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Optimal Approximation for Functions Prescribed at Equally Spaced Points

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Explicit upper and lower bounds for the value $F(u)$ of a linear functional F applied to a function $u(x)$ defined on the interval $0 \leq x \leq 1$ are given when u is prescribed at the $N+1$ points i/N, $i=0, \ldots, N$, and a bound for the integral of $u^{[k]2}$ is known. These bounds are optimal in the sense that they are attained for functions satisfying the prescribed conditions. Their computation involves the inversion of a matrix of size $k-1$ rather than N, which means that N is permitted to be large.

1. Introduction

Many problems in numerical analysis can be reduced to approximating the value $F(u)$ of a given linear functional F operating on an unknown element u of a linear vector space. The approximation is to be made in terms of a finite set of data concerning u. Thus, the values $F_1(u), \ldots, F_n(u)$ of N linear functionals acting on u may be given. For example, the $F_i(u)$ may be values of the function u at certain points x_i. If $F(u)$ is the value of u at another point ξ, we have the problem of linear interpolation. If $F(u)$ is an integral of u, we have the problem of numerical quadratures. If $F(u)$ is the value of a derivative, we have numerical differentiation.

It was shown by M. Golomb and the author [2] that in order to obtain a finite interval in which the value $F(u)$ must lie, one must be given the value of at least one nonlinear functional operating on u. The simplest case is that in which one is given a bound for a quadratic functional (u,u). In this case Golomb and the author [2] showed how to obtain the exact interval in which $F(u)$ must lie when the values $F_1(u), \ldots, F_N(u)$, and (u,u) are given. That is, upper and lower bounds for $F(u)$ which are attained for some elements u satisfying the given conditions are found. The construction of these bounds requires the inversion of a matrix depending upon the functionals $F, F_1, \ldots, F_N$, and (u,u).

In this paper we restrict our attention to a very simple case. We deal with a function $u(x)$ of a single variable on the interval $[0,1]$. The given functionals F_i are the values $u(i/N)$ of u at the $N+1$ equally spaced points i/N, $i=0, \ldots, N$. The quadratic functional is taken to be the integral of the square of the k^{th} derivative of u.

We think of the number of points N as large, while the number k of the derivative will usually be small, say two or three. The matrix to be inverted is of size N. By making use of the equal spacing of our points, we shall reduce its inversion to that of a $(k-1)\times(k-1)$ matrix. Thus the problem of obtaining best formulas for interpolation, quadratures, and numerical differentiation is made manageable even when the number of points involved becomes large.

When $F(u)=\int_0^1 u\,dx$, our results yield as special cases the best quadrature formulas of Sard [3] for $k \leq 3$, $N \leq 6$.

2. Approximation Problem

Let the values of the unknown function $u(x)$ be given at the $N+1$ evenly spaced points i/N, $i=0, \ldots, N$. Let M^2 be a given bound for the integral of the kth derivative of u.

$$(u,u)=\int_0^1 u^{[k]2}dx \leq M^2. \tag{2.1}$$

We assume that $N \geq 2k$, $k \geq 2$.

Our problem is to approximate the value $F(u)$ of a certain linear functional F applied to u. According to the theory in footnote 2, this is possible if and only if the functional F is bounded in the norm (2.1) for functions vanishing at the points i/N. That is, we must assume that there is a constant c such that

$$F(v)^2 \leq c\int_0^1 v^{[k]2}dx \tag{2.2}$$

for all k times differentiable functions $v(x)$ such that

$$v\left(\frac{i}{N}\right)=0, \qquad i=0, \ldots, N. \tag{2.3}$$

Any linear combination of pointwise values or integrals of v and its derivatives up to order k-1 will satisfy this condition.

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[2] M. Golomb and H. F. Weinberger, Optimal approximations and error bounds, Symp. on Numer. Approx. (Univ. Wisconsin Press, Madison, Wis., 1959).
[3] A. Sard, Best approximate integration formulas; best approximation formulas, Am. J. Math. **71**, 80–91 (1949).

