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Cylindrical Antenna Theory [1]

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A partial survey of cylindrical antenna theory pertaining to a tubular model with a narrow gap is presented. The survey includes discussion of the theories of Hallén, King and Middleton, Storm, and Zuhrt. A conceptual relation between theory and experiment is described. The latter part of the article is concerned with a new Fourier series solution of the Hallén equation. This solution is developed in such a way that the expansion coefficients are the unknowns of a system of linear equations. The elements of the coefficient matrix are given by a highly convergent series. Numerical results are given for half and full wavelength antennas with half length to radius ratios of 60 and 500π. These results compare quite closely with those obtained from King-Middleton theory.

1. Introduction

Existing solutions to Hallén's integral equation for the current distribution on cylindrical antennas fall into two main categories [1, 2]: [2]

1. Iterative solutions which use an approximation to cylindrical antenna current as a starting point. Successive iterations generate improvements in the original assumption. Approximations are required at some stage in the process if tractable integrals are to be obtained. The approximations are not severe if h/a, the ratio of antenna half length to radius, is large. Impedances obtained from iterative solutions are in good agreement with experiments performed on thin antennas. Although successful, iterative solutions become laborious beyond second or third order, and the approximations become suspect in the case of thick structures.

2. Solutions in which the integral equation is converted into a set of linear simultaneous equations with Fourier coefficients of the current distribution as unknowns. Typical of these are the theories of Storm and Zuhrt. (Strictly speaking, Zuhrt did not solve Hallén's integral equation, but one derivable from a somewhat different point of view.) Storm approximated matrix elements in his set of equations so that his theory is limited to thin structures. In addition, Storm's theory contains two rather fundamental errors which, acting in concert, produce fortuitous results. Zuhrt obtained matrix elements by graphical integration, a sufficiently tedious process to limit his calculations to low order. Neither of these solutions fully exploits the Fourier series technique for obtaining the current distribution.

We have obtained a solution similar to Storm's in which computation of matrix elements can be easily done with high accuracy even for h/a ratios as low as 8 or 10. Matrix inversion is easily accomplished with modern digital computers so that solutions of high order are feasible. We have carried out calculations to 25th order for half and full wavelength antennas with h/a ratios of 60 and 500π. Results from the King-Middleton iterative solution compare favorably to ours so that their work for $h/a \geq 60$ has been checked by comparison with an exact theory.

Although this paper is principally concerned with developing a Fourier series solution, a reasonably complete treatment of iterative solutions is included in an attempt to provide a self-contained account of cylindrical antenna theory. Even with this aim in mind, the treatment of iterative solutions, as well as the theories of Storm and Zuhrt, is sufficiently involved that the reader is referred to the original work for many of the details.

2. Statement of the Problem

Vector potential as a function of position, $\overline{A}(\overline{r})$, is given in terms of a current distribution, $\overline{J}(\overline{r})$, by the expression

$$\overline{A}(\overline{r})=\frac{\mu}{4\pi}\iiint \overline{J}(\overline{r}_0)\,\frac{e^{-jk|\overline{r}-\overline{r}_0|}}{|\overline{r}-\overline{r}_0|}\,dV_0. \tag{2.1}$$

The vector integration in (2.1) must be taken over all sources of the vector potential field and it is this requirement that causes a direct solution of a realistic antenna problem to be exceedingly difficult. In the case of a cylindrical antenna the integration would have to include currents in the antenna, the feeding transmission line, and the oscillator which supplies power to the antenna-transmission line assembly. Every possible combination of antenna, transmission line and driving generator would have to be treated as a special case and a mathematical solution of any given case would by itself be formidable. An idealized problem can be extracted from this situation by consideration of an extremely thin walled tube of infinite conductivity with a narrow circumferential gap corresponding to the antenna terminal zone.

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[2] Figures in brackets indicate the literature references at the end of this paper.

Choosing the conventional cylindrical coordinates, (ρ, ϕ, z), the antenna is defined by $\rho=a$ and $|z| \leq h$.

A purely hypothetical generator is assumed such that the electric field in the gap is azimuthally symmetric. The voltage across the gap is defined by

$$V=-\int_{\text{gap}} E_s dz, \tag{2.2}$$

where E_s is the z-component of electric field at $\rho=a$. E_s is zero outside the gap since the simplifying assumption of perfectly conducting tube walls has been made. If the gap width is decreased as V is held constant, we must express E_s by

$$E_s=-V\delta(z) \text{ for } |z| \leq h, \tag{2.3}$$

where $\delta(z)$ is the usual delta function. Center-fed models will be considered here although this restriction can be removed.

