and t at some ρ-perigee and Φ_i and t_i are their values i ρ-perigees later. Then, again omitting the proof, we have

$$\dot{\bar{\Phi}}=2\pi(|\nu_3|-\nu_1)\ \mathrm{sgn}\ \alpha_3, \tag{7.33}$$

Now let ω be the arc on the celestial sphere from an ascending node (when it exists) to the next ρ-perigee. Then, since the mean relative motion of these points must be equal to the mean relative motion of their projections in the equatorial plane, we find that the mean motion $\dot{\bar{\omega}}$ of ρ-perigee relative to the node is given by

$$\dot{\bar{\omega}}=(\dot{\bar{\Phi}}-\dot{\bar{\Omega}})\ \mathrm{sgn}\ \alpha_3, \tag{7.34}$$

$$=2\pi(\nu_2-\nu_1), \tag{7.35}$$

with use of (7.31) and (7.33), for any orbit, direct or retrograde.

From (7.31), (7.25), and (7.29) it follows that

$$\dot{\bar{\Omega}}=-\tfrac{3}{2}kn_0\cos i_0+O(k^2) \tag{7.36}$$

and from (7.35), (7.24), and (7.25) that

$$\dot{\bar{\omega}}=\tfrac{3}{4}kn_0(5\cos^2 i_0-1)+O(k^2) \tag{7.37}$$

Here $n_0/2\pi$ is the frequency in an elliptic orbit with the same total energy. Equation (7.36) agrees with results found by many other authors, as does (7.37) when $\dot{\bar{\omega}}$ is the mean motion of r-perigee relative to the node. By (2.1) and (2.2), however, $r^2=\rho^2+kp^2\ (1-\eta^2)$, so that r and ρ differ by a variable quantity of $O(k)$. It is thus a little surprising that the mean motions of ρ-perigee and of r-perigee relative to the node should be equal through $O(k)$. This means that the mean ρ-frequency is equal to the mean r-frequency to this order.

8. Solution of the Kinetic Equations

Before solving the kinetic equations (2.5) through (2.7) it is convenient to have several relations connecting the uniformising variables E and v. From (5.12) we obtain

$$\cos v = \frac{\cos E - e}{1 - e\cos E}. \tag{8.1a}$$

The requirements that $dv/dt > 0$, $dE/dt > 0$ for all t lead to the result that $dv/dE > 0$ for all t. Because of this result, (8.1a) leads to

$$\sin v = +\frac{(1-e^2)^{\frac{1}{2}}\sin E}{1 - e\cos E}, \tag{8.1b}$$

without ambiguity in sign. For a given value of E, eqs (8.1a,b) determine v modulo 2π. On imposing the further requirement that v shall always equal E whenever the latter is a multiple of π, we find that E determines v completely. Two other relations are often useful, viz,

$$\tan\frac{v}{2} = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \tan\frac{E}{2} \tag{8.1c}$$

and

$$\tan\frac{v-E}{2} = \frac{\gamma\sin E}{1-\gamma\cos E}, \qquad \gamma \equiv e^{-1}[1-(1-e^2)^{\frac{1}{2}}] < 1. \tag{8.1d}$$

Before beginning the solution of the kinetic equations it is desirable to assemble the results already obtained. By (2.5), (2.6), (5.1), (5.2), (5.30) through (5.33), (5.6), (5.28), (5.35) through (5.40), (6.1), (6.2), (6.6), and (6.38) through (6.41), the equations for ρ and η are

$$t+\beta_1 = (-2\alpha_1)^{-\frac{1}{2}}[b_1E + a(E - e\sin E) + A_1v + A_{11}\sin v + A_{12}\sin 2v]$$
$$+c^2(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\left[B_1\psi - \frac{1}{8}(2+q^2)\sin 2\psi + \frac{q^2}{64}\sin 4\psi\right] + \text{periodic terms of } O(k^3), \tag{8.2}$$

$$\beta_2/\alpha_2 = -(-2\alpha_1)^{-\frac{1}{2}}[A_2v + A_{21}\sin v + A_{22}\sin 2v + A_{23}\sin 3v + A_{24}\sin 4v]$$
$$+(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0\left[B_2\psi - \frac{q^2}{32}(4+3q^2)\sin 2\psi + \frac{3q^4}{256}\sin 4\psi\right] + \text{periodic terms of } O(k^3). \tag{8.3}$$

Here

$$\rho = a(1-e\cos E) = a(1-e^2)(1+e\cos v)^{-1}, \tag{8.4}$$

$$\eta = \eta_0 \sin\psi, \tag{8.5}$$

with E and v connected by any of the eqs (8.1).

$$A_1 = (1-e^2)^{\frac{1}{2}}p\sum_{n=2}^{\infty}(b_2/p)^nP_n(b_1/b_2)R_{n-2}[(1-e^2)^{\frac{1}{2}}] = O(k), \tag{8.6}$$

$$A_2 = (1-e^2)^{\frac{1}{2}}p^{-1}\sum_{n=0}^{\infty}(b_2/p)^nP_n(b_1/b_2)R_n[(1-e^2)^{\frac{1}{2}}] = O(k^0), \tag{8.7}$$

$$A_{11} = \tfrac{3}{4}(1-e^2)^{\frac{1}{2}}p^{-3}e(-2b_1b_2^2p + b_2^4) = O(k^2), \tag{8.8}$$

$$A_{12} = \frac{3}{32}(1-e^2)^{\frac{1}{2}}p^{-3}b_2^4e^2 = O(k^2), \tag{8.9}$$

$$B_1 = \frac{2}{\pi q^2}[K(q) - E(q)] = O(k^0), \tag{8.10}$$

$$B_2 = \frac{2}{\pi}K(q) = O(k^0), \tag{8.11}$$

$$q = \eta_0/\eta_2 = O(k^{\frac{1}{2}}). \tag{8.12}$$

$K(q)$ and $E(q)$ are the complete elliptic integrals of the first and second kinds.

