JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 65B, No. 2, April–June 1961

Mean Motions In Conditionally Periodic Separable Systems[1]

John P. Vinti

(November 10, 1960)

A search of the literature failed to disclose any general statement or proof of a theorem informally current among dynamical astronomers. The present paper gives a proof of the theorem, which states that, in any conditionally periodic separable system the mean frequency n_k of any separation coordinate q_k is equal to $\nu_k \equiv \partial\alpha_1/\partial J_k$. Here α_1 is the energy and J_k is the k'th action variable. The proof is carried out for nonsingular Staeckel systems, so that it is applicable to any nonpolar orbit of an artificial satellite, when the potential leads to separability.

1. Introduction

Conditionally periodic separable systems are commonly illustrated in works on advanced dynamics by the motion of a particle under the joint action of harmonic oscillator forces at right angles. This happens to be a very special case, for which each rectangular coordinate q_k has a constant frequency, equal to the corresponding "fundamental frequency" $\nu_k \equiv \partial\alpha_1/\partial J_k$, where α_1 is the energy and J_k the corresponding action variable.

In a more general system of this type, each generalized coordinate q_k may have a variable frequency, but it appears to be generally believed among dynamical astronomers that the mean frequency of q_k must be equal to ν_k, if the conditionally periodic system is separable. Needing to refer to such a theorem in solving a specific problem, I have searched the literature but have found no explicit statement or proof of it. The present paper is an attempt to furnish such a reference, with a proof sufficiently general to be applicable to all the separable problems that may arise in the gravitational theory of the orbit of a satellite of an oblate planet.

Little of the analysis in this paper can claim to be really new. Much of the pertinent material in the literature, however, is discursive, relatively unavailable, and expressed in notations now unfamiliar to most mathematical physicists. Some of it is inadequate, if not incorrect, especially in the treatment of the periodicity of the q's as functions of the angle variables. Moreover, none of it seems to have been carried out in the shortest and most appropriate way to prove the theorem in question. The present paper attempts to give a concise and correct treatment that will serve this purpose.

It is easy to see why such a theorem should have escaped formal statement and proof. Physicists have not been concerned with mean frequencies of this kind. Dynamical astronomers have been, but ordinarily for nonseparable systems. Until 1957 their only separable problems were the Kepler problem [1][2] for which the coordinate frequencies are all equal to the constant value $\nu_1=\nu_2=\nu_3$, and the problem of two centers, which has remained a curiosity to be found in Charlier's famous book [2], but without application. Since 1957 various potentials have been suggested, by Sterne [3], by Garfinkel [4], and by the author [5], for the gravitational field of an oblate planet, all of which lead to separability and to intermediary satellite orbits with $\nu_1 \neq \nu_2 \neq \nu_3$. Since the solution for these orbits is greatly facilitated by knowledge of the mean coordinate frequencies, it now becomes desirable to have a formal and general proof of the theorem.

It is here convenient to discuss briefly the general plan of the paper, without definitions. The difficulty in proving the theorem arises only when the fundamental frequencies $\nu_1, \ldots, \nu_n$ are incommensurable. If $w_1, \ldots, w_n$ are the angle variables, the plan is first to show that for a conditionally periodic system there exist infinitely many values of the time, with no upper bound, at which the orbit in w-space passes arbitrarily closely to points separated from the initial point $w_1(0), \ldots, w_n(0)$ by integer intervals $\Delta w_k = m_k$, $k=1, \ldots, n$. This fact follows directly from a theorem of Dirichlet, which is easy to understand and to apply.

To convert this result to q-space, it is necessary to know that q_k is a single-valued, continuous, periodic function of the w's. To show this the author restricts considerations to nonsingular Staeckel systems, proving that for them each q_k is a single-valued differentiable function of $v_k \equiv \int dq_k/p_k$ and each v_k of the angle variables $w_1, \ldots, w_n$. With a careful use of the single-valuedness, the periodic property then follows.

Application of these properties of the q's as functions of the w's shows that, at the values of the time mentioned above, the orbit in q-space then passes arbitrarily closely to points where each q_k would have gone through exactly m_k cycles. The proof of the theorem then follows.