The value of electric field anywhere in space is given by

$$E=-\text{grad } \Phi - j\omega \overline{A}, \tag{2.4}$$

where Φ is the scalar potential. (Time dependence proportional to $e^{j\omega t}$ is assumed in all of the relationships used here.) Scalar potential can be found from the Lorentz condition

$$\Phi=\frac{j}{\omega\mu\epsilon} \text{ div } \overline{A}. \tag{2.5}$$

The combination of a tubular model without end caps and the assumption of a symmetric field leads to the conclusion that only the z-component of $\overline{A}$ is different from zero since all of the current sources on the tubular surface will be in the z-direction. Equations (2.4) and (2.5) can be combined to give

$$E_z=-\frac{j}{\omega\mu\epsilon}(\partial^2 A/\partial z^2+k^2 A), \tag{2.6}$$

where the symbol A without the vector bar simply stands for A_z, and $k^2=\omega^2\mu\epsilon$. In general A is a function of both z and ρ. If the operator $(\partial^2/\partial z^2+k^2)$ is applied to $A(z, \rho)$ and then ρ is taken at the antenna surface, we obtain

$$\partial^2 A_s/\partial z^2+k^2 A_s=\frac{\omega\mu\epsilon}{j} V\delta(z), \tag{2.7}$$

an equation for the surface value of vector potential valid for $|z| \leq h$. Solutions of the homogeneous equation, $\partial^2 A_s/\partial z^2+k^2 A_s=0$, are simply $\cos kz$, $\sin kz$, e^{jkz} or e^{-jkz}. Linear combinations of these solutions may be used to build a solution to (2.7). Thus,

$$A_s=C_1 \cos kz+D_1 \sin k|z|, \tag{2.8}$$

or

$$A_s=C_2 \cos kz+D_2 e^{-jk|z|}. \tag{2.9}$$

These solutions are completely equivalent. However, we prefer (2.9) as a basis for studying certain properties of the infinite cylinder. D_1 and D_2 are evaluated by substituting these solutions into (2.7) and performing the indicated operations with due account being taken of the discontinuous derivatives of $\sin k|z|$ and $e^{-jk|z|}$. The results are $D_1=(\omega\mu\epsilon V)/2jk$ and $D_2=(\omega\mu\epsilon V)/2k$. Then

$$A_s=C_1 \cos kz+\frac{\omega\mu\epsilon}{jk}\frac{V}{2} \sin k|z|, \tag{2.10}$$

$$A_s=C_2 \cos kz+\frac{\omega\mu\epsilon}{k}\frac{V}{2} e^{-jk|z|}. \tag{2.11}$$

Since $e^{-jk|z|}$ can be written as $\cos kz - j \sin k|z|$, it is easily shown that $C_1=C_2+\omega\mu\epsilon V/2k$. Either expression may be used, the choice between them being only a matter of taste.

Under the assumptions being made, the current distribution of the general formula (2.1) degenerates to a surface distribution, $K(z_0)$. It must be borne in mind that the tube has both inner and outer surfaces, and $K(z_0)$ is the sum of current densities on both surfaces. The field point, $\overline{r}$, can be taken at the surface of the cylinder $\rho=a$ so that an alternate formula for A_s is

$$A_s=\frac{\mu}{4\pi}\int_{-h}^{+h}\int_{-\pi}^{\pi} K(z_0)\frac{e^{-jk|\overline{r}_s-\overline{r}_0|}}{|\overline{r}_s-\overline{r}_0|} a dz_0 d\phi_0, \tag{2.12}$$

where $\overline{r}_0$ ranges over the antenna surface during the course of the integration.