To obtain a best estimate for $F(u)$ we construct an auxiliary function $\bar{u}$ defined by the properties

$$\bar{u}^{[2k]}(x)=0, \qquad 0<x<1, \quad Nx\neq 0,1,\ldots, N \tag{2.4}$$

and

$$\begin{aligned} &\bar{u}\left(\frac{i}{N}\right)=u\left(\frac{i}{N}\right) && i=0,1,\ldots, N, \\ &\bar{u}^{[l]}(0)=\bar{u}^{[l]}(1)=0 && l=k,\ldots, 2k-2. \end{aligned} \tag{2.5}$$

The function u and its first $2k$-2 derivatives are continuous, while $\bar{u}^{[2k-1]}$ is allowed to have jump discontinuities at the points i/N.

The Green's function $G(x;\ \xi)$ is defined by the properties

$$\left.\begin{aligned} &\frac{\partial^{2k}G}{\partial x^{2k}}=0, && 0<x<1,\ Nx\neq N\xi, 0,1,\ldots N, \\ &G\left(\frac{i}{N};\ \xi\right)=0, && i=0,1,\ldots, N, \\ &\frac{\partial^{l}G}{\partial x^{l}}=0, && \text{at } x=0,1,\ l=k,\ldots 2k-2, \\ &\frac{\partial^{2k-1}G(\xi+0;\xi)}{\partial x^{2k-1}}-\frac{\partial^{2k-1}G(\xi-0;\xi)}{\partial x^{2k-1}}=(-1)^{k}. \end{aligned}\right\} \tag{2.6}$$

Again G and its first $2k$-2 derivatives are continuous, while the $(2k-1)$st derivative may have discontinuities at i/N. By integration by parts we find that

$$\bar{u}(\xi)=\sum_{i=0}^{N} g_i(\xi)u\left(\frac{i}{N}\right), \tag{2.7}$$

where

$$g_i(\xi)=(-1)^{k-1}\left\{\frac{\partial^{2k-1}G\left(\frac{i}{N}+0;\ \xi\right)}{\partial x^{2k-1}}-\frac{\partial^{2k-1}G\left(\frac{i}{N}-0;\ \xi\right)}{\partial x^{2k-1}}\right\}. \tag{2.8}$$

Also by integration by parts we have

$$u(\xi)-\bar{u}(\xi)=\int_0^1\{u^{[k]}-\bar{u}^{[k]}\}\frac{\partial^k G(x;\ \xi)}{\partial x^k}dx. \tag{2.9}$$

Applying the functional F and using Schwarz's inequality, we find that

$$\begin{aligned} |F(u)-F(\bar{u})|^2 &\leq \int_0^1|u^{[k]}-\bar{u}^{[k]}|^2dx\int_0^1\left|F_\xi\left[\frac{\partial^k G(x;\xi)}{\partial x^k}\right]\right|^2dx \\ &=\int_0^1|u^{[k]}-\bar{u}^{[k]}|^2dxF_\xi\{F_\eta[G(\xi;\ \eta)]\}. \end{aligned} \tag{2.10}$$

(The symbol $F\eta\ [G\ (\xi;\ \eta)]$ means that the functional F is applied to G considered as a function of η for fixed ξ. $F_\xi\{F_\eta[G\ (\xi;\ \eta)]\}$ means that F is then applied to the function $F_\eta[G]$). We have used the property

$$G(\xi;\ \eta)=\int_0^1\frac{\partial^k G(\xi;\ x)}{\partial x^k}\frac{\partial^k G(\eta;\ x)}{\partial x^k}dx, \tag{2.11}$$

which follows from integration by parts.

Another integration by parts shows that

$$\int_0^1\bar{u}^{[k]}\{u^{[k]}-\bar{u}^{[k]}\}dx=0. \tag{2.12}$$

Hence, we can rewrite (2.10) as

$$|F(u)-F(\bar{u})|^2\leq\left\{M^2-\int_0^1\bar{u}^{[k]2}dx\right\}F_\xi\{F_\eta[G(\xi;\ \eta)]\}. \tag{2.13}$$

Once $\bar{u}$ and G are found, this inequality provides upper and lower bounds for $F(u)$. These bounds are sharp in the sense that we can construct functions u satisfying (2.1) and having the given values $u(i/N)$ for which the bounds for $F(u)$ are attained.

We write $\bar{u}$ in the form

$$\bar{u}(x)=\sum_{i=-k+1}^{N+k-1}a_i\Delta^{2k}|Nx-i|^{2k-1}. \tag{2.14}$$

The centered difference operator Δ^{2k} is defined by

$$\Delta^{2k}c_i=\sum_{l=-k}^{k}\binom{2k}{k+l}(-1)^{k+l}c_{i+l} \tag{2.15}$$

It is easily seen that $\Delta^{2k}|Nx-i|^{2k-1}$ vanishes for $|Nx-i|\geq k$. Hence the sum in (2.14) has at most $2k$–1 nonzero terms.

