**Bound for the P-Condition Number of Matrices With Positive Roots**

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Let a matrix $A$ have positive roots $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Upper and lower bounds for the $P$-condition number of $A$, $P = \lambda_n/\lambda_1$, are given in terms of $\det A$ and one other symmetric function of the roots.

1. **Introduction**

Suppose $A = (a_{ij})$ is a nonsingular $n \times n$ matrix with roots, $\lambda_i (i = 1, \ldots, n)$, ordered so that

$$0 < |\lambda_1| \leq |\lambda_2| \leq \ldots \leq |\lambda_n|.$$  

For a wide class of matrices $A$, the ratio $|\lambda_n|/|\lambda_1|$ gives a rough measure of the probable accuracy of the computation of the inverse of $A$, or the solution of the system of equations for which $A$ is the matrix of coefficients.

This measure was evaluated in some detail by von Neumann and Goldstine [5] and has been called by J. Todd [2, 3, 4] the $P$-condition number of the matrix, i.e.,

$$P = \frac{|\lambda_n|}{|\lambda_1|}.$$  

In general the accuracy of the results is in proportion to the reciprocal of $P$.

In this paper we shall show that, if the roots of $A$ are all positive with

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$  

and the determinant of $A$, $\det A$, is known, together with one other symmetric function of the roots, upper and lower bounds for $P$ can be given in terms of these two known quantities.

More precisely, if the characteristic polynomial for the matrix $A$ is

$$p(x) = x^n - c_1 x^{n-1} + c_2 x^{n-2} - \ldots + (-1)^n c_n, \quad c_i \geq 0 \quad (i = 1, \ldots, n)$$  

and we know $c_k$ and $c_n = (\det A)$, there exist positive upper and lower bounds for the ratio

$$P = \frac{\lambda_n}{\lambda_1},$$  

in terms of $c_k$ and $c_n$.

To simplify the notation in the statement and proof of the theorem we use the following device.

Let

$$c_k = c_k(\lambda_1, \ldots, \lambda_n) = \binom{n}{k} s_k \lambda_1 \ldots \lambda_n = \binom{n}{k} s_k;$$  

and divide each root of $p(x)$ by $(s_k)^{1/k}$. (This is equivalent to dividing each element of the matrix by $(s_k)^{1/k}$.) We will call the new matrix "normalized with respect to $k" or simply the normalized matrix. Since the condition number, $P$, is the ratio of two roots, $P$ will not be affected by such a transformation. Let $D_k$ be the determinant of the normalized matrix. Then

$$D_k = \frac{s_k}{(s_k)^{n/k}} = \frac{c_n}{(s_k)^{n/k}}.$$  

From Hardy-Littlewood-Pólya [1] the numbers $s_k$ as defined in (4) satisfy the inequalities

$$s_1 \geq s_2^{1/2} \geq s_3^{1/3} \geq \ldots \geq s_n^{1/n}$$  

(where equality holds only if all the $\lambda_i$ are equal) hence, by (5), $D_k \leq 1$.

2. **Statement and Proof of Theorem**

**Theorem:** If we have a matrix $A = (a_{ij})$ with characteristic polynomial (2) and if the constant term and the $k$th coefficient are known, the following bounds hold for $P$;

$$\frac{1}{D_k^{(n-k)}} \leq P \leq \frac{1+\sqrt{1-D_k^{k-1}}}{1-\sqrt{1-D_k^{k-1}}}, \quad (k = 1, \ldots, n-1)$$  

where $D$ is defined by (4) and (5).

If $k = 1$ (i.e., the trace of the matrix is given) we can improve the upper bound:

$$P \leq \frac{1+\sqrt{1-D_1}}{1-\sqrt{1-D_1}}; \quad D_1 = \frac{n^2 c_n}{c_1^n}.$$  

**Proof:** Part 1—Upper bound. We prove (8) first, as the method used for this can be applied in the proof of (7).

Since $P = \lambda_n/\lambda_1$, eq (8) is equivalent to

$$D_1 \leq \frac{4\lambda_1 \lambda_n}{(\lambda_1 + \lambda_n)^2}.$$  

1 Figures in brackets indicate the literature references at the end of this paper.
so from (5), we must show that
\[
\frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n} \leq \left( \frac{s_1}{n \sqrt{s_n}} \right)^n.
\] (10)

But the right side of (10) is \((A_n/G_n)^n\) where \(A_n\) is the arithmetic mean, and \(G_n\) the geometric mean, of \(\lambda_1, \ldots, \lambda_n\). Thus, we must show that
\[
is the arithmetic mean, and
\[
of the following:
\]
\[
\text{inequality (c}, \text{Hardy, Littlewood, Pólya [1]), lemma.}
\]
\[
must prove metric means of
\[
\text{where}
\]
\[
\text{if}
\]
\[
\text{But (12) follows from the arithmetic-geometric mean}
\]
\[
\text{using finite induction on}
\]
\[
\text{so it suffices to show that}
\]
\[
\text{if we let}
\]
\[
\text{Thus (11) holds and (10) follows immediately}
\]
\[
\text{To prove (7) we note that, by the inequalities (6),}
\]
\[
\text{so it suffices to show that}
\]
\[
\text{Now let}
\]
\[
\text{and we note that}
\]
\[
\text{But (14) is the same as (10) with the \(\mu\)'s substituted}
\]
\[
\text{Proof of lemma: Simplifying (11) we see we}
\]
\[
\text{LEMMA:}
\]
\[
\text{LEMMA:}
\]
\[
\text{PROOF OF}
\]
\[
\text{PROOF:}
\]
\[
\text{Now let}
\]
\[
\text{where, by the definition (4)}
\]
\[
\text{Hence}
\]
\[
\text{which proves the lower bound in (7).}
\]

3. Some Remarks

In reference [6] T. Kato gives an upper bound for the \(P\)-condition number which in our notation is
\[
P < \frac{4}{D_1}.
\]
This is larger than the bound given in (8).

We derive several consequences from (7), (8). Suppose that we have a matrix with positive roots, which is normalized so that \(e_i = \left( \frac{1}{k} \right)\), i.e., \(s_k = 1\). Then
\[
\frac{1}{(\det A)^{(n-1)/n}} \leq P < \frac{1 + \sqrt{1 - (\det A)^{n-1}}}{1 - \sqrt{1 - (\det A)^{n-1}}}
\]
(15)
This implies that
\[
det A \rightarrow 0 \quad \text{if and only if} \quad P \rightarrow \infty,
\]
\[
det A \rightarrow 1 \quad \text{if and only if} \quad P \rightarrow 1.
\]

Hence, the \(\det A\) behaves essentially as the reciprocal of the \(P\)-condition number and may be said to constitute a reasonable condition number of its own for such matrices. This lends substance to the popular feeling that for a properly defined class of normalized matrices the smallness of its determinant is accompanied by difficulty in inversion.

4. References


(Paper 65B1–42)