Several changes in notation are convenient at this point. The field point is (a, z, ϕ) and the source point has the coordinates (a, z_0, ϕ_0). Only the difference $\phi-\phi_0$ is of any significance because of the azimuthal symmetry. Therefore, we may set ϕ to zero, change ϕ_0 to ϕ and indicate the angular integration as already performed. The quantity z_0 is changed to ζ so that the subscripts may be avoided in future formulas. Total current is given by $I(\zeta)=2\pi a K(\zeta)$. The quantity $|\overline{r}_s-\overline{r}_0|$ is replaced simply by R. With these changes

$$A_s=\frac{\mu}{4\pi}\int_{-h}^{+h} I(\zeta) g(z-\zeta) d\zeta, \tag{2.13}$$

where

$$g(z-\zeta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-jkR}}{R} d\phi \tag{2.14}$$

and

$$R=[4a^2 \sin^2 \phi/2+(z-\zeta)^2]^{1/2}. \tag{2.15}$$

We now have two formulas for the surface value of vector potential. One has been derived from the generic formula for vector potential in terms of current sources by specializing to the geometry of the problem under consideration. As such it is a general formula for vector potential generated by a ϕ-symmetric current in the z-direction on a tubular conductor no matter what other conditions are to be imposed on the problem. On the other hand, (2.10) provides a vector potential which leads to the boundary values of electric field desired in the present problem. If these expressions are equated, an integral equation for the current distribution associated with the chosen boundary conditions results:

$$\frac{\mu}{4\pi}\int_{-h}^{+h} I(\zeta)g(z-\zeta)d\zeta = C_1 \cos kz + \frac{\omega\mu\epsilon}{jk}\frac{V}{2}\sin k|z|. \tag{2.16}$$

It is convenient to multiply the previous equation through by $jk/\omega\mu\epsilon$ and make the following definitions:

$$C=\frac{jkC_1}{\omega\mu\epsilon},\ \frac{jk}{\omega\mu\epsilon}\frac{\mu}{4\pi}=\frac{j}{4\pi}(\mu/\epsilon)^{1/2}=\frac{jZ_0}{4\pi}$$

and

$$\frac{jZ_0}{4\pi}I(\zeta)=f(\zeta).$$

No generality is lost by letting $V=1$. This completes the mathematical formulation of the problem. We are to consider the solution of

$$\int_{-h}^{+h} f(\zeta)g(z-\zeta)d\zeta = C\cos kz + \frac{1}{2}\sin k|z|,\ |z|\leq h. \tag{2.17}$$

The constant C must be determined by the boundary condition $f(\pm h)=0$.

There is no doubt that the integral equation corresponds exactly to the chosen model. It has already been pointed out that the model does not correspond to any physically realizable antenna. Physical antennas may be either solid or tubular conductors, and they may or may not be fed in such a way as to preserve ϕ-symmetry. Lack of symmetry (as exemplified by a linear antenna fed by a two wire line) can be rationalized to some extent if ka is small. However, the most serious point is the highly idealized nature of the generator region of the mathematical model. The infinitesimal gap is really a short circuit across which a hypothetical but finite voltage has been impressed. Thus, the input current and admittance of the model are certainly infinite.

It is not immediately clear that an infinite admittance model can be managed mathematically in such a way as to yield a physically significant finite result. Wu and King have discussed this point in a recent paper [3]. They have shown that the singularity in $I(z)$ near $z=0$ is logarithmic and of very short range. Thus, according to Wu and King, "since . . . the singularity actually gives a contribution to the current distribution only in an exceedingly small and physically meaningless distance of the order of magnitude $h\exp(-1/ka)$, it may, in principle simply be subtracted out." According to this line of reasoning iterative solutions of the integral equation are successful because they are started with a continuous approximating function and are carried to such low order that the singularity does not develop.

An important aspect of any theory is its relationship to experiment. It is possible to avoid the inherent singularity in the current distribution which is a solution to Hallén's integral equation and obtain a finite answer for the theoretical input admittance to a cylindrical antenna. The finiteness of the result does not, however, guarantee that it is physically significant. Experimental antennas must be provided with a realistic terminal zone which is connected to a transmission line of some sort. Measured impedances are then complex combinations of antenna and terminal zone effects. One way of extricating these effects is to make a theoretical correction for terminal zone effects on a sequence of experimental impedances and extrapolate the corrected data to the limit of a small terminal zone. The residual impedance is then supposed to be characteristic of the antenna itself and it is this idealized inference from experiment rather than raw data which is to be compared with theory.