Clearly the function (2.14) satisfies (2.4) and has the required continuity properties.

The coefficients a_i are to be determined by the conditons (2.5). Thus we must have

$$\sum_{i=-k+1}^{N+k-1}a_i\Delta^{2k}|j-i|^{2k-1}=u\left(\frac{j}{N}\right), \qquad j=0,1,\ldots, N. \tag{2.16}$$

In order to apply the last line of (2.5), we first use partial summation to write

$$\bar{u}(x)=(-1)^k\sum_{i=-\infty}^{\infty}D^ka_iD^k|Nx-i|^{2k-i}, \tag{2.17}$$

where we have put $a_i=0$ for $i\leq -k$, $i\geq N+k$, and where D^k is the kth forward difference operator:

$$D^kc_i=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}c_{i+l} \tag{2.18}$$

Since $\Delta^{2k}|Nx-i|^{2k-1}=0$ for $|Nx-i|\geq k$, (2.14) and (2.17) are independent of the values of a_i for $i\leq -k$, $i\geq N+k$ when $0\leq x\leq 1$. We note that $D^k|Nx-i|^{2k-1}$

is a polynomial of degree $k-1$ for $Nx \leq i$ or $Nx \geq i+k$. Hence the last line of (2.5) will be satisfied if

$$D^k a_i = 0, \qquad i = -k+1, \ldots, -1, N-k+1, \ldots, N-1. \quad (2.19)$$

The boundary value problem (2.5) is thus replaced by the system (2.16), (2.19) of $N+2k-1$ linear equations in the $N+2k-1$ unknowns a_i.

We turn to this problem of matrix inversion. *Remark:* We have assumed that $k \geq 2$. The case $k=1$ is easily treated. Let $\bar{u}$ be the broken linear function coinciding with $\bar{u}$ at the points i/N and having its breaks at these points. Let $G(x;\xi)$ be the broken linear function with breaks at ξ and the two neighboring points of the form i/N which vanishes at the points i/N, and whose derivative decreases by -1 at ξ. Then (2.13) gives optimal bounds for $F(u)$ when $k=1$.

3. Matrix Inversion

We consider the system of linear equations (2.16), (2.19). Since $\Delta^{2k}|j-i|^{2k-1}$ depends only upon $|j-i|$ and vanishes for $|j-i| \geq k$, (2.16) is a finite difference equation or order $2k-2$. We solve it by means of a system of $2k-2$ independent solutions of the homogeneous equation. To find these solutions, we note that for any number z

$$\sum_{i=-k+1}^{N+k-1} e^{iz}\Delta^{2k}|j-i|^{2k-1} = -2e^{(j-k)z}(e^z-1)^{2k} \left(\frac{d}{dz}\right)^{2k-1} [(e^z-1)^{-1}] \quad (3.1)$$

for $0 \leq j \leq N$.

We define the polynomial in e^z

$$Q_l(e^z) \equiv 2(-1)^{l+1} e^{-z}(e^z-1)^{l+2} \left(\frac{d}{dz}\right)^{l+1} [(e^z-1)^{-1}] \quad (3.2)$$

It is easily seen to be of degree l with leading coefficient 2.

The Q_l can be generated by the recursion

$$Q_l(y) = (ly+1)Q_{l-1}(y) - y(y-1)Q_{l-1}(y), \qquad Q_0(y) \equiv 2 \quad (3.3)$$

The first few of these polynomials are

$$\left.\begin{aligned} Q_1(y) &= 2(y+1), \\ Q_2(y) &= 2(y^2+4y+1), \\ Q_3(y) &= 2(y^3+11y^2+11y+1), \\ Q_4(y) &= 2(y^4+26y^3+66y^2+26y+1). \end{aligned}\right\} \quad (3.4)$$

The coefficients of Q_{2k-2} are the coefficients of the finite difference equation (2.16). It can be shown by induction that they are symmetric in the sense that

$$Q_l\left(\frac{1}{y}\right) = y^{-l} Q_l(y) \quad (3.5)$$

This means that the zeros of Q_l occur in reciprocal pairs. When l is odd, one of the zeros is -1. When l is even, Q can be written as a polynomial in $(y+y^{-1})$. Thus, the zeros of Q_{2k-2} can be found by solving an equation of degree $k-1$ and a quadratic equation.