$$A_{21}=(1-e^2)^{\frac{1}{2}}ep^{-5}\left[-\frac{1}{2}Ap^3+\left(\frac{3}{4}A^2-B\right)p^2+\frac{9}{4}ABp\left(1+\frac{e^2}{4}\right)+\frac{3}{8}B^2(4+3e^2)\right]=O(k), \tag{8.13}$$

$$A_{22}=(1-e^2)^{\frac{1}{2}}p^{-5}\left[\left(\frac{3}{8}A^2-\frac{1}{2}B\right)\frac{e^2}{4}p^2+\frac{9}{16}ABe^2p+\frac{3}{8}B^2\left(\frac{3}{2}e^2+\frac{1}{4}e^4\right)\right]=O(k), \tag{8.14}$$

$$A_{23}=(1-e^2)^{\frac{1}{2}}p^{-5}e\left[\frac{ABe^2p}{16}+\frac{B^2e^2}{8}\right]=O(k^2), \tag{8.15}$$

$$A_{24}=\frac{3}{256}(1-e^2)^{\frac{1}{2}}p^{-5}B^2e^4=O(k^2). \tag{8.16}$$

In the above equations $c^2=r_e^2J_2$ is regarded as known, as are the orbital elements a, e, η_0, β_1, and β_2. Then $p=a(1-e^2)$ and η_2, A, B, α_1, α_2, and α_3 are given by section 4. Also $b_1=-\frac{1}{2}A$, $b_2^2=B$, $R_m(x)=x^mP_m(1/x)$, with $|x|<1$.

To solve (8.2) and (8.3), place

$$E=E_s+E_p, \qquad v=v_s+v_p, \qquad \psi=\psi_s+\psi_p, \tag{8.17}$$

where the subscript s means "secular" and the subscript p "periodic." Then if ρ goes through N_1 cycles in time T_1 and if η goes through N_2 cycles in time T_2,

$$\bar{\dot{E}}=\bar{\dot{v}}=\dot{E}_s=\dot{v}_s=\lim_{T_1\to\infty}\frac{2\pi N_1}{T_1}=2\pi\nu_1 \tag{8.18}$$

$$\bar{\dot{\psi}}=\dot{\psi}_s=\lim_{T_2\to\infty}\frac{2\pi N_2}{T_2}=2\pi\nu_2. \tag{8.19}$$

Since we have already obtained exact expressions in section 7 for ν_1 and ν_2, it is clear that we can obtain the secular terms exactly. We shall also obtain the periodic terms through $O(k^2)$.

By (8.18) we can write

$$E_s=v_s=M_s, \tag{8.20}$$

where M_s will play the role of the secular part of a mean anomaly.
Then

$$E=M_s+E_p, \qquad v=M_s+v_p, \qquad \psi=\psi_s+\psi_p. \tag{8.21}$$

We may obtain the secular solution of (8.2) and (8.3) independently of section 7, by dropping all the sines in those equations, placing $E=v=M_s$ and $\psi=\psi_s$, and solving the resulting equations for M_s and ψ_s. The resulting equations are

$$(-2\alpha_1)^{-\frac{1}{2}}(a+b_1+A_1)M_s+c^2(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3B_1\psi_s=t+\beta_1, \tag{8.22}$$

$$-(-2\alpha_1)^{-\frac{1}{2}}A_2M_s+(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0B_2\psi_s=\beta_2/\alpha_2, \tag{8.23}$$

with the following solution.

8.1. Secular Solution

$$M_s=(-2\alpha_1)^{\frac{1}{2}}\frac{B_2(t+\beta_1)-c^2\beta_2\alpha_2^{-1}\eta_0^2B_1}{(a+b_1+A_1)B_2+c^2\eta_0^2A_2B_1}, \tag{8.24}$$

$$\psi_s=(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}\frac{A_2(t+\beta_1)+\beta_2\alpha_2^{-1}(a+b_1+A_1)}{(a+b_1+A_1)B_2+c^2\eta_0^2A_2B_1}. \tag{8.25}$$

Comparison of these results with (7.22) and (7.23) shows that $\dot{M}_s=2\pi\nu_1$ and $\dot{\psi}_s=2\pi\nu_2$, as expected. We may now rewrite these expressions more conveniently as

$$M_s=2\pi\nu_1(t+\beta_1-c^2\beta_2\alpha_2^{-1}\eta_0^2B_1B_2^{-1}), \tag{8.26}$$

$$\psi_s = 2\pi\nu_2[t+\beta_1+\beta_2\alpha_2^{-1}(a+b_1+A_1)A_2^{-1}]. \tag{8.27}$$

As a check, note that to $O(k^0)$, $2\pi\nu_1 = 2\pi\nu_2 = n_0$, so that

$$M_s = n_0(t+\beta_1) + O(k),$$

$$\psi_s = n_0(t+\beta_1) + \frac{n_0 a\beta_2}{\alpha_2 A_2} + O(k).$$

But, to order k^0, $\alpha_2 = (\mu p)^{\frac{1}{2}}$ and $A_2 = (1-e^2)^{\frac{1}{2}}p^{-1}$, so that $n_0 a(\alpha_2 A_2)^{-1} = n_0 ap[\mu p(1-e^2)]^{-\frac{1}{2}} = 1$, since $n_0^2 a^3 = \mu$ to $O(k^0)$.
Thus

$$\psi_s = M_s + \beta_2 + O(k),$$

as is to be expected, with β_2 replacing ω.