An outline of a feasible experimental program will be helpful at this point. Consider the experimental arrangement shown in figure 1. Symmetry allows

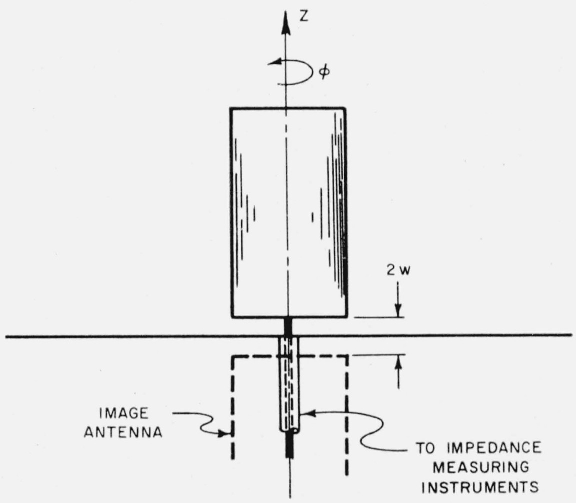

FIGURE 1. *Experimental model of a cylindrical antenna.*

us to place a large ground plane at $z=0$ and simulate half of a center fed antenna. Measuring instruments are located below the ground plane so that the antenna is shielded from extraneous interaction with the apparatus. Appropriate measurements are made so that the experimenter can determine Z_T, the equivalent impedance terminating the coaxial line. It should be possible to relate Z_T to parameters of the radial transmission line and to $Z_a(w)$, the input impedance of a cylindrical antenna with gap width 2 w. If so, one should be able to calculate $Z_a(w)$ given measurements of Z_T. $Z_a(w)$ can be extrapolated to obtain $Z_a(0)$. $Z_a(0)$ obtained in this way is independent of the terminal configuration actually used. We will not attempt to give a mathematical treatment of the experimental situation. It is sufficient for our purposes to establish a conceptual foundation for relating theory to experiment as a prelude to subsequent theoretical development.

Consider now the sense in which a Fourier series solution for $I(z)$ "subtracts the singularity." Since the unknown function in (2.17) is even, it is developed to finite order as

$$f_N(\zeta)=\frac{F_0}{2}+\sum_{n=1}^{n=N} F_n \cos n\pi\zeta/h. \tag{2.18}$$

A procedure for determining the F_n will be exhibited later. At this point it is sufficient to recall that the sum of a finite number of terms of a Fourier series fits the function being described in a least squares sense. Specifically,

$$q^2=\int_{-h}^{+h} |f_N(\zeta)-f(\zeta)|^2 d\zeta \tag{2.19}$$

is a minimum when the coefficients of $f_N(\zeta)$ are Fourier coefficients. Intense, short range variations cannot contribute appreciably to $f_N(\zeta)$ unless N is made so large that even the singular behavior of $f(\zeta)$ begins to develop.

A numerical estimate of the range of the singularity is worthwhile at this point. Consider an antenna for which $kh=\pi/2$ and $h=60a$. Then $ka\sim.025$ and $1/ka\sim40$. Substitution of these numbers into the range estimate of Wu and King leads to the conclusion that a continuous function can fit the current distribution except in a small distance equal to about he^{-40}. Even if the thickness is increased until $h=10a$ the singularity is important only over a range of about $he^{-6.5}\sim.0015h$. Thus it can be seen that $f(0)$ as given by

$$f(0)=\frac{F_0}{2}+\sum_{n=1}^{n=N} F_n \tag{2.20}$$

will be apparently well-behaved in calculations of practically feasible order even though theoretical considerations indicate that the infinite series must diverge.

An extraneous feature of the theoretical model becomes apparent when it is compared to a proposed experimental model. Current on the inner surface of the theoretical model near the feed point is replaced by current associated with a realizable terminal zone in the experimental model. If ka is less than $2.61a$, tube modes are below cutoff and are rapidly damped out [4]. Formulas for removing the inner current near $z=0$ from the total current, which is a solution to the integral equation, will be presented later. As might be expected, the correction makes a small difference in theoretical antenna admittance.

Once the singularity near $z=0$ and the tube current are understood and properly removed from the theory the remaining possibilities for refinement are somewhat limited and consist, for the most part, of removing the assumption of an infinitesimally thin, perfectly conducting tube. Such considerations would be supererogatory in view of the many successes of infinite conductivity models in electromagnetic theory.