It is shown by induction that the coefficients of Q_l are positive and that the zeros of Q_l are real and negative. The zeros of Q_l separate thos of Q_{l+1}.

Let $y_1 < y_2 < \ldots < y_{2k-2} < 0$ be the zeros of Q_{2k-2}:

$$Q_{2k-2}(y_\nu) = 0, \qquad \nu = 1, \ldots, 2k-2. \quad (3.6)$$

Then by (3.7)

$$y_{2k-1-\nu} = \frac{1}{y_\nu}. \quad (3.7)$$

Because of (3.1) and (3.2), the functions $a_i = y_\nu$ satisfy the homogeneous equations corresponding to (2.16). We now define

$$\psi_i = \begin{cases} 0 & i \leq k-2. \\ \displaystyle\sum_{\nu=1}^{2k-2} \frac{y_\nu^{i+k-2}}{Q'_{2k-2}(y_\nu)} & i \geq -k+2, \end{cases} \quad (3.8)$$

It follows from the Lagrange interpolation formula [4] that the two definitions of ψ_i coincide for $-k+2 \leq i \leq k-2$. Using (3.1) and (3.2), we find that

$$\sum_{i=-k-1}^{N+k-1} \psi_{i-p}\Delta^{2k}|j-i|^{2k-1} = \delta_{jp}, \qquad j,p = 0, \ldots, N. \quad (3.9)$$

The same equation is satisfied by ψ_{i-p} plus any linear combination of the y_ν^i. We add a linear combination such that the new function satisfies (2.19). Since ψ_{i-p} vanishes for $i \leq p+k-2$ these conditions are already satisfied for $i = -k+1, \ldots, -2$. Therefore we add only functions which also satisfy these conditions. The Lagrange interpolation formula [4] shows that such linear combinations are furnished by the functions $\eta_{i+\alpha}$, $\alpha = 1, \ldots, k$, where

$$\eta_i = \sum_{\nu=1}^{2k-2} \frac{y_\nu^{i+k-2}}{(y_\nu-1)^k Q'_{2k-2}(y_\nu)}. \quad (3.10)$$

Note that

$$D^k \eta_i = \psi_i, \qquad i \geq -k+2. \quad (3.11)$$

We let

$$\Gamma_{ip} = \psi_{i-p} - \sum_{\alpha=1}^{k} c_\alpha \eta_{i+\alpha} \quad (3.12)$$

and determine the coefficients c_α in such a way that

[4] J. F. Steffensen, Interpolation (Chelsea Press, New York, N. Y., 1950.

$D_i^k \Gamma_{ip}=0$ for $i=-1,\ N-k+1,\ \ldots,\ N-1$. This gives $c_k=\delta_{0p}$, and the equations

$$\sum_{\alpha=1}^{k-1} c_\alpha \psi_{N-k+\alpha+\beta}=D^k \psi_{N-k+\beta-p}$$
$$-\delta_{0p}\psi_{N+\beta}, \qquad \beta=1,\ldots,k-1. \quad (3.13)$$

We let $A_{\alpha\beta}$ be the inverse of the symmetric $(k-1)\times(k-1)$ matrix $\psi_{N-k+\alpha+\beta}$ so that [5]

$$\sum_{\beta=1}^{k-1} A_{\alpha\beta}\psi_{N-k+\beta+\gamma}=\delta_{\alpha\gamma}, \qquad \alpha,\gamma=1,\ldots,k-1. \quad (3.14)$$

Then Γ_{ip} is given by

$$\Gamma_{ip}=\psi_{i-p}-\sum_{\alpha,\beta=1}^{k-1} A_{\alpha\beta}\eta_{i+\alpha} D^k \psi_{N-k-p+\beta}$$
$$-\delta_{0p}\{\eta_{i+k}-\sum A_{\alpha\beta}\eta_{i+\alpha}\psi_{N+\beta}\}, \qquad i=$$
$$-k+1,\ldots,N+k-1, \qquad p=0,\ldots,N. \quad (3.15)$$

This function satisfies

$$\sum_{i=-k+1}^{N+k-1} \Gamma_{ip}\Delta^{2k}|i-j|^{2k-1}=\delta_{jp}, \qquad j,p=0,\ldots,N \quad (3.16)$$

and

$$D_i^k \Gamma_{ip}=0, \qquad i=-k+1,\ldots,-1,$$
$$N-k+1,\ldots,N-1; \qquad p=0,\ldots,N. \quad (3.17)$$

The solution of (2.16), (2.19) is given by

$$a_i=\sum_{p=0}^{N} \Gamma_{ip}\, u\left(\frac{p}{N}\right). \quad (3.18)$$

Therefore Γ_{ip} is the inverse matrix for the problem (2.16), (2.19).