As a later aid in reducing the solution for the periodic terms to Kepler's equation, it is here convenient to rewrite (8.22), by transposition of some of its terms:

$$t+\beta_1-(-2\alpha_1)^{-\frac{1}{2}}(a+b_1)M_s = (-2\alpha_1)^{-\frac{1}{2}}A_1 M_s + c^2(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3 B_1\psi_s. \tag{8.28}$$

8.2. Periodic Terms

We shall put, successively,

$$E_p = E_0, \qquad v_p = v_0, \qquad \psi_p = \psi_0; \tag{Step 0}$$

$$E_p = E_0 + E_1, \qquad v_p = v_0 + v_1, \qquad \psi_p = \psi_0 + \psi_1; \tag{Step 1}$$

$$E_p = E_0 + E_1 + E_2, \qquad v_p = v_0 + v_1 + v_2, \qquad \psi_p = \psi_0 + \psi_1 + \psi_2. \tag{Step 2}$$

In step 0, we retain in the equations (8.2) and (8.3) only those periodic terms which are of (Ok^0), viz, sin E. In step 1, we retain in these equations all periodic terms of order k^0 or k, but none of higher order. In step 2, we retain in the equations all periodic terms of order k^0, k, or k^2, but none of higher order. In carrying out each step, however, we shall suppose that each quantity involved is calculated to such an accuracy that the error is of order k^3. Then, effectively, E_0, v_0, and ψ_0 will all contain terms of order k and k^2, as well as terms of order k^0. E_1, v_1, and ψ_1 will contain no terms of order k^0, but will contain terms of order k and k^2. E_2, v_2, and ψ_2 will be of order k^2. Such a procedure will greatly simplify the resulting equations.

8.3. The Periodic Contributions E_0, v_0, and ψ_0

On placing $E = M_s + E_0$, $v = M_s + v_0$, and $\psi = \psi_s + \psi_0$ in (8.2) and (8.3) and retaining only the term sin E of the periodic terms, we find

$$t+\beta_1 = (-2\alpha_1)^{-\frac{1}{2}}[(a+b_1)(M_s+E_0) - ae\sin(M_s+E_0) + A_1 M_s] + c^2(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3 B_1\psi_s, \tag{8.29}$$

$$\beta_2/\alpha_2 = -(-2\alpha_1)^{-\frac{1}{2}}A_2(M_s+v_0) + (\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0 B_2(\psi_s+\psi_0). \tag{8.30}$$

On subtracting (8.28) from (8.29) and dividing the resulting equation by $(a+b_1)(-2\alpha_1)^{-\frac{1}{2}}$, we find

$$M_s + E_0 - e'\sin(M_s+E_0) = M_s, \tag{8.31}$$

where

$$e' \equiv \frac{ae}{a+b_1} < 1, \tag{8.32}$$

since $b_1 > 0$. Equation (8.31) is Kepler's equation for $M_s + E_0$, with an effective eccentricity e'. Let us suppose it to be solved by the most appropriate method, which will depend on the value of e'. We then have $M_s + E_0$ and can then find $v = M_s + v_0$ by use of eqs (8.1).

On subtracting (8.23) from (8.30) we then obtain ψ_0 as a function of v_0:

$$\psi_0=(-2\alpha_1)^{-\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}A_2B_2^{-1}v_0. \tag{8.33}$$

Here the coefficient ψ_0/v_0 is unity to $O(k^0)$, but to follow the procedure outlined above we must not make such an approximation. Instead we must calculate it so accurately that the error is of order k^3.

8.4. The Periodic Contributions E_1, v_1, and ψ_1

Now, knowing M_s, ψ_s, E_0, v_0, and ψ_0, we place $E=M_s+E_0+E_1$, $v=M_s+v_0+v_1$, and $\psi=\psi_s+\psi_0+\psi_1$ into eqs (8.2) and (8.3), discarding only the periodic terms of order k^2. Then

$$t+\beta_1=(-2\alpha_1)^{-\frac{3}{2}}[(a+b_1)(M_s+E_0+E_1)-ae\sin(M_s+E_0+E_1)+A_1(M_s+v_0)]$$

$$+c^2(\alpha_2^2-\alpha_3^2)^{-\frac{3}{2}}\eta_0^3[B_1(\psi_s+\psi_0)-\tfrac{1}{4}\sin(2\psi_s+2\psi_0)], \tag{8.34}$$

$$\beta_2/\alpha_2=-(-2\alpha_1)^{-\frac{1}{2}}[A_2(M_s+v_0+v_1)+A_{21}\sin(M_s+v_0)+A_{22}\sin(2M_s+2v_0)]$$

$$+(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0\left[B_2(\psi_s+\psi_0+\psi_1)-\frac{q^2}{8}\sin(2\psi_s+2\psi_0)\right]. \tag{8.35}$$

Subtraction of (8.28) from (8.34) and division of the result by $(a+b_1)(-2\alpha_1)^{-\frac{3}{2}}$ now gives a Kepler equation for $M_s+E_0+E_1$:

$$M_s+E_0+E_1-e'\sin(M_s+E_0+E_1)=M_s+M_1, \tag{8.36}$$

where

$$M_1\equiv(a+b_1)^{-1}\left[-(A_1+c^2\eta_0^2A_2B_1B_2^{-1})v_0+\frac{c^2}{4}(-2\alpha_1)^{\frac{3}{2}}(\alpha_2^2-\alpha_3^2)^{-\frac{3}{2}}\eta_0^3\sin(2\psi_s+2\psi_0)\right]. \tag{8.37}$$

Here we have used (8.33) to combine terms in v_0 and ψ_0. The quantity M_s+M_1 is then a mean anomaly whose secular part is exact and whose periodic part is correct through order k. It has no periodic part of order k^0; this is characteristic of a mean anomaly.

It is not necessary to solve the Kepler equation all over again. If in (8.36) we put

$$\sin(M_s+E_0+E_1)=(1-\tfrac{1}{2}E_1^2)\sin(M_s+E_0)+E_1\cos(M_s+E_0)+O(E_1^3), \tag{8.38}$$

the error is of order k^3. Then (8.36) and (8.38) yield a quadratic equation for E_1, whose solution through terms of $O(k^2)$ is given by

$$E_1=\frac{M_1}{1-e'\cos(M_s+E_0)}-\frac{e'}{2}\frac{M_1^2\sin(M_s+E_0)}{[1-e'\cos(M_s+E_0)]^3}. \tag{8.39}$$

To find v_1 insert $v=M_s+v_0+v_1$ and $E=M_s+E_0+E_1$ into eqs (8.1) and solve for v_1.

On subtracting (8.23) from (8.35) and eliminating terms in v_0 and ψ_0 by use of (8.33), we then find

$$\psi_1=(-2\alpha_1)^{-\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}B_2^{-1}[A_2v_1+A_{21}\sin(M_s+v_0)+A_{22}\sin(2M_s+2v_0)] \tag{8.40}$$

$$+\frac{q^2}{8}B_2^{-1}\sin(2\psi_s+2\psi_0).$$

(Note that the elimination of v_0 and ψ_0 would not have been possible if they had been carried only through order k^0.)