3. Solution for the Infinite Cylinder

In subsequent work the current distribution on a finite antenna is expanded in a Fourier series. Such an expansion cannot be valid for a function containing a singularity unless the singularity is integrable. One expects that $I(z)$ is singular in the neighborhood of the delta generator and that the nature of the singularity is independent of antenna length. It can be imagined that the current on a finite antenna is composed of waves emanating from the feedpoint and waves reflected from the ends of the antenna. Only the outgoing waves are expected to be singular at $z=0$ since they are directly associated with the delta generator. It is instructive to consider the case of the infinite cylinder before proceeding with the solution for a finite antenna. This case can be solved exactly in integral form and the Fourier transform of the current identified from the solution. Asymptotic behavior of the transform gives a clue to the nature of $I(z)$ near $z=0$ even for the finite antenna.

Consider now the integral equation formed by equating (2.11) and (2.13),

$$\frac{\mu}{4\pi}\int_{-h}^{h} I(\zeta)g(z-\zeta)d\zeta=C_2\cos kz+\frac{\omega\mu\epsilon}{k}\frac{V}{2}e^{-jk|z|}. \tag{3.1}$$

If h becomes infinite there is no mechanism for the formation of standing waves on the cylinder. In that event C_2 may be set to zero. The integral equation becomes

$$\frac{Z_0}{2\pi}\int_{-\infty}^{+\infty} I(\zeta)g(z-\zeta)d\zeta=Ve^{-jk|z|};|z|<\infty. \tag{3.2}$$

Solution of this equation is easy if we are armed with the identities

$$g(z-\zeta)=-\frac{j}{2}\int_{-\infty}^{+\infty} J_0(\beta a)H_0^{(2)}(\beta a)e^{j\alpha(z-\zeta)}d\alpha, \tag{3.3}$$

and

$$e^{-jk|z|}=\frac{j}{2\pi}\int_{-\infty}^{+\infty}\frac{2ke^{j\alpha z}}{(\alpha+k)(\alpha-k)}d\alpha. \tag{3.4}$$

The parameter β is given by

$$\beta=(k^2-\alpha^2)^{1/2}. \tag{3.5}$$

Each integral is taken in the complex plane of α along the real axis from $-\infty$ to $+\infty$ with a downward identation at $\alpha=-k$ and an upward indentation at $\alpha=+k$.

The Fourier transform of $I(z)$ is

$$\hat{I}(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} I(\zeta)e^{-j\alpha\zeta}d\zeta. \tag{3.6}$$

This definition and the identities for $g(z-\zeta)$ and

$e^{-jk|z|}$ lead to

$$\int_{-\infty}^{+\infty} J_0(\beta a)H_0^{(2)}(\beta a)\hat{I}(\alpha)e^{j\alpha z}d\alpha=\frac{4kV}{\sqrt{2\pi}Z_0}\int_{-\infty}^{+\infty}\frac{e^{-j\alpha z}d\alpha.}{\beta^2} \tag{3.7}$$

If (3.7) is to be true for all z,

$$\hat{I}(\alpha)=\frac{4kV}{\sqrt{2\pi}Z_0}\frac{1}{\beta^2 J_0(\beta a)H_0^{(2)}(\beta a)}. \tag{3.8}$$

Then the solution of (3.2) is

$$I(z)=\frac{2kV}{\pi Z_0}\int_{-\infty}^{+\infty}\frac{e^{j\alpha z}}{\beta^2 J_0(\beta a)H_0^{(2)}(\beta a)}d\alpha\cdot \tag{3.9}$$

Since $e^{j\alpha z}$ is the only factor in the integrand containing an odd part, (3.9) becomes

$$I(z)=\frac{4kV}{\pi Z_0}\int_0^{+\infty}\frac{\cos\alpha z}{\beta^2 J_0(\beta a)H_0^{(2)}(\beta a)}d\alpha\cdot \tag{3.10}$$

We have defined β as follows:

$$\begin{aligned}&\beta^2=k^2-\alpha^2,\\ &\beta=|\beta| \text{ for } \alpha<k \text{ and}\\ &\beta=-j|\beta| \text{ for } \alpha>k.\end{aligned} \tag{3.11}$$

