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Another Extension of Heinz's Inequality

Marvin Marcus

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A recent result of Heinz gives bounds on the bilinear form associated with a matrix Q in terms of bounds on the two Hermitian parts of Q. This is extended to certain determinants associated with Q by use of the Grassmann algebra.

Let A and B be non-negative Hermitian (n.n.h.) n-square matrices and let Q be an arbitrary n-square matrix. A recent inequality due to E. Heinz [3][1] states that *if* $A^2-Q^*Q\geq 0$ *and* $B^2-QQ^*\geq 0$, *then*

$$|(Qu, v)|\leq ||A^pu||\,||B^{1-p}v||, \qquad 0\leq p\leq 1 \tag{1}$$

for any u and v. Here $X\geq 0$ means simply that X is non-negative Hermitian. The most recent proof of (1) is found in [1] along with several references to other proofs and extensions.

In this paper the following generalization of (1) will be presented.

THEOREM. *If $A^2-Q^*Q\geq 0$ and $B^2-QQ^*\geq 0$ and $u_1, \ldots, u_k, v_1, \ldots, v_k$ are any $2k$ vectors, $1\leq k\leq n$, then*

$$|\det\{(Qu_i,v_j)\}|^2\leq \det\{(A^pu_i, A^pu_j)\}$$
$$\times\det\{(B^{1-p}v_i,B^{1-p}v_j)\}$$
$$\leq\prod_1^k ||A^pu_i||^2||B^{1-p}v_i||^2, \tag{2}$$

where $0\leq p\leq 1$.

In what follows we use certain elementary properties of the kth compound matrix of A, $C_k(A)$, which are found in [5].

LEMMA. *If $H\geq 0$ and $K\geq 0$ and $H-K\geq 0$, then* $C_k(H)-C_k(K)\geq 0$.

PROOF. We may assume K is nonsingular (and hence H will be) by the standard continuity argument. Now $H-K\geq 0$ if and only if $H^{-1/2}(H-K)H^{-1/2}=I-H^{-1/2}KH^{-1/2}\geq 0$. This latter inequality holds if and only if every eigenvalue of $H^{-1/2}KH^{-1/2}$ is at most 1. Now the eigenvalues of $C_k(H^{-1}K)$ are the $\binom{n}{k}$ products taken k at a time of the eigenvalues of $H^{-1}K$. Moreover $H^{-1}K$ has non-negative eigenvalues and hence every eigenvalue of $C_k(H^{-1}K)$ is bounded by 1. Now $C_k(H^{-1}K)=C_k(H^{-1})C_k(K)$ and this latter matrix is similar to $[C_k(H)]^{-1/2}C_k(K)[C_k(H)]^{-1/2}$. Thus

$$C_k(I)-[C_k(H)]^{-1/2}C_k(K)[C_k(H)]^{-1/2}\geq 0$$

and

$$C_k(H)-C_k(K)\geq 0$$

For completeness we next include a very brief and elementary proof of (1) which relies on the fact [4] that $\varphi(\lambda)=\lambda^p$, $0\leq p\leq 1$, is monotone of every order for non-negative λ. A scalar function φ is said to be *monotone of order n* on $a\leq\lambda\leq b$ if whenever $H-K\geq 0$ it follows that $\varphi(H)-\varphi(K)\geq 0$, where H and K are n-square Hermitian with eigenvalues in the interval $a\leq\lambda\leq b$. To see (1) let $Q=UH$ be the polar factorization of Q, $H\geq 0$, U unitary. Then the hypotheses are equivalent to $A^2-H^2\geq 0$, $B^2-(UHU^*)^2\geq 0$. Setting $w=U^*v$ we compute that

$$|(Qu, v)|^2=|(Hu, w)|^2=|(H^pH^{1-p}u,w)|^2$$
$$=|(H^pu, H^{1-p}w)|^2\leq (H^{2p}u, u)(H^{2(1-p)}w, w)$$
$$=(H^pu, H^pu)(UH^{1-p}U^*v, UH^{1-p}U^*v)$$
$$\leq (A^pu, A^pu)(B^{1-p}v, B^{1-p}v)$$
$$=||A^pu||^2||B^{1-p}v||^2.$$