8.5. The Periodic Contributions E_2, v_2, and ψ_2

Finally, knowing M_s, ψ_s, E_0, v_0, ψ_0, E_1, v_1, and ψ_1, we place $E=M_s+E_0+E_1+E_2$, $v=M_s+v_0+v_1+v_2$, and $\psi=\psi_s+\psi_0+\psi_1+\psi_2$ in (8.2) and (8.3), discarding only the periodic terms of order greater than k^2. The equations become

$$t+\beta_1=(-2\alpha_1)^{-\frac{1}{2}}[(a+b_1)(M_s+E_0+E_1+E_2)-ae\sin(M_s+E_0+E_1+E_2)+A_1(M_s+v_0+v_1)$$

$$+A_{11}\sin(M_s+v_0)+A_{12}\sin(2M_s+2v_0)]+c^2(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\Big[B_1(\psi_s+\psi_0+\psi_1)$$

$$-\frac{1}{8}(2+q^2)\sin(2\psi_s+2\psi_0+2\psi_1)+\frac{q^2}{64}\sin(4\psi_s+4\psi_0)\Big], \tag{8.41}$$

$$\beta_2/\alpha_2=-(-2\alpha_1)^{-\frac{3}{2}}[A_2(M_s+v_0+v_1+v_2)+A_{21}\sin(M_s+v_0+v_1)+A_{22}\sin(2M_s+2v_0+2v_1)$$

$$+A_{23}\sin(3M_s+3v_0)+A_{24}\sin(4M_s+4v_0)]+(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0\Big[B_2(\psi_s+\psi_0+\psi_1+\psi_2)$$

$$-\frac{q^2}{32}(4+3q^2)\sin(2\psi_s+2\psi_0+2\psi_1)+\frac{3}{256}q^4\sin(4\psi_s+4\psi_0)\Big]. \tag{8.42}$$

On subtracting (8.34) from (8.41) and discarding periodic terms of order k^3 or higher, we find

$$0=(-2\alpha_1)^{-\frac{1}{2}}[(a+b_1)E_2-aeE_2\cos(M_s+E_0+E_1)+A_1v_1+A_{11}\sin(M_s+v_0)$$

$$+A_{12}\sin(2M_s+2v_0)]+c^2(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\Big[B_1\psi_1-\frac{1}{2}\psi_1\cos(2\psi_s+2\psi_0)$$

$$-\frac{q^2}{8}\sin(2\psi_s+2\psi_0)+\frac{q^2}{64}\sin(4\psi_s+4\psi_0)\Big]. \tag{8.43}$$

Then

$$E_2=\frac{M_2}{1-e'\cos(M_s+E_0+E_1)}, \tag{8.44}$$

where

$$M_2\equiv-(a+b_1)^{-1}\Big[A_1v_1+A_{11}\sin(M_s+v_0)+A_{12}\sin(2M_s+2v_0)$$

$$+c^2(-2\alpha_1)^{\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\Big\{B_1\psi_1-\frac{1}{2}\psi_1\cos(2\psi_s+2\psi_0)-\frac{q^2}{8}\sin(2\psi_s+2\psi_0)+\frac{q^2}{64}\sin(4\psi_s+4\psi_0)\Big\}\Big]. \tag{8.45}$$

It is easy to show that, to order k^2,

$$M_s+E_0+E_1+E_2-e'\sin(M_s+E_0+E_1+E_2)=M_s+M_1+M_2, \tag{8.46}$$

so that M_2 is the second-order periodic term of a total mean anomaly

$$M=M_s+M_1+M_2+\ldots \tag{8.47}$$

corresponding to the effective eccentricity e'.

To find v_2 insert $v=M_s+v_0+v_1+v_2$ and $E=M_s+E_0+E_1+E_2$ into eqs (8.1) and solve for v_2.

To find ψ_2, subtract (8.35) from (8.42) and discard periodic terms of order k^3 or higher. The result is

$$\psi_2=(-2\alpha_1)^{-\frac{3}{2}}(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}B_2^{-1}[A_2v_2+A_{21}v_1\cos(M_s+v_0)+2A_{22}v_1\cos(2M_s+2v_0)+A_{23}\sin(3M_s+3v_0)$$

$$+A_{24}\sin(4M_s+4v_0)]+\frac{q^2}{4}B_2^{-1}\Big[\psi_1\cos(2\psi_s+2\psi_0)+3\frac{q^2}{8}\sin(2\psi_2+2\psi_0)-3\frac{q^2}{64}\sin(4\psi_s+4\psi_0)\Big]. \tag{8.48}$$

This completes the solution, with exact secular terms and periodic terms correct through order k^2, for E, v, and ψ and thus for the spheroidal coordinates ρ and η.

8.6. The Right Ascension ϕ

From (2.7), (5.3), and (6.3) we obtain

$$\phi=\beta_3+\alpha_3 N_3-c^2\alpha_3 R_3. \tag{8.49}$$

Then, from the equations in sections 5 and 6

$$\phi=\beta_3+\alpha_3(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0\Big[(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}}\chi+B_3\psi$$
$$+\frac{3}{32}\eta_0^2\eta_2^{-4}\sin 2\psi\Big]-c^2\alpha_3(-2\alpha_1)^{-\frac{1}{2}}\Big[A_3 v+\sum_{n=1}^{4}A_{3n}\sin nv\Big], \tag{8.50}$$

where χ is an angle that always equals ψ whenever ψ is a multiple of $\pi/2$ and which also satisfies

$$\tan\chi=(1-\eta_0^2)^{\frac{1}{2}}\tan\psi.$$

The expressions for A_3 and the A_{3n}'s are given in (5.61) to (5.65) and that for B_3 is given in (6.65). With the secular parts of v and ψ exact and their periodic terms correct through order k^2, the right ascension ϕ, as given by (8.50), has a secular part that is exact and a periodic part correct through order k^2. To check the secular part of ϕ, note that one can obtain it from (8.50) by placing $\chi=\psi=\psi_s$ and $v=v_s$ and discarding the sines. If we do so and also use $\dot{v}_s=2\pi\nu_1$ and $\dot{\psi}_s=2\pi\nu_2$, we find

$$\dot{\phi}_s=-2\pi c^2\alpha_3(-2\alpha_1)^{-\frac{1}{2}}A_3\nu_1+2\pi\alpha_3(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0[B_3+(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}}]\nu_2=-\nu_1 J_{13}-\nu_2 J_{23} \tag{8.51}$$

on comparison with (7.18) and (7.21). Thus we find

$$\bar{\dot{\phi}}=\dot{\phi}_s=2\pi\nu_3 \tag{8.52}$$

by (8.51) and (7.14), a result known to be correct [8].