[1] This work was supported by the U.S. Air Force, through the Office of Scientific Research of the Air Research and Development Command.

[2] Figures in brackets indicate the literature references at the end of this paper.

2. Staeckel Systems

If $q_1, \ldots, q_n$ and $p_1, \ldots, p_n$ denote the generalized coordinates and momenta of a dynamical system of n degrees of freedom, the system is said to be of the Staeckel type [2, 6, 7, 8, 9] if the Hamiltonian H is given by

$$H=\frac{1}{2}\sum_{k=1}^{n} A_k(q_1, \ldots, q_n)p_k^2+V(q_1, \ldots, q_n),$$
$$A_k>0, \quad (k=1, \ldots, n) \tag{1}$$

and if there exist functions $\phi_{ij}(q_i), \psi_i(q_i)$, $i,j=1, \ldots, n$, such that

$$A_k=M_{k1}/\det(\phi_{ij}), \tag{2}$$

$$V=\sum_{k=1}^{n} \psi_k(q_k)A_k, \tag{3}$$

M_{k1} being the cofactor of $\phi_{k1}(q_k)$ in the determinant $\det(\phi_{ij})$.

Conditions (2) and (3) are necessary and sufficient for the separability of a system with such a Hamiltonian.

If we next define the domain Q of the q's as the totality of real values of $q_1, \ldots, q_n$ for which $p_k^2 \geqq 0$, $k=1, \ldots, n$, we may then define a *non-singular* Staeckel system as one for which $\psi_k(q_k)$ and $\phi_{ki}(q_k)$, i, $k=1, \ldots, n$, exist and are single-valued and for which $\det(\phi_{ij}) \neq 0$ anywhere in Q. If we put

$$\Phi \equiv (\phi_{ij}(q_i)) \tag{4}$$

for the Staeckel matrix, then Φ^{-1} exists and is single-valued anywhere in Q; in particular,

$$(\Phi^{-1})_{1k}=A_k \qquad (k=1, \ldots, n) \tag{5}$$

all exist anywhere in Q. (This restriction thus rules out polar orbits from consideration if the right ascension φ is one of the coordinates, since A_3 then becomes infinite on the polar axis.)

The momenta p_k are then given by

$$p_k^2=-2\psi_k(q_k)+2\sum_{i=1}^{n} \phi_{ki}(q_k)\alpha_i, \qquad (k=1, \ldots, n), \tag{6}$$

where the α's are separation constants, α_1 being the energy. (For satellite problems, where $n=3$, $2\alpha_2$ and $2\alpha_3$ are usually denoted by α_2^2 and α_3^2.)

The Hamilton-Jacobi function W is then given by

$$W=\sum_{k=1}^{n}\int_{q_{k0}}^{q_k} p_k dq_k$$
$$=\sum_{k=1}^{n}(\pm)\int_{q_{k0}}^{q_k}\left[-2\psi_k(q_k)+2\sum_{i=1}^{n}\phi_{ki}(q_k)\alpha_i\right]^{1/2} dq_k, \tag{7}$$

where the sign is $\pm$ respectively as $dq_k \gtrless 0$.

3. Conditionally Periodic Staeckel Systems

We call a Staeckel system conditionally periodic if each coordinate is either rotational or librational. A coordinate q_k is rotational if: (i) it is an angle, (ii) with $p_k^2 \equiv F_k(q_k)$ there exist positive real numbers c_{1k} and c_{2k} such that $c_{2k} \geqq F_k(q_k) \geqq c_{1k} > 0$ for all real values of q_k, and (iii) if

$$F_k(q_k+2\pi)=F_k(q_k). \tag{8}$$

Note that $c_{1k}>0$ rules out asymptotic motions and that the periodicity implied by (8) may reduce to simple constancy. The latter holds, e.g., when q_k is the right ascension ϕ of an artificial satellite, since there does not exist any potential, depending on the right ascension ϕ, which both leads to separability and remains finite on the polar axis.[3] For such a rotational coordinate q_k either $p_k > c_{1k}^{1/2}$ for all q_k or $p_k < -c_{1k}^{1/2}$ for all q_k. In either case

$$v_k(q_k) \equiv \int_{q_{k0}}^{q_k} dq_k/p_k \tag{9}$$

is a single-valued function of q_k, with derivative dv_k/dq_k existing and differing from zero for all values of q_k. Thus q_k is a single-valued differentiable[4] function of v_k.