To proceed to the proof of (2) let $u_1\wedge\ldots\wedge u_k$ denote the Grassmann (outer) product [2] of the vectors $u_1, \ldots, u_k$. Then, by the lemma,

$$0\leq C_k(A^2)-C_k(Q^*Q)=C_k^2(A)-C_k^*(Q)C_k(Q)$$

and

$$0\leq C_k^2(B)-C_k(Q)C_k^*(Q).$$

Hence, applying (1) to $C_k(Q)$, $C_k(A)$, and $C_k(B)$ in place of Q, A, and B respectively, we have

$$|\det\{(Qu_i,v_j)\}|^2=|(C_k(Q)u_1\wedge\ldots\wedge u_k, v_1\wedge\ldots\wedge v_k)|^2$$
$$\leq ||[C_k(A)]^pu_1\wedge\ldots\wedge u_k||^2$$
$$||[C_k(B)]^{1-p}v_1\wedge\ldots\wedge v_k||^2$$
$$=||C_k(A^p)u_1\wedge\ldots\wedge u_k||^2$$
$$||C_k(B^{1-p})v_1\wedge\ldots\wedge v_k||^2$$
$$=||A^pu_1\wedge\ldots\wedge A^pu_k||^2$$
$$||B^{1-p}v_1\wedge\ldots\wedge B^{1-p}v_k||^2$$
$$=\det\{(A^pu_i, A^pu_j)\}$$
$$\times\det\{(B^{1-p}v_i, B^{1-p}v_j)\}$$
$$\leq\prod_{i=1}^k ||A^pu_i||^2||B^{1-p}v_i||^2.$$

[1] Figures in brackets indicate the literature references at the end of this paper.

This last inequality is an application of the Hadamard determinant inequality to $\{(A^p u_i, A^p u_j)\}$ and $\{(B^{1-p} v_i, B^{1-p} v_j)\}$ and completes the proof.

If $1 \leq i_1 < \ldots < i_k \leq n$ and $1 \leq j_1 < \ldots < j_k \leq n$ are sequences of integers, then $A[i_1, \ldots, i_k | j_1, \ldots, j_k]$ will denote the k-square submatrix of A, $(a_{i_s j_t})$, $s, t = 1, \ldots, k$.

Corollary 1. *If $A^2 - Q^*Q \geq 0$ and $B^2 - QQ^* \geq 0$ and $0 \leq p \leq 1$, then*

$$|\det Q[j_1, \ldots, j_k | i_1, \ldots, i_k]|^2 \leq \det A^{2p}[i_1, \ldots, i_k | i_1, \ldots, i_k] \det B^{2(1-p)}[j_1, \ldots, j_k | j_1, \ldots, j_k].$$

Proof. Let $u_s = e_{i_s}$, $v_s = e_{j_s}$, $s = 1, \ldots, k$ where e_t is the unit vector with 1 in position t, 0 elsewhere.

Let A and B have eigenvalues $\alpha_1 \geq \ldots \geq \alpha_n$ and $\beta_1 \geq \ldots \geq \beta_n$ respectively. A bound for the left hand member of (2) may be given in terms of these eigenvalues as follows.

Corollary 2. *Under the conditions of the theorem*

$$|\det\{(Qu_i, v_j)\}| \leq \prod_1^k \alpha_i^p \beta_i^{1-p} (\det\{(u_i, u_j)\})^{1/2} (\det\{(v_i, v_j)\})^{1/2}. \quad (3)$$

Proof. Note that

$$\det\{(A^{2p}u_i, u_j)\} = \| C_k(A^p) u_1 \wedge \ldots \wedge u_k \|^2 \leq \prod_{i=1}^k \alpha_i^{2p} \| u_1 \wedge \ldots \wedge u_k \|^2 = \prod_{i=1}^k \alpha_i^{2p} \det\{(u_i, u_j)\}.$$

Applying this to (2) we get (3).

References

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