A summary of the principal results of the paper follows in section 9.

9. Summary of the Solution

We assume that μ and c are known, where μ is the product of the gravitational constant and the mass of the planet and where $c^2=r_e^2 J_2$, r_e being the equatorial radius and J_2 the coefficient of the second zonal harmonic of the planet's gravitational potential. For the earth $J_2\approx(1.08)10^{-3}$.

If X, Y, Z are the usual rectangular coordinates of an artificial satellite and if r, θ, ϕ are respectively its planetocentric distance, declination, and right ascension, its oblate spheroidal coordinates ρ, η, ϕ are given by

$$X+iY=r\cos\theta\exp i\phi=[(\rho^2+c^2)(1-\eta^2)]^{\frac{1}{2}}\exp i\phi,$$

$$Z=r\sin\theta=\rho\eta, \qquad (-1\leqq\eta\leqq 1).$$

The potential

$$V_a=-\mu\rho\,(\rho^2+c^2\eta^2)^{-1}$$

then fits the even zonal harmonics exactly through the second and, in the case of the earth, approximately through the fourth. Solution for the motion with such a potential thus furnishes a very accurate intermediary orbit. Since this potential leads to separability of the Hamilton-Jacobi equation, the solution is given implicitly by the quadratures of eqs (2.5) through (2.7). The integration constants are the Jacobi α's and β's.

If the initial conditions are known, one can readily evaluate the α's. Then if one can evaluate the integrals in (2.5) through (2.7) one can also evaluate the β's. Evaluating the integrals depends on factoring the quartics $F(\rho)$ and $G(\eta)$. The factoring of $G(\eta)$ is immediate, since it is quadratic in η^2. To discuss the factoring of $F(\rho)$ we introduce, in place of the α's, the orbital elements $a_0 \equiv -\mu/2\alpha_1$, $e_0 \equiv [1+2\alpha_1\alpha_2^2/\mu^2]^{\frac{1}{2}}$, and $i_0 \equiv \cos^{-1}(\alpha_3/\alpha_2)$.

With ρ varying in the range $\rho_1 \leqq \rho \leqq \rho_2$, we write

$$F(\rho) \equiv c^2\alpha_3^2 + (\rho^2+c^2)(-\alpha_2^2+2\mu\rho+2\alpha_1\rho^2) = (-2\alpha_1)(\rho-\rho_1)(\rho_2-\rho)(\rho^2+A\rho+B)$$

and find that A, B, a, p, and e occur in the ρ-integrals.
Here

$$a \equiv \frac{1}{2}(\rho_1+\rho_2), \qquad 1-e^2 = \frac{4\rho_1\rho_2}{(\rho_1+\rho_2)^2}, \qquad p \equiv a(1-e^2) = \frac{2\rho_1\rho_2}{\rho_1+\rho_2}, \qquad e \equiv \frac{\rho_2-\rho_1}{\rho_2+\rho_1} = [1-(1-e^2)]^{\frac{1}{2}}$$

Thus A, B, $\rho_1+\rho_2$, and $\rho_1\rho_2$ are needed. If we put

$$p_0 \equiv a_0(1-e_0^2), \qquad k_0 \equiv c^2/p_0^2, \qquad x \equiv (1-e_0^2)^{\frac{1}{2}}, \qquad y \equiv \alpha_3/\alpha_2 = \cos i_0,$$

then through $O(k_0^2)$

$$A = -2k_0p_0y^2[1+k_0(2x^2-3x^2y^2-4+8y^2)+\ldots],$$

$$B = k_0p_0^2(1-y^2)[1+k_0(4y^2-x^2y^2)+\ldots],$$

$$2a \equiv \rho_1+\rho_2 = 2p_0x^{-2}[1-k_0x^2y^2-k_0^2x^2y^2(2x^2-3x^2y^2-4+8y^2)+\ldots],$$

$$ap \equiv \rho_1\rho_2 = p_0^2x^{-2}[1+k_0y^2(x^2-4)-k_0^2y^2(12x^2-x^4-20x^2y^2-16+32y^2+x^4y^2)+\ldots].$$

If a_0, e_0, i_0, β_1, β_2, β_3 are used as orbital elements, we then assume that the quartic $F(\rho)$ is factored numerically, to as great an accuracy as may be desired, with the aid of the above relations as a starting point.

With η varying in the range $-1 \leqq -\eta_0 \leqq \eta \leqq \eta_0 \leqq 1$, we write $G(\eta) \equiv -\alpha_3^2+(1-\eta^2)(\alpha_2^2+2\alpha_1c^2\eta^2) = -2\alpha_1c^2(\eta_0^2-\eta^2)(\eta_2^2-\eta^2)$ and find

$$\begin{pmatrix}\eta_0^{-2}\\ \eta_2^{-2}\end{pmatrix} = \frac{\alpha_2^2-2\alpha_1c^2}{2(\alpha_2^2-\alpha_3^2)}\left[1 \pm \left\{1+\frac{8\alpha_1c^2(\alpha_2^2-\alpha_3^2)}{(\alpha_2^2-2\alpha_1c^2)^2}\right\}^{\frac{1}{2}}\right].$$

Here $\eta_2 \gg 1$. Then all the quantities a, p, e, A, B, η_0, and η_2 are known in terms of the orbital elements a_0, e_0, and i_0.