Therefore, for large α

$$\beta=-j\alpha. \tag{3.12}$$

Certain identities involving cylindrical functions are required:

$$\begin{aligned}&J_0(-j\alpha a)=I_0(-\alpha a)=I_0(\alpha a),\\ &H_0^{(2)}(-j\alpha a)=-H_0^{(1)}(j\alpha a)=-\frac{2}{j\pi}K_0(\alpha a),\\ &I_0(\alpha a)\to\frac{e^{\alpha a}}{(2\pi\alpha a)^{1/2}}, \text{ and}\\ &K_0(\alpha a)\to(\pi/2\alpha a)^{1/2}e^{-\alpha a}.\end{aligned} \tag{3.13}$$

For the definition of cosine transforms we take

$$\hat{I}_c(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^{+\infty} I(z)\cos\alpha\cdot z dz. \tag{3.14}$$

Equations (3.10) through (3.14) imply that, for large α

$$\hat{I}_c(\alpha)\to 2j\frac{\sqrt{2\pi}kaV}{Z_0}\frac{1}{\alpha}. \tag{3.15}$$

Inspection of tables of cosine transforms [5] reveals that when a function behaves as $\ln z$ for small z, its cosine transform behaves as $-\sqrt{\pi/2}\ \alpha^{-1}$ for large α. It follows that for small enough z

$$I(z)\simeq -j\frac{4kaV}{Z_0}\ln kz\cdot \tag{3.16}$$

In the above expression k has been selected as a multiplier for $|z|$ to make the argument of the logarithmic function a pure number. The asymptotic behavior of $\hat{I}_c(\alpha)$ is of no help in deciding whether or not k is a proper choice for this parameter. However, k seems attractive since it is given by $k=2\pi/\lambda$, and the wavelength is a natural unit of length in radiation problems.

There is also a somewhat obscure reason for choosing k as a parameter to convert $|z|$ to nondimensional form. Consider another version of (3.15) in which the parameter k is retained as α is allowed to become large. As soon as α becomes greater than k, $\hat{I}_c(\alpha)$ becomes a pure imaginary, $J_0(\beta a)$ and $H_0^{(2)}(\beta a)$ go over to $I_0(|\beta|a)$ and $(-2/j\pi)\ K_0(|\beta|a)$. Then using asymptotic forms,

$$\hat{I}(\alpha>k)\to j\frac{2\sqrt{2\pi}kaV}{Z_0}\frac{1}{(\alpha^2-k^2)^{1/2}}. \tag{3.17}$$

$\hat{I}_c(\alpha)$ can now be regarded as having been separated into two parts: $\hat{I}_c(\alpha<k)$ which is zero if $\alpha>k$ and $\hat{I}_c(\alpha>k)$ which is zero if $\alpha<k$. Only $\hat{I}_c(\alpha>k)$ is of interest at present since it is responsible for the singularity in $I(z)$. Now $\hat{I}_c(\alpha>k)$ is asymptotically proportional to the transform of the Neumann function, $Y_0(kz)$. The latter contains a singular part which is proportional to $\ln k\,z$.

If the above arguments in favor of choosing (3.16) for the form of the singularity are acceptable, we can proceed with an estimate of the range over which this expression is a good approximation to antenna current. For this purpose we require an estimate of $I(z)$ outside the range in which the logarithmic function is dominant. It is difficult to obtain such an estimate from (3.10). However, an alternative argument can be constructed. It is known that an expression of the form $I_m \sin k(h-|z|)$ is a fair approximation to current on a finite antenna over most of its length. It is convenient, rather than necessary, to use knowledge of the finite length antenna in estimating the range of the singularity. We now arbitrarily establish the criterion that the logarithmic function is to be used in the range $|z|<\eta$ where η is given by

$$-\frac{4kaV}{Z_0}\ln k\eta\sim|I_m|. \tag{3.18}$$

$|I_m|$ is of the order of 0.01 when V=1. Z_0 is of the order of 400. An order of magnitude estimate for η is

$$\eta\sim\frac{\lambda}{2\pi}e^{-1/ka}. \tag{3.19}$$

This is a small fraction of a wavelength even for quite thick antennas. Moderately large changes in

$|I_m|$ do not affect this conclusion appreciably.

Further examination of this question would require analytic or numerical inversion of (3.10). Either approach is apparently formidable.

4. Iterative Solutions for Finite Antennas

A brief discussion of iterative solutions is provided here as background for the general reader. We are primarily interested in presenting a critique of the approach rather than compiling a report on the voluminous literature of the cylindrical antenna problem. Consequently, the presentation omits many points which are essential to a detailed understanding of the iterative method. It does present a few features which require comment by way of justifying additional consideration of finite antenna theory.