A coordinate q_i is librational if there exist real numbers a_i, b_i, C_{1i}, and C_{2i} and a real function $G_i(q_i)$ such that

$$p_i^2=(q_i-a_i)(b_i-q_i)G_i(q_i), \tag{10.1}$$

with

$$C_{2i} \geqq G_i(q_i) \geqq C_{1i} > 0 \qquad (a_i \leqq q_i \leqq b_i) \tag{10.2}$$

and

$$a_i \leqq q_i(0) \leqq b_i, \tag{10.3}$$

$q_i(0)$ being the initial value of q_i. If we again define v_i by (9), then

$$v_i(q_i)=\pm\int_{q_{i0}}^{q_i}[(q_i-a_i)(b_i-q_i)G_i(q_i)]^{-1/2} dq_i, \tag{11}$$

where the sign is $\pm$ accordingly as $dq_i \gtrless 0$, respectively. Then

$$v_i=\int_{E_{i0}}^{E_i} G_i^{-1/2} dE_i, \tag{12}$$

where the uniformizing variable E_i is defined by the equation

$$2q_i=a_i+b_i+(a_i-b_i)\cos E_i \tag{13}$$

and the requirement that E_i shall always increase as q_i varies. By (12) and (10.2) v_i is then a single-valued function of E_i, with derivative existing and

[3] See the tables in [9], pp. 656, 658, 660.
[4] Hereafter abbreviated to "s.v.d."

nonvanishing for all E_i, so that E_i must be a s.v.d. function of v_i. By (13), however, q_i is a s.v.d. function of E_i, so that, finally, q_i is a s.v.d. function of v_i. Thus in a conditionally periodic Staeckel system any coordinate q_k is a s.v.d. function of the corresponding v_k.

4. The v_k's as Functions of the Angle-Variables w_k

If we now let an increase of 2π be one cycle of a rotational coordinate and a single round trip from a_k to b_k be one cycle of a librational coordinate, we may define the action and angle variables J_k and w_k by

$$J_k \equiv \oint p_k \, dq_k \tag{14}$$

and

$$w_k \equiv \partial W / \partial J_k, \tag{15}$$

where W is now to be considered a function of the q's and the J's, rather than of the q's and the α's. It is well known [10, 11] that J_k and w_k are canonically conjugate, so that

$$\dot{w}_k = \partial \alpha_1 / \partial J_k \equiv \nu_k. \tag{16}$$

If we also define the Jacobi variables B_i by

$$B_i \equiv \partial W / \partial \alpha_i, \tag{17}$$

we obtain

$$B_i = \sum_{k=1}^{n} \frac{\partial W}{\partial J_k} \frac{\partial J_k}{\partial \alpha_i} = \sum_{k=1}^{n} w_k \omega_{ki}, \tag{18}$$

where

$$\omega_{ki} \equiv \partial J_k / \partial \alpha_i \tag{19}$$

Increments $dw_1, \ldots, dw_n$ then lead to

$$dB_i = \sum_{k=1}^{n} (dw_k)\, \omega_{ki} \qquad (i=1, \ldots, n). \tag{20}$$

But, by (7) and (17),

$$dB_i = \sum_{k=1}^{n} \frac{\partial p_k}{\partial \alpha_i} dq_k = \frac{1}{2} \sum_{k=1}^{n} \frac{\partial p_k^2}{\partial \alpha_i} dq_k / p_k \tag{21}$$

or

$$dB_i = \sum_{k=1}^{n} \phi_{ki}(q_k)\, dv_k, \tag{22}$$

by (21), (6), and (9). Also, by (19), (14), (6), and (9)

$$\omega_{ki} = \oint \phi_{ki}(q_k) dv_k. \tag{23}$$

If we now introduce the matrix Φ, the matrix $\Omega \equiv (\omega_{ki})$, and the row matrices $dv \equiv (dv_1, \ldots, dv_n)$ and $dw \equiv (dw_1, \ldots, dw_n)$, we find from (20) and (22)