If we assume that the orbital elements are to be determined by an iterated least-square fitting of the solution to many revolutions in the orbit and not by initial conditions, there is a better set of elements, introduced by Izsak [4]. These, viz, a, e, $I \equiv \sin^{-1}\eta_0$, β_1, β_2, and β_3, although not easily found from the initial conditions, result in immediate factoring of the quartics $F(\rho)$ and $G(\eta)$. We therefore give the solution in terms of these quantities, with the understanding that they are to be determined either by the least-square fitting or from initial conditions by numerical factoring of $F(\rho)$.

Given μ, c, and the elements a, e, I, β_1, β_2, β_3,
compute

$$\eta_0 = \sin I,$$

$$p = a(1-e^2), \qquad D = (ap-c^2)(ap-c^2\eta_0^2)+4a^2c^2\eta_0^2, \qquad D' = D+4a^2c^2(1-\eta_0^2),$$

$$A = -2ac^2D^{-1}(1-\eta_0^2)(ap-c^2\eta_0^2), \qquad B = c^2\eta_0^2D^{-1}D',$$

$$b_1 = -\frac{1}{2}A, \qquad b_2 = B^{\frac{1}{2}}, \qquad -2\alpha_1 = \mu(a+b_1)^{-1},$$

$$-\frac{\alpha_2^2}{2\alpha_1} = a_0p_0 = -c^2(1-\eta_0^2)+apD^{-1}D', \qquad \alpha_2 = (-2\alpha_1)^{\frac{1}{2}}(a_0p_0)^{\frac{1}{2}} > 0,$$

$$\alpha_3 = \alpha_2\left(1-\frac{c^2\eta_0^2}{a_0p_0}\right)^{\frac{1}{2}}\cos I, \qquad \eta_2^{-2} = \frac{c^2D}{apD'}, \qquad k = c^2/p^2, \qquad q = \eta_0/\eta_2.$$

Restrict considerations to the case $b_1/b_2 \leqq 1$. Then

$$I_c \leqq I \leqq 180° - I_c,$$

where $I_c = 1°54'$, approximately. Equatorial and almost equatorial orbits are thus ruled out.

Then compute

$$A_1 = (1-e^2)^{\frac{1}{2}} p \sum_{n=2}^{\infty} (b_2/p)^n P_n(b_1/b_2) R_{n-2}[(1-e^2)^{\frac{1}{2}}],$$

$$A_2 = (1-e^2)^{\frac{1}{2}} p^{-1} \sum_{n=0}^{\infty} (b_2/p)^n P_n(b_1/b_2) R_n[(1-e^2)^{\frac{1}{2}}],$$

where $P_n(x)$ is the Legendre polynomial of degree n and where $R_n(x) \equiv x^n P_n(x^{-1})$, always a polynomial of degree $[n/2]$ in x^2.

$$A_3 = (1-e^2)^{\frac{1}{2}} p^{-3} \sum_{m=0}^{\infty} D_m R_{m+2}[(1-e^2)^{\frac{1}{2}}],$$

where

$$D_{2i} = \sum_{n=0}^{i} (-1)^{i-n} (c/p)^{2i-2n} (b_2/p)^{2n} P_{2n}(b_1/b_2),$$

$$D_{2i+1} = \sum_{n=0}^{i} (-1)^{i-n} (c/p)^{2i-2n} (b_2/p)^{2n+1} P_{2n+1}(b_1/b_2),$$

$$B_1 = 2\pi^{-1} q^{-2} [K(q) - E(q)], \qquad B_2 = 2\pi^{-1} K(q),$$

where $K(q)$ and $E(q)$ are the complete elliptic integrals of the first and second kinds, respectively.

$$B_3 = 1 - (1-\eta_2^{-2})^{\frac{1}{2}} - \sum_{m=2}^{\infty} \gamma_m \eta_2^{-2m}, \qquad \text{where} \qquad \gamma_m \equiv \frac{(2m)!}{2^{2m}(m!)^2} \sum_{n=1}^{m-1} \frac{(2n)!\eta_0^{2n}}{2^{2n}(n!)^2}.$$

(The above series all converge rapidly.)

$$A_{11} = \frac{3}{4}(1-e^2)^{\frac{1}{2}} p^{-3} e(-2b_1 b_2^2 p + b_2^4), \qquad A_{12} = \frac{3}{32} p^{-3} (1-e^2)^{\frac{1}{2}} b_2^4 e^2,$$

$$A_{21} = (1-e^2)^{\frac{1}{2}} p^{-1} e \left[b_1 p^{-1} + (3b_1^2 - b_2^2) p^{-2} - \frac{9}{2} b_1 b_2^2 \left(1 + \frac{e^2}{4}\right) p^{-3} + \frac{3}{8} b_2^4 (4+3e^2) p^{-4} \right],$$

$$A_{22} = (1-e^2)^{\frac{1}{2}} p^{-1} \left[\frac{e^2}{8} (3b_1^2 - b_2^2) p^{-2} - \frac{9}{8} e^2 b_1 b_2^2 p^{-3} + \frac{3}{32} b_2^4 (6e^2 + e^4) p^{-4} \right],$$

$$A_{23} = (1-e^2)^{\frac{1}{2}} p^{-1} \frac{e^3}{8} (-b_1 b_2^2 p^{-3} + b_2^4 p^{-4}),$$

$$A_{24} = \frac{3}{256} (1-e^2)^{\frac{1}{2}} p^{-5} b_2^4 e^4,$$

$$A_{31} = (1-e^2)^{\frac{1}{2}} p^{-3} e \, [2 + b_1 p^{-1} (3 + \tfrac{3}{4} e^2) - p^{-2} (\tfrac{1}{2} b_2^2 + c^2)(4+3e^2)],$$

$$A_{32} = (1-e^2)^{\frac{1}{2}} p^{-3} \left[\frac{e^2}{4} + \frac{3}{4} b_1 p^{-1} e^2 - p^{-2} \left(\frac{e^4}{4} + \frac{3}{2} e^2 \right) \left(\frac{1}{2} b_2^2 + c^2 \right) \right],$$

$$A_{33} = (1-e^2)^{\frac{1}{2}} p^{-3} e^3 \left[\frac{1}{12} b_1 p^{-1} - \frac{1}{3} p^{-2} \left(\frac{1}{2} b_2^2 + c^2 \right) \right],$$

$$A_{34} = -\frac{1}{32} (1-e^2)^{\frac{1}{2}} p^{-5} e^4 \left(\frac{1}{2} b_2^2 + c^2 \right),$$

$$2\pi\nu_1 = (-2\alpha_1)^{\frac{1}{2}}(a+b_1+A_1+c^2\eta_0^2 A_2 B_1 B_2^{-1})^{-1},$$

$$2\pi\nu_2 = (\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}A_2 B_2^{-1}(a+b_1+A_1+c^2\eta_0^2 A_2 B_1 B_2^{-1})^{-1}.$$