Our problem of matrix inversion has thus been reduced to the solution of polynomial equations of degree $k-1$ and two in order to find the y_ν and the inversion of the $(k-1)$-dimensional symmetric matrix $\psi_{N-k+\alpha+\beta}$.

Example: We consider the case $k=2$. The zeros of $Q_2(y)$ are

$$y_1=-2-\sqrt{3},$$
$$y_2=-2+\sqrt{3}. \quad (3.19)$$

The function ψ_i defined by (3.10) is

$$\psi_i=\begin{cases} 0 & i\leq 0, \\ \dfrac{\sqrt{3}}{12}(y_1^{-i}-y_1^{i}) & i\geq 0. \end{cases} \quad (3.20)$$

[5] The fact that $\psi_{N-k+\alpha+\beta}$ is nonsingular follows from the uniqueness of $\overline{u}$ defined by (2.4), (2.5), and the linear independence of the functions $\Delta^{2k}|Nx-i|^{2k-1}$ and $\eta_{i+\alpha}$, $\alpha=1,\ \ldots,\ k$.

Since $k-1=1$, the matrix $A_{\alpha\beta}$ has the single element ψ_N^{-1}. Moreover,

$$D^2\psi_i=-\frac{1}{2}\sqrt{3}(y_1^{-i-1}-y_1^{i+1}), \qquad i\geq 0,$$
$$D^2\psi_{-1}=\frac{1}{2},$$
$$\eta_i=-\frac{1}{72}\sqrt{3}(y_1^{1-i}-y_1^{i-1}). \quad (3.21)$$

Thus (3.17) gives

$$\Gamma_{ip}=\begin{cases} \dfrac{\sqrt{3}(1-y_1^{-2i})(1-y_1^{-2(N-p)})y_1^{i-p}}{12(1-y_1^{-2N})} & -1\leq i\leq p,\quad 1\leq p\leq N-1 \\ \dfrac{\sqrt{3}(1-y_1^{-2p})(1-y_1^{-2(N-i)})y_1^{p-i}}{12(1-y_1^{-2N})} & p\leq i\leq N+1,\quad 1\leq p\leq N-1 \\ \dfrac{(1-y_1^{-2i})y_1^{i-N}}{12(1-y_1^{-2N})} & i\leq N,\quad p=N \\ \dfrac{(1-y_1^{-2(N-i)})y_1^{-i}}{12(1-y_1^{-2N})} & i=0,\quad p=0 \\ \dfrac{4-\sqrt{3}-(4+\sqrt{3})y_1^{-2N}}{12(1-y_1^{-2N})} & \begin{cases} i=-1, & p=0 \\ i=N+1, & p=N \end{cases} \end{cases} \quad (3.22)$$

4. Bounds

We now return to the consideration of section 2. The function $\overline{u}$ is given by

$$\overline{u}(x)=\sum_{\substack{i=-k+1 \\ |Nx-i|<k}}^{N+k-1} a_i\Delta^{2k}|Nx-i|^{2k-1}, \quad (4.1)$$

with

$$a_i=\sum_{j=0}^{N}\Gamma_{ij}\, u\left(\frac{j}{N}\right). \quad (4.2)$$

Thus, we may write

$$\overline{u}(x)=\sum_{j=0}^{N} g_j(x)\, u\left(\frac{j}{N}\right), \quad (4.3)$$

where

$$g_j(x)=\sum_{\substack{i=-k+1 \\ |Nx-i|<k}}^{N+k-1}\Gamma_{ij}\Delta^{2k}|Nx-i|^{2k-1} \quad (4.4)$$

is the optimal approximation function corresponding to $u(i/N)=\delta_{ij}$.