$$dv\Phi = dw\Omega. \tag{24}$$

For a nonsingular Staeckel system[5] Φ, Φ^{-1}, and Ω all exist at every point of Q, so that

$$dv = dw \Omega \Phi^{-1} \tag{25}$$

or

$$\partial v_i / \partial w_k = (\Omega \Phi^{-1})_{ki}. \tag{26}$$

Thus each derivative $\partial v_i / \partial w_k$ exists and is single-valued everywhere in Q. Now the J's are all real, by (14). Thus, by (7) and (15), if the p's are all real, then W and the w's are all real; if some of the p's are nonreal, then W is nonreal and so are some of the w's. If all the w's are real, it then follows that all the p's are real, else we should have a contradiction. Thus the domain Q, corresponding to the totality of all real values of the q's for which the p's are all real, also corresponds exactly to the set of all possible real values for all the w's. It therefore follows that each derivative $\partial v_i / \partial w_k$ exists and is single-valued at any point in w-space. Thus each v_k must be a s.v.d. function of $w_1, \ldots, w_n$. In section 3, however, we showed that each q_k is a s.v.d. function of the corresponding v_k. Thus for a conditionally periodic nonsingular Staeckel system each q_k is a s.v.d. function $f_k(w_1, \ldots, w_n)$.

5. Periodic Properties of $q_k = f_k(w_1, \ldots, w_n)$

By (7) and (15)

$$dw_k = \sum_{i=1}^{n} \frac{\partial p_i}{\partial J_k} dq_i. \tag{27}$$

If now each coordinate q_i goes through an integral number m_i of cycles, then by a familiar argument

$$\Delta w_k = \sum_{i=1}^{n} m_i \oint (\partial p_i / \partial J_k) dq_i = \frac{\partial}{\partial J_k} \sum_{i=1}^{n} m_i \oint p_i dq_i = m_k. \tag{28}$$

Thus if each q_k goes through exactly m_k cycles, each w_k increases by the integer m_k. (Note, however, that such simultaneous increases are not always physically possible: this section is thus concerned only with the mathematical properties of the functions $f_k(w_1, \ldots, w_n)$.)

But we are really interested in the inverse problem where each w_k has increased by an integer m_k and we ask what has happened to the q's. Now the q's are uniquely determined by the w's, because of the single-valued property. In the situation of the preceding paragraph where each librational coordinate returns to its initial value and each rotational coordinate q_i increases by $2\pi m_i$, each angle variable w_k increases by m_k. Since the w's determine the q's uniquely, this has the result that whenever $\Delta w_k = m_k$, $k=1, \ldots, n$, each librational coordinate returns to its initial value and each rotational coordinate q_i increases by $2\pi m_i$.

[5] See appendix for examples.

Thus, in the inverse problem, whenever we are given $\Delta w_k = m_k$, $k=1, \ldots, n$, we find that each rotational coordinate q_i must go through exactly m_i cycles and that each librational coordinate q_j must go through some integral number of cycles, τ_j, say. By (28), however, we then find $\tau_j = m_j$. Thus whenever the angle variables w_k are all increased by integer amounts $\Delta w_k = m_k$, each of the functions $q_k = f_k(w_1, \ldots, w_n)$ must go through exactly m_k cycles.

6. Mean Motions

If in a time interval T the number of complete cycles passed through by any coordinate q_k is N_k, the corresponding mean frequency n_k is, by definition

$$n_k = \lim_{T\to\infty} (N_k/T), \tag{29}$$

if the limit exists. We shall now prove that $n_k = \nu_k \equiv \partial\alpha_1/\partial J_k$, $k=1, \ldots, n$, for any conditionally periodic nonsingular Staeckel system.