The first task is to cast eq (2.17) into a form suitable for iteration. To that end the quantity

$$f(z)\psi(z)=\int_{-h}^{h} f(z)w(z,\zeta)d\zeta \tag{4.1}$$

is added and subtracted to the left hand side of (2.17). It is convenient to abbreviate by letting

$$C\cos kz+\frac{1}{2}\sin k|z|=P(z). \tag{4.2}$$

With these changes and some elementary transpositions, (2.17) can be written

$$f(z)=\frac{1}{\psi(z)}\left\{P(z)-\int_{-h}^{h}[f\zeta)g(z-\zeta)-f(z)w(z,\zeta)]d\zeta\right\}. \tag{4.3}$$

Assuming that $w(z,\zeta)$ has been selected, a program for obtaining a solution is:

1. Substitute an approximation to $f(z)$ under the integral on the right hand side of (4.3). Denote this zeroth order approximation by $f_0(z)$. Carry out the indicated integrations to obtain $f_1(z)$ and adjust C so that $f_1(+h)=0$.
2. Repeat, using $f_1(z)$ to generate $f_2(z)$. Readjust C so that $f_2(+h)=0$.

In principle this process may be continued indefinitely with the formula for the Nth approximation being

$$f_N(z)=\frac{1}{\psi(z)}\left\{P(z)-\int_{-h}^{h}[f_{N-1}(\zeta)g(z-\zeta)-f_{N-1}(z)w(z,\zeta)]d\zeta\right\} \tag{4.4}$$

subject to $f_N(+h)=0$, which defines C_N, the Nth approximation to C.

Equation (4.4) is formally true for any $w(z,\zeta)$. However, it will clearly be to the advantage of the investigator to make a choice which results in rapid convergence of the iterative process. This matter is the raison d'etre of much of the literature of linear antenna theory. A choice which leads to manageable integrals is

$$w_H(z,\zeta)=\frac{1}{[a^2+(z-\zeta)^2]^{1/2}}. \tag{4.5}$$

Then

$$\psi_H(z)=\int_{-h}^{+h}\frac{d\zeta}{[a^2+(z-\zeta)^2]^{1/2}}. \tag{4.6}$$

The kernel $g(z\text{-}\zeta)$ as given by (2.14) and (2.15) is difficult to handle. A manageable but crude approximation is

$$g(z-\zeta)=\frac{e^{-jk|z-\zeta|}}{|z-\zeta|}. \tag{4.7}$$

If $g(z\text{-}\zeta)$ is approximated to this order it is appropriate to approximate $w_H(z\text{-}\zeta)$ to the same order when it is used in the integrals on the right hand side of (4.4), but not in the calculation of $\psi_H(z)$. Then

$$f_N(z)=\frac{1}{\psi_H(z)}\left\{P(z)-\int_{-h}^{h}\frac{f_{N-1}(\zeta)e^{-jk|z-\zeta|}-f_{N-1}(z)}{|z-\zeta|}d\zeta\right\} \tag{4.8}$$

becomes the fundamental equation of the iterative program. Equation (4.8) has been used by Hallén in the investigation of linear antenna theory [6].

Equation (4.8) can be criticized on two counts. First, the approximation involved in replacing R by simpy $|z\text{-}\zeta|$ is quite severe. Secondly, $w(z,\zeta)$ was chosen for its simplicity rather than according to the requirement that the iterative program produce good results in low order. King and Middleton improved the iterative procedure outlined above in that they used a kernel distance

$$R_1=\left[a^2+(z-\zeta)^2\right]^{1/2}. \tag{4.9}$$

This is a better approximation to R than is the quantity $|z\text{-}\zeta|$.