In order to find the bounds (2.13) we need the integral of $\overline{u}^{[k]2}$ and $G(x,\xi)$. Integrating by parts we find that

$$\int_0^1 \overline{u}^{[k]2}dx=(-1)^k\sum_{j=0}^{N} u\left(\frac{j}{N}\right)[\overline{u}^{[2k-1]}]_{j/N}, \quad (4.5)$$

where $[\ ]_{j/N}$ denotes the discontinuity in the function at j/N, with the convention $\overline{u}^{[2k-1]}(0-)=\overline{u}^{[2k-1]}(1+)=0$. Applying partial summation to (4.1), we find that

$$\overline{u}(x)=\sum_{i=-k+1}^{N+k-1}|Nx-i|^{2k-1}\Delta^{2k}a_i. \tag{4.6}$$

Consequently,

$$\int_0^1 \overline{u}^{[k]\,2}dx=(-1)^k 2N^{2k-1}(2k-1)!\sum_{i=0}^{N}u\left(\frac{i}{N}\right)\Delta^{2k}a_i$$
$$=(-1)^k 2N^{2k-1}(2k-1)!\sum_{i,j=0}^{N}u\left(\frac{i}{N}\right)u\left(\frac{j}{N}\right)\Delta_i^{2k}\Gamma_{ij} \tag{4.7}$$

with the convention that

$$\Delta_i^{2k}\Gamma_{ij}=D^k\Gamma_{ij}\qquad\text{for}\quad i=0,\ i=N. \tag{4.8}$$

We now construct the Green's function $G(x,\xi)$. We begin with a function $H(x,\xi)$ having the proper jump at $x=\xi$ and satisfying the condition that the derivatives of orders $k, \ldots 2k\text{-}2$ vanish at the end points. Let p be any integer satisfying

$$0\leq p<N\xi<p+2k-1\leq N. \tag{4.9}$$

It follows from the Lagrange interpolation formula (see footnote 4) that the function

$$H(x;\xi)=|x-\xi|^{2k-1}-\sum_{\mu=0}^{2k-1}\left|\frac{p+\mu}{N}-x\right|^{2k-1}b_\mu(\xi), \tag{4.10}$$

where

$$b_\mu(\xi)=\frac{(-1)^{\mu+1}N^{2k-1}}{(2k-1)!}\binom{2k-1}{\mu}\left(\xi-\frac{p+\mu}{N}\right)^{-1}\prod_{\nu=0}^{2k-1}\left(\xi-\frac{p+\nu}{N}\right) \tag{4.11}$$

has the property

$$H(x;\xi)\equiv 0\qquad\text{for}\quad x\leq\frac{p}{N},\quad x\geq\frac{p+2k-1}{N}. \tag{4.12}$$

Therefore, the function

$$G(x;\xi)=\frac{(-1)^k}{2(2k-1)!}\left\{H(x;\xi)-\sum_{j=0}^{N}g_j(x)H\left(\frac{j}{N};\xi\right)\right\} \tag{4.13}$$

has all the properties (2.6). Thus, it is the Green's function.[6] We not that the sums in (4.4) and (4.13) involve at most $2k-1$ terms for each x and ξ.

The bounds (2.13) for $F(u)$ are now given explicitly by

$$\left|F(u)-\sum_{i=-k+1}^{N+k-1}\sum_{j=0}^{N}\Gamma_{ij}F(\Delta^{2k}|Nx-i|^{2k-1})u\left(\frac{j}{N}\right)\right|^2$$
$$\leq\frac{(-1)^k}{2(2k-1)!}\left\{M^2-2(-1)^kN^{2k-1}(2k-1)!\sum_{i,j=0}^{N}\Delta_i^{2k}\Gamma_{ij}u\left(\frac{i}{N}\right)u\left(\frac{j}{N}\right)\right\}$$
$$\left\{F_x[F_\xi(H(x;\xi))]-\sum_{i=-k+1}^{N+k-1}\sum_{j=0}^{N}\Gamma_{ij}F(\Delta^{2k}|Nx-i|^{2k-1})F\left[H\left(\frac{j}{N};\xi\right)\right]\right\} \tag{4.14}$$

where we again use the convention (4.8). If F is local in the sense that $F(u)$ only involves the values of u in the neighborhood of a point, the sums in the second term on the right involve at most $2k-1$ values of i and j. *Example: Let* $k=2$. Γ_{ij} is given by (3.22). We find that

$$\Delta^4|Nx-i|^3=\begin{cases}2\{4-6|Nx-i|^2+3|Nx-i|^3\} & |Nx-i|\leq 1\\ 2\{2-|Nx-i|^3\} & 1\leq|Nx-i|\leq 2\\ 0 & |Nx-i|\geq 2.\end{cases} \tag{4.15}$$