The uniformising variables E, v, and ψ are then given by $E=M_s+E_p$, $v=M_s+v_p$, and $\psi=\psi_s+\psi_p$. If t is the time, their secular parts M_s and ψ_s are given exactly by

$$M_s = 2\pi\nu_1(t+\beta_1-c^2\beta_2\alpha_2^{-1}\eta_0^2 B_1 B_2^{-1}),$$

$$\psi_s = 2\pi\nu_2[t+\beta_1+\beta_2\alpha_2^{-1}(a+b_1+A_1)A_2^{-1}].$$

Let the periodic parts be split as follows: $E_p=E_0+E_1+E_2$, $v_p=v_0+v_1+v_2$, and $\psi_p=\psi_0+\psi_1+\psi_2$, where, e.g., E_0 contains terms of order k^0, k, and k^2, E_1 contains terms of order k and k^2, and E_2 contains only terms of order k^2.

Then E_0 is given by the Kepler equation

$$M_s+E_0-e'\sin(M_s+E_0)=M_s,$$

where $e'\equiv a(ae+b_1)^{-1}<e$. The term v_0 is then given by placing $v=M_s+v_0$ and $E=M_s+E_0$ in the *anomaly connections*

$$\cos v=(\cos E-e)(1-e\cos E)^{-1} \qquad \sin v=+(1-e^2)^{\frac{1}{2}}(1-e\cos E)^{-1}\sin E$$

or equivalent relations. (Note that the e here is the original e and not the e' in the Kepler equation.) Then

$$\psi_0=(-2\alpha_1)^{-\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}A_2 B_2^{-1}v_0.$$

The term E_1 is now given by

$$E_1=[1-e'\cos(M_s+E_0)]^{-1}M_1-\tfrac{1}{2}e'[1-e'\cos(M_s+E_0)]^{-3}M_1^2\sin(M_s+E_0),$$

where

$$M_1\equiv(a+b_1)^{-1}\left[-(A_1+c^2\eta_0{}^2A_2B_1B_2^{-1})v_0+\frac{c^2}{4}(-2\alpha_1)^{\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\sin(2\psi_s+2\psi_0)\right].$$

The term v_1 is then given by placing $v=M_s+v_0+v_1$ and $E=M_s+E_0+E_1$ in the *anomaly connections*. Then

$$\psi_1=(-2\alpha_1)^{-\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}B_2^{-1}[A_2v_1+A_{21}\sin(M_s+v_0)+A_{22}\sin(2M_s+2v_0)]+\frac{q^2}{8}B_2^{-1}\sin(2\psi_s+2\psi_0)$$

Finally

$$E_2=[1-e'\cos(M_s+E_0+E_1)]^{-1}M_2,$$

where

$$M_2\equiv-(a+b_1)^{-1}\Big[A_1v_1+A_{11}\sin(M_s+v_0)+A_{12}\sin(2M_s+2v_0)$$
$$+c^2(-2\alpha_1)^{\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0^3\left\{B_1\psi_1-\frac{1}{2}\psi_1\cos(2\psi_s+2\psi_0)-\frac{q^2}{8}\sin(2\psi_s+2\psi_0)+\frac{q^2}{64}\sin(4\psi_s+4\psi_0)\right\}\Big].$$

Then v_2 is found by placing $v=M_s+v_0+v_1+v_2$ and $E=M_s+E_0+E_1+E_2$ in the *anomaly connections* and

$$\psi_2=(-2\alpha_1)^{-\frac{1}{2}}(\alpha_2^2-\alpha_3^2)^{\frac{1}{2}}\eta_0^{-1}B_2^{-1}[A_2v_2+A_{21}v_1\cos(M_s+v_0)$$
$$+2A_{22}v_1\cos(2M_s+2v_0)+A_{23}\sin(3M_s+3v_0)+A_{24}\sin(4M_s+4v_0)]$$
$$+\frac{q^2}{4}B_2^{-1}\left[\psi_1\cos(2\psi_s+2\psi_0)+\frac{3q^2}{8}\sin(2\psi_s+2\psi_0)-\frac{3q^2}{64}\sin(4\psi_s+4\psi_0)\right].$$

The spheroidal coordinates ρ and η are then given by

$$\rho=a(1-e\cos E)=(1+e\cos v)^{-1}p, \qquad \eta=\eta_0\sin\psi, \text{ where}$$
$$E=M_s+E_0+E_1+E_2, \qquad v=M_s+v_0+v_1+v_2, \qquad \psi=\psi_s+\psi_0+\psi_1+\psi_2.$$

The right ascension is

$$\phi=\beta_3+\alpha_3(\alpha_2^2-\alpha_3^2)^{-\frac{1}{2}}\eta_0\left[(1-\eta_0^2)^{-\frac{1}{2}}(1-\eta_2^{-2})^{-\frac{1}{2}}\chi+B_3\psi+\frac{3}{32}\eta_0^2\eta_2^{-4}\sin 2\psi\right]$$
$$-c^2\alpha_3(-2\alpha_1)^{-\frac{1}{2}}\left[A_3v+\sum_{n=1}^{4}A_{3n}\sin nv\right].$$

Here χ is an angle that equals ψ whenever ψ is a multiple of $\pi/2$ and which also satisfies $\tan\chi=(1-\eta_0^2)^{\frac{1}{2}}\tan\psi$. If α_3 is positive or negative, the orbit is respectively direct or retrograde.

