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Computational Problems Concerning the Hilbert Matrix[1]

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The interaction between theoretical mathematics and practical computational expirement is illustrated by a discussion of recent work, at various centers, concerning the Hilbert matrix.

1. Introduction

Recent work, in various centers, concerning the Hilbert matrix shows very well the interaction between (theoretical) mathematics and practical computational experiments. Some aspects of this work will be discussed here in an expository manner. Although we shall be concerned mainly with the finite segments H_n of the (infinite) Hilbert matrix

$$H=\{h_{i,j}\}=\{(i+j+1)^{-1}\}, \qquad i,j=0,1,2,\ldots,$$

many of the results can be generalized, e.g., to matrices of the form

$$H=\{(a_i+b_j)^{-1}\}, \qquad i,j=0,1,2,\ldots;$$

or

$$H(\theta)=\{(i+j+\theta)^{-1}\}, \qquad i,j=0,1,2,\ldots, \qquad 0<\theta<1.$$

2. Theory

The literature of the last half-century contains many theoretical results concerning the Hilbert matrix. Of these only the basic inequality is immediately relevant [8, p. 234]:

2.1. *If* $\sum_0^\infty a_m^2$ *is convergent, then*

$$\sum_0^\infty \sum_0^\infty a_m a_n/(m+n+1)<\pi \sum_0^\infty a_m^2.$$

We note the following consequence of this.

2.2. *The equation* $Hx=\pi x$ *cannot be satisfied by any vector* x *in Hilbert space,* l^2.

Suppose

$$Hx=\pi x$$

is satisfied for a vector $x \in l^2$. Then it is permissible to multiply both sides by x', i.e., take inner products, to get

$$x'Hx=\pi x'x.$$

If $x=(x_0,x_1,\ldots,\ldots)$ so that Σx_m^2 is convergent this gives

$$\Sigma\Sigma x_m x_n/(m+n+1)=\pi\Sigma x_m^2,$$

in contradiction with (2.1).

The question as to whether there is an x, necessarily not in l^2, for which the equation

$$Hx=\pi x$$

is true, was raised by O. Taussky [20]. This question will be discussed in sections 5 and 6 below.

3. Applications

Among the areas in applied mathematics in which the Hilbert matrix or related matrices have turned up are aerodynamics (A. R. Collar [3]), diffraction of electromagnetic waves (W. Magnus and F. Oberhettinger [12]), and statistics (I. R. Savage and E. Lukacs [17]).

4. Computational Problems—Inversion

Theoretically, the problem of inverting the finite matrices H_n is solved because of the existence of an explicit representation of the inverse, a result which is, essentially, due to Cauchy. The matrix is of the form known as a double-alternant, and the inversion has been discussed e.g., by A. C. Aitken [1, p. 134], A. R. Collar [3], and G. Pòlya and G. Szegö [15, pp. 98–99, 299–300]. The explicit form of the inverse, and the actual numerical values for $n=1(1)10$, have been given by I. R. Savage and E. Lukacs [17].

Practically, however, the problem remains interesting. For instance, although the Hilbert matrix itself turns up in idealized situations, it may be expected that slightly different matrices turn up in the practical circumstances which are approximated by these. It is, therefore, desirable to study the mechanical inversion of the Hilbert matrices H_n, especially as they enjoyed a reputation for ill-condition i.e., "numerical instability," long before the advent of high-speed computers and the basic papers of von Neumann and Goldstine [25] and Turing [24], in order to compare the observed inverse with the theoretical one.

The Hilbert matrix and related ones (e.g., Lotkin [10]) have been often used to test matrix inversion

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programs prepared for high-speed computers. All the experiments of which we are aware [2, 16, 23]—and in these several different schemes for inversion were used—completely confirm the bad reputation. The results quoted by Todd [23] are fully representative. For machines with 10 to 12 decimals, with floating point arithmetic to about 8 significant figures, there is rapid deterioration of the quality of the alleged inverses which are produced—and usually complete failure to produce any inverse for H_n with n about 8.

A detailed examination of the condition of H_n has been given by Todd [23]. Given a matrix A and a specific process of inversion, the error in computing A^{-1} can be found, in theory. In practice what is wanted is a cheap estimate of this, in terms of quantities associated with the matrix and e.g., the word-length of the machine. Various condition-numbers have been introduced to measure the condition of the matrix. One of these, the P-condition number is defined as $P(A)=|\lambda|/|\mu|$ where $|\lambda|$ and $|\mu|$ are the greatest and least among the absolute values of the characteristic values, of A. It has been shown [23], that α being a certain positive constant,

$$P(H_n)=O(e^{\alpha n})$$

while for the $n\times n$ matrix

$$C_n=\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

associated with a second-order differential equation, we have

$$P(C_n)=O(n^2).$$

Using these results in the general error-estimate obtained by von Neumann and Goldstine [25], for an elimination process, and comparing these theoretical error-estimates with actual observed estimates, shows that the two are in reasonable agreement (see Todd [22]).

5. Computation Problems—Characteristic Roots and Vectors

As soon as programs for the computation of characteristic roots and vectors were constructed, it was natural to test them on the H_n, since input programs for this had already been constructed. There was no reason to suppose that it would be particularly difficult to handle—since it is considered that the presence of multiple roots, or roots close in absolute value would be the most likely source of trouble. The theory of positive matrices due to Perron and Frobenius (see e.g., Wielandt [26]) guarantees the existence of a single dominant characteristic root with a characteristic vector all components of which are positive, so that the power method should be satisfactory. This is indeed the case and the dominant root, and the corresponding vector have been obtained for values of n up to 200 at various centers [5, 18].