Thus the interpolation function $g_i(x)$ is given by

$$g_i(x)=12(q+1-Nx)(Nx-q)[(q+2-Nx)\Gamma_{qi}+(Nx+1-q)\Gamma_{q+1,\,i}]+(q+1-Nx)^3\delta_{qj})+(Nx-q)^3\delta_{q+1,\,j}, \tag{4.16}$$

where the integer q is defined by

$$q<Nx\leq q+1. \tag{4.17}$$

If $1<N\xi<N-1$, we let p be the integer such that

$$p+1<N\xi<p+2. \tag{4.18}$$

[6] Since Green's function is uniquely defined by (2.6), it does not depend upon the integer p used in the construction of H. The function H does depend upon p.

Then we have

$$\frac{3N^3H(x;\xi)}{(N\xi-p)(N\xi-p-1)(p+2-N\xi)(p+3-N\xi)}=\begin{cases} 0 & Nx\leq p,\ Nx\geq p+3 \\ \dfrac{(Nx-p)^3}{N\xi-p} & p\leq Nx\leq p+1 \\ \dfrac{(Nx-p)^3}{N\xi-p}-\dfrac{3(Nx-p-1)^3}{N\xi-p-1} & p+1\leq Nx\leq N\xi \\ \dfrac{(p+3-Nx)^3}{p+3-N\xi}-\dfrac{3(p+2-Nx)^3}{p+2-N\xi} & N\xi\leq Nx\leq p+2 \\ \dfrac{(p+3-Nx)^3}{p+3-N\xi} & p+2\leq Nx\leq p+3. \end{cases} \tag{4.19}$$

If $0<N\xi<1$, the definition (4.18) gives $p=-1$, so that the function H defined by (4.19) does not vanish at $x=0$. In this case we simply subtract the function

$$\frac{1}{2}(N\xi+1)^{-1}\Delta^4|Nx-1|^3 \tag{4.20}$$

from the right-hand side of (4.19) with $p=-1$ to obtain $H(x,\xi)$. Similarly, if $N-1<N\xi<N$, we subtract

$$\frac{1}{2}(N+1-N\xi)^{-1}\Delta^4|Nx-(N+1)|^3 \tag{4.21}$$

from the right-hand side of (4.19) with $p=N-2$.

Thus we can evaluate the bounds 4.14 explicitly. In the special case of linear interpolation we have

$$F(u)=u(\zeta). \tag{4.22}$$

If

$$1\leq q<N\zeta<q+1\leq N-1, \tag{4.23}$$

we find

$$F_x[F_\xi(H(x;\xi))]-\sum_{i=-1}^{N+1}\sum_{j=0}^{N}\Gamma_{ij}F(\Delta^4|Nx-i|^3)F\left[H\left(\frac{j}{N};\xi\right)\right]$$
$$=N^{-3}(N\zeta-q)^2(q+1-N\zeta)^2\left\{4-\frac{\sqrt{3}}{3(1-y_1^{-2N})}\left[(q+2-N\zeta)^2(1-y_1^{-2q})(1-y_1^{-2(N-q)})\right.\right.$$
$$+2(q+2-N\zeta)(N\zeta-q+1)(1-y_1^{-2q})(1-y_1^{-2(N-q-1)})y_1^{-1}$$
$$\left.\left.+(N\zeta-q+1)^2(1-y_1^{-2(q+1)})(1-y_1^{-2(N-q-1)})\right]\right\}, \tag{4.24}$$

where

$$y_1=-2-\sqrt{3}.$$

The first factor on the right of (4.14) does not approach zero as $N\rightarrow\infty$ unless M^2 happens to be the exact value of the integral of $u^{[k]2}$. Thus the difference between the best upper and lower bounds in the linear interpolation problem with $N+1$ equally spaced points and with a given bound for $\int_0^1 u''^2dx$ is of the order $N^{-3/2}$. If a uniform bound for $|u''|$ is given, one can obtain bounds for $u(\zeta)$ which differ by a term of order N^{-2}. This shows that a bound for the square integral gives considerably less information than a bound for the maximum. The problem of finding best bounds when the maximum of $|u^{[k]}|$ is bounded is much more difficult than the problem treated here.

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