To do so, note that if $\nu_1, \ldots, \nu_n$ are all commensurable, there exist a positive ν_0 and positive integers $m_1, \ldots, m_n$ such that

$$\nu_k = m_k \nu_0, \qquad (k=1, \ldots, n), \tag{30}$$

where we may choose ν_0 to be the greatest common divisor of the ν_k's. Then, from (16) and (30), during the actual motion,

$$w_k = w_k(0) + m_k \nu_0 t \qquad (k=1, \ldots, n) \tag{31}$$

and in the time interval $T \equiv 1/\nu_0$ we have

$$\Delta w_k = m_k \qquad (k=1, \ldots, n). \tag{32}$$

By section 5 each q_k goes through exactly m_k cycles in this time, so that in this case the motion is truly periodic, with period $1/\nu_0$. The mean frequency of q_k is thus

$$n_k \equiv m_k/T = m_k \nu_0 = \nu_k. \tag{33}$$

If the frequencies $\nu_1, \ldots, \nu_n$ are not all commensurable, we may let

$$\xi_k \equiv \nu_k/\nu_1 \qquad (k=1, \ldots, n) \tag{34}$$

and then at least one of the ξ's will be irrational. Then by (16) and (34), during the actual motion

$$w_k = w_k(0) + \xi_k \nu_1 t. \tag{35}$$

We now use a theorem of Dirichlet [12], which states that if the set of real numbers $\xi_1, \ldots, \xi_n$ has at least one irrational member, then the system of inequalities

$$|\xi_k - m_k/P| < P^{-1-\frac{1}{n}} \qquad (k=1, \ldots, n) \tag{36}$$

has an infinite number of integer solutions for P and the m's. Note that the solutions for P have no upper bound.

To apply this theorem, consider only those values of the time interval T such that $\nu_1 T = P$, where P is an integer that satisfies (36). In this time each w_k increases from its initial value $w_k(0)$ to a final value given by

$$w_k(T) = w_k(0) + P\xi_k, \tag{37}$$

by (35). But by (36)

$$P\xi_k = m_k + \eta_k, \qquad |\eta_k| < P^{-\frac{1}{n}} \tag{38}$$

so that

$$w_k(T) = w_k(0) + m_k + \eta_k. \tag{39}$$

As $\nu_1 T = P$ takes on those larger and larger integer values corresponding to solutions of (36), each η_k approaches zero, by (38). Then, by (39), there exist infinitely many values of T, with no upper bound, at which the orbit in w-space passes arbitrarily closely to points where $\Delta w_k = m_k$, $k=1, \ldots, n$, the m_k's being solutions of (36).

If the initial q's are given by

$$q_k(0) = f_k[w_1(0), \ldots, w_n(0)], \qquad (k=1, \ldots, n), \tag{40}$$

then the values of the q's at any of these times T are given by

$$q_k(T) = f_k[w_1(0) + m_1 + \eta_1, \ldots, w_n(0) + m_n + \eta_n] \qquad (k=1, \ldots, n). \tag{41}$$

As we let $T = P/\nu_1$ assume those larger and larger values already referred to, the q's then approach arbitrarily closely to the values

$$q_k^*(T) = f_k[w_1(0) + m_1, \ldots, w_n(0) + m_n] \qquad (k=1, \ldots, n). \tag{42}$$

This conclusion follows from (38) and the single-valuedness and differentiability of the functions f_k.

Comparison of (40) and (42) then shows that the values $q_k^*(T)$ correspond to $\Delta w_k = m_k$, $k=1, \ldots, n$, and are thus, by section 5, the values that would be reached after each q_k had gone through exactly m_k cycles. Now, by the definition (29), it follows that the mean frequency

$$n_k = \lim_{T\to\infty} (m_k/T), \tag{43}$$

if the limit exists. But $m_k/T = \nu_1 m_k/P$ and, as $T\to\infty$, $\lim(m_k/P) = \xi_k$, by (36). Thus,

$$n_k = \nu_1 \xi_k = \nu_k, \tag{44}$$

by (43) and (34).

Thus, for each coordinate q_k of a conditionally periodic nonsingular Staeckel system, the mean frequency n_k is equal to the corresponding fundamental frequency $\nu_k \equiv \partial\alpha_1/\partial J_k$.