The above solution gives secular terms of the intermediary orbit exactly and periodic terms correctly through order k^2. There are no long-period terms.

10. List of Symbols

We here list only the symbols that are most frequently used, giving for each a short definition or the number of the equation in which it first appears. Note that the first digit in each such number is the number of the section.

Symbol	*Definition or equation number*
a	3.23; element of 2d set,
a_0	3.3; element of 1st set,
A	3.11 (also 4.12),
A_1	5.31,
A_2	5.36,
A_{11}	5.32,
A_{12}	5.33,
A_{2j}	5.37 through 5.40 for $j=1, 2, 3, 4$,
A_3	5.61,
A_{3j}	5.62 through 5.65 for $j=1, 2, 3, 4$,
b_1,b_2	5.6,
B	3.11 (also 4.13)
B_1	6.40,
B_2	6.41,
B_3	6.65,
B_{3s}	6.64 (also 6.68)
c	2.4 (fundamental distance in potential and in definitions of coordinates),
d_j	5.46,
D_m	5.44 (also 5.50 and 5.53),
e	3.24; element of 2d set,
e_0	3.4; element of 1st set,
e'	8.32,
E	5.12; "eccentric anomaly,"
E_s	8.17; secular part of $E(=M_s)$,
E_p	8.17; periodic part of E,
E_0,E_1,E_2	various terms of E_p,
$E(q)$	complete elliptic integral of second kind, modulus q,
$E(\psi,q)$	incomplete elliptic integral of second kind; 6.9 and 6.10,
$f(\rho)$	2.10,
$F(\rho)$	2.8,
$F(\psi,q)$	incomplete elliptic integral of the first kind; 6.9 and 6.10,
$G(\eta)$	2.9,
h	5.7,
i_0	3.5; element of 1st set,
I	4.7; element of 2d set,
I_c	value of I for which $\lambda=1$,
J_2	2.4; coefficient of second zonal harmonic of potential,
J_1,J_2,J_3	action variables; (7.1) through (7.3),
J_{mn}	7.8,
k_0	3.16; c^2/p_0^2,
k	4.17; c^2/p^2,
$K(q)$	6.15; complete elliptic integral of first kind, modulus q,
L_0	6.46,
L_m	6.45,
L_{1n}	6.49,
M_s	8.20; secular part of "mean anomaly,"
M_1,M_2	8.37, 8.45; periodic parts of "mean anomaly," of orders k and k^2,
N_1,N_2,N_3	6.1 through 6.3; η-integrals,
p_0	$a_0(1-e_0^2)$,
p	$a(1-e^2)$,
P_n	Legendre polynomial of degree n,
q	η_0/η_2,
r	2.1 and 2.2; geocentric distance,
r_e	equatorial radius,
r_1,r_2	3.1, 3.2; zeros of $f(\rho)$,
R_1,R_2,R_3	5.1 through 5.3; ρ-integrals,
$R_m(z)$	$z^mP_m(z^{-1})$,
sgn α_3	sign of α_3,
t	time,
v	5.12; "true anomaly,"
v_s	8.17; secular part of $v(=M_s)$
v_p	8.17; periodic part of v,
v_0,v_1,v_2	various terms of v_p,
V_a	2.3; the potential of this paper,
X,Y,Z	2.1 and 2.2; rectangular coordinates,
x	3.17; $(1-e_0^2)^{1/2}$,
y	3.18; $\alpha_3/\alpha_2=\cos i_0$,
α_1	total energy, first Jacobi const, <0 for a satellite orbit,
α_2	second Jacobi constant,
α_3	Z-component of angular momentum, third Jacobi constant,
β_1,β_2,β_3	Jacobi β's,
γ	8.1 d,
γ_m	6.66,
δ_n	5.47,
ϵ	6.81,

Symbol	Definition or equation number
η	—2.1 and 2.2; a spheroidal coordinate $\rightarrow \sin\theta$ as $r \rightarrow \infty$,
η_0	—3.33, 3.36, 3.39; during motion $-1 \leqq -\eta_0 \leqq \eta \leqq \eta_0 \leqq 1$,
η_2	—3.33, 3.37, $\eta_2 >> 1$,
θ	—2.1 and 2.2; geocentric declination,
λ	—5.6; b_1/b_2,
ν_1,ν_2,ν_3	—7.4 through 7.6,
ρ	—2.1 and 2.2; a spheroidal coordinate $\rightarrow r$ as $r \rightarrow \infty$,
ϕ	—2.1 and 2.2; geocentric right ascension; the third spheroidal coordinate,
χ	—6.51,
ψ	—6.4; a uniformizing variable analogous to the argument of latitude,
ψ_s	—8.17; secular part of ψ,
ψ_p	—8.17; periodic part of ψ,
ψ_0,ψ_1,ψ_2	—various terms of ψ_p,
$\dot{\bar{\omega}}$	—mean motion of ρ-perigee relative to the ascending node,
$\dot{\bar{\Omega}}$	—mean motion of the ascending node relative to OX.

11. References

[1] J. P. Vinti, J. Research NBS **63B,** 105–116 (1959).

[2] T. E. Sterne, Astron. J. **62,** 96 (1957); **63,** 28–40 (1958).

[3] B. Garfinkel, Astron. J. **63,** 88–96 (1958); **64,** 353–367 (1959).

[4] I. Izsak, Smithsonian Institution Astrophysical Observatory, Research in Space Science, Special Rept. No. 52, (1960).

[5] E. T. Whittaker and G. N. Watson, Modern Analysis, 4th ed., p. 312 (Cambridge Univ. Press, Cambridge, 1952).

[6] E. T. Whittaker and G. N. Watson, Modern Analysis, 4th ed., p. 29 (Cambridge Univ. Press, Cambridge, 1952).

[7] E. T. Whittaker and G. N. Watson, Modern Analysis, 4th ed., p. 49 (Cambridge Univ. Press, Cambridge, 1952).

[8] J. P. Vinti, J. Research NBS **65B,** 131–135 (1961).

(Paper 65B3–56)

596686—61——3