If more characteristic roots and vectors are needed, the Gantmakher and Krein [6] theory of completely positive matrices guarantees that the characteristic roots are all simple and positive and the components of the vectors exhibit a characteristic pattern: the first has all its components positive and each succeeding one has one more change of sign.

The last criteria shows up the weakness of the results obtained. For instance, a reasonably efficient code using floating point with 8 significant figures which gives the dominant root λ_5 of H_5 correct to 8S gives the smallest root negative! Actually,

$$\det H_5 \doteq 3.75\times 10^{-12}$$

and

$$5.58\times 10^{-6}\geq \mu_5 \geq 1.12\times 10^{-6}$$

so the results are not too surprising.

The usual method for handling this case is an approximate diagonalization of the matrix using the Jacobi process, or the variation on this due to W. Givens [7] in which the matrix is reduced to triple diagonal form and the roots located by a Sturmian process. So far few actual results of the use of these methods have been published.

6. More Theory

Following experimental computations, O. Taussky [19] showed that, if λ_n denotes the dominant characteristic root of H_n, then

$$\lambda_n=\pi+O(1/\log n)$$

so that $\lambda_n\rightarrow\pi$. The actual approach to π is very slow, e.g., it is known [5,18] that: $\lambda_{50}\doteq 2.08$, $\lambda_{60}\doteq 2.11$, $\lambda_{75}\doteq 2.14$, $\lambda_{100}\doteq 2.18$, $\lambda_{125}\doteq 2.21$, $\lambda_{200}\doteq 2.27$; the exact order of magnitude of $\pi-\lambda_n$ does not appear to be known.

A constructive attack on the problem, already mentioned in section 2, of the existence of a vector v satisfying

$$Hv=\pi v$$

appeared difficult and an affirmative solution to it was obtained by T. Kato [9], who used an indirect approach. His method, which will now be described, was suggested by observations of the monotonic behavior of the coordinates of the dominant characteristic vectors of H_n, when these are normalized so that the first coordinate is unity.

Suppose the characteristic vector v_n of H_n corresponding to λ_n is

$$v_n=(v_1^n, v_2^n, \ldots, v_n^n), \qquad v_1^n=1.$$

Numerical evidence suggests that both the sequence $\{\lambda_n\}$ and all the sequences $v_i^i, v_i^{i+1}, v_i^{i+2}, \ldots$ ($i=1,2, \ldots$) are monotone. The first fact is known (e.g., Collatz [4]) and the second is a special case of a more general result regarding the dominant vectors a, b of two positive matrices A, B where A is $\alpha\times\alpha$ and B is $\beta\times\beta$. If B dominates A in the sense that $\alpha\leq\beta$ and if $c_{ij}=b_{ij}/a_{ij}$ is a nondecreasing function of i, j (as long as it is significant), then b dominates a in a similar sense, provided further that B has all its second-order minors positive. This is proved using the fact (already noted) that in these circumstances the dominant vectors can be obtained by the "power" method. The special case used is that in which $A=H_n$, $B=H_{n+1}$.

Now assume that λ_n is bounded—that this is the case when we are dealing with H_n, follows from (2.1). Then $\lim \lambda_n=\lambda$, where $\lambda=\pi$ in the Hilbert case. It follows that the sequences $v_i^i, v_i^{i+1}, v_i^{i+2}, \ldots$ are bounded, for comparing the first coordinates in the equation

$$H_n v_\eta=\lambda_n v_n$$

we have

$$h_{0,i}\, v_i^n\leq\sum h_{0,j}\, v_j^n=\lambda_n\, v_1^n=\lambda_n\leq\lambda,$$

which gives, for all n,

$$v_i^n\leq\lambda(h_{0,i})^{-1}.$$

In the Hilbert case this is

$$v_i^n\leq\pi(i+1).$$

If we write $v_i=\lim\limits_n v_i^n$, it can then be shown that the infinite vector $v=(v_1,v_2, \ldots)$ satisfies

$$Hv=\pi v.$$

The result obtained by Kato [9], while answering the original question, itself suggests many more. For instance, is the vector v the only (linearly independent) vector corresponding to $\lambda=\pi$? Are these characteristic values which exceed π?

7. Recent Developments

1. During the last few years the theory described in section 6 has been developed considerably by Tosio Kato and Marvin Rosenblum, and some of the questions raised there have been answered. Among the relevant results are the following:

M. Rosenblum [28] has shown that every complex number with a positive real part is a characteristic root of $H(\theta)$ for θ fixed, $\theta>0$. This was accomplished by using the theory of special functions to obtain a characteristic vector explicitly. In [29] Rosenblum completely determined the spectrum of $H(\theta)$, for any real $\theta\neq 0, -1, -2, \ldots$; this was accomplished using the Titchmarsh-Kodaira theory of singular differential operators.

T. Kato [30] shows that if $M(\theta)$ is the Hilbert bound of $H(\theta)$, i.e. $M(\theta)=\pi$ cosec $\pi\theta$, $0<\theta\leq\frac{1}{2}$; $M(\theta)=\pi$, $\theta\geq\frac{1}{2}$, then every $\lambda\geq M(\theta)$ is a characteristic value of $H(\theta)$, with a positive characteristic vector and there is no characteristic value $\lambda<M(\theta)$ which has a positive characteristic vector.

2. A reference to Hilbert's own work about H_n is [27] where he evaluates det H_n. For some related results see R. B. Smith [32], W. W. Sawyer [33]. The problem of matrix inversion is discussed from the experimental point of view in [31], which includes further results on the Hilbert matrix.

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