7. Appendix

For theories of satellite orbits, appropriate coordinates are spherical or oblate spheroidal. The corresponding Staeckel matrices and their inverses are, if $x=r \sin \theta \cos \phi$, $y=r \sin \theta \sin \phi$, $z=r \cos \theta$:

Spherical

$$\Phi=\begin{pmatrix} 1 & -r^{-2} & 0 \\ 0 & 1 & -\csc^2 \theta \\ 0 & 0 & 1 \end{pmatrix} \qquad \Phi^{-1}=\begin{pmatrix} 1 & r^{-2} & r^{-2}\csc^2 \theta \\ 0 & 1 & \csc^2 \theta \\ 0 & 0 & 1 \end{pmatrix}$$

or, if $x=c[(\xi^2+1)(1-\eta^2)]^{1/2} \cos \phi$, $y=c[(\xi^2+1)(1-\eta^2)]^{1/2} \sin \phi$, $z=c\xi\eta$:

Oblate Spheroidal

$$\Phi=\begin{pmatrix} c^2\xi^2(\xi^2+1)^{-1} & -(\xi^2+1)^{-1} & (\xi^2+1)^{-2} \\ c^2\eta^2(1-\eta^2)^{-1} & (1-\eta^2)^{-1} & -(1-\eta^2)^{-2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Phi^{-1}=\begin{pmatrix} c^{-2}(\xi^2+1)(\xi^2+\eta^2)^{-1} & c^{-2}(1-\eta^2)(\xi^2+\eta^2)^{-1} & c^{-2}(\xi^2+1)^{-1}(1-\eta^2)^{-1} \\ -\eta^2(\xi^2+1)(\xi^2+\eta^2)^{-1} & \xi^2(1-\eta^2)(\xi^2+\eta^2)^{-1} & (\xi^2+\eta^2)^{-1}[\xi^2(1-\eta^2)^{-1}+\eta^2(1+\xi^2)^{-1}] \\ 0 & 0 & 1 \end{pmatrix}$$

For spherical coordinates Φ is most easily written down from the expressions for p_r^2, p_θ^2, and p_ϕ^2 in the Kepler problem [1], with replacement of α_2^2 and α_3^2 by $2\alpha_2$ and $2\alpha_3$. For oblate spheroidal coordinates Φ may be found by comparing eqs (53) and (59.1) of [5] with eq (6) of the present paper.

Note that Φ or Φ^{-1} could fail to exist only when $\sin \theta=0$ or when $\eta^2=1$. This could happen only when the satellite goes over a pole and thus only in a polar orbit. Such a singularity in a polar orbit, however, is to be expected, since nonsingularity of a Staeckel system leads to the q's being differentiable functions of the w's and thus of the time. In a polar orbit, on the other hand, the right ascension $q_3 \equiv \phi$ is a discontinuous function of time, being constant except at polar crossings, where it changes by π.

(Paper 65B2–52)

8. References

[1] W. M. Smart, Celestial Mechanics, pp. 143–148 (Longmans, Green, & Co., New York, N.Y., 1953).
[2] C. L. Charlier, Die Mechanik des Himmels **I** (Veit and Co., Leipzig, Germany, 1902).
[3] T. E. Sterne, Astron. J. **62,** 96 (1957); **63,** 28–40 (1958).
[4] B. Garfinkel, Astron. J. **63,** 88–96 (1958); **64,** 353–367 (1959).
[5] J. P. Vinti, Phys. Rev. Letters **3,** 8 (1959); J. Research NBS **63B,** 105–116 (1959).
[6] P. Staeckel, Habilitationsschrift (Halle, 1891).
[7] L. P. Eisenhart, Annals of Math. **35,** 284–305 (1934).
[8] L. P. Eisenhart, Phys. Rev. **74,** 87–89 (1948).
[9] P. M. Morse and H. Feshbach, Methods of Theoretical Physics **I,** 509–518, 655–664 (McGraw Hill Book Co., Inc., New York, N.Y., 1953).
[10] P. Frank and R. v. Mises, Die Differential-und Integralgleichungen der Physik **II,** 94–101 (Vieweg & Sohn, Braunschweig, Germany, 1935).
[11] M. Born, The Mechanics of the Atom, ch. 1, 2 (G. Bell and Sons, Ltd., London, England, 1927).
[12] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, p. 169, theorem 200 (Clarendon Press, Oxford, England, 1938).