The second modification introduced by King and Middleton comes from considering the combination of integrals on the right hand side of (4.3). Denote these by

$$Q(z)=\int_{-h}^{h}[f(\zeta)g(z-\zeta)-f(z)w(z,\zeta)]d\zeta. \tag{4.10}$$

$Q(z)$ can be rewritten as

$$Q(z)=\int_{-h}^{h}[f(\zeta)-f(z)W(z,\zeta)]g(z-\zeta)d\zeta. \tag{4.11}$$

A $W(z,\zeta)$ which makes the integrand of (4.11) vanish is

$$W(z,\zeta)=f(\zeta)/f(z). \tag{4.12}$$

Of course, one cannot know the desired $W(z, \zeta)$ because $f(z)$ is not yet known. One can, however, approximate $W(z, \zeta)$ by making use of a fair low order approximation to $f(z)$. A suitable choice of $f(z)$ in this case is the sinusoidal approximation to antenna current.[3]

Instead of $W(z, \zeta)$ King and Middleton introduce

$$W_K(z, \zeta)=\frac{\sin k(h-|\zeta|)}{\sin k(h-|z|)}, \tag{4.13}$$

with the attendant

$$\psi_K(z)=\int_{-h}^{h} W_K(z, \zeta)\frac{e^{-jkR_1}}{R_1}d\zeta. \tag{4.14}$$

Using these definitions the interative program is based on

$$f_N(z)=\frac{1}{\psi_K(z)}\left\{P(z)-\int_{-h}^{h}[f_{N-1}(\zeta)-f_{N-1}(z)W_K(z, \zeta)]\frac{e^{-jkR_1}}{R_1}d\zeta\right\}. \tag{4.15}$$

Equations (4.8) and (4.15) are not quite the forms used by Hallén and King-Middleton in their computational programs. To appreciate the need for some improvement consider the boundary condition $f(+h)=0$ applied to (4.4) from which (4.8) and (4.15) were developed. The result of applying the boundary condition is

$$0=P(h)-\int_{-h}^{h} f_{N-1}(\zeta)g(h-\zeta)d\zeta. \tag{4.16}$$

(The term involving $f_{N-1}(z)$ drops out because the boundary condition is applied at each iteration.) Now, $P(h)$ contains $C \cos kh$ and if $kh=\pi/2$, the constant C disappears completely and (4.16) cannot be satisfied. A revision of the theory to overcome this defect can be made by subtracting (4.16) from (4.4) to obtain

$$f_N(z)=\frac{1}{\psi(z)}\left\{P(z)-P(h)+\int_{-h}^{h} f_{N-1}(\zeta)g(h-\zeta)d\zeta - \int_{-h}^{h}[f_{N-1}(\zeta)g(z-\zeta)-f_{N-1}(z)w(z, \zeta)]d\zeta\right\}. \tag{4.17}$$

The condition $f_N(+h)=0$ is always automatically satisfied by (4.17). Equation (4.16) is forced to hold for all N so that C is determined by

$$0=P(h)-\int_{-h}^{h} f_N(\zeta)g(h-\zeta)d\zeta. \tag{4.18}$$

A zeroth order approximation which satisfies boundary conditions can be obtained directly from (4.17) by omitting the integrals. Then

$$f_0(z)=\frac{P(z)-P(h)}{\psi(z)}. \tag{4.19}$$

Additional notation and definitions may be invented so that the result of the iterative process can be cast into series form. The series can be designed to lead off with $f_0(z)$ as given by (4.19). A sufficient basis for the remaining part of our discussion has been displayed at this point.

A noteworthy criticism of the use of R_1 has been present by Gans [7]. Gans correctly points out that

$$\int_{-h}^{h} f(\zeta)\frac{e^{-jkR_1}}{R_1}d\zeta=C\cos kz+\frac{1}{2}\sin k|z| \tag{4.20}$$

is not a true equation because the right hand side has discontinuous derivatives in z whereas the derivatives of the left hand side are continuous.

Hallén claims immunity from Gans' criticism on the grounds that he uses the distance $|z-\zeta|$. It seems to us that this practice raises another difficulty. A development similar to that used in examining the singularity in $f(\zeta)$ shows that $g(z-\zeta)$ is logarithmic near $z-\zeta=0$. The approximate kernel $|z-\zeta|^{-1}\exp(-jk|z-\zeta|)$ has an entirely different kind of singularity. It is extremely doubtful if the Hallén theory develops a solution to the original integral equation.

These considerations lead to a definite statement that

$$f(z)\neq\lim_{N\to\infty} f_N(z) \tag{4.21}$$

where $f_N(z)$ is taken from either the Hallén or King Middleton form of the theory and $f(z)$ is the correct solution of (4.3). Even though one is compelled to this conclusion, it is completely irrelevant because it is entirely possible for

$$f(z)\simeq f_N(z) \tag{4.22}$$

in low order. If any confusion exists about this matter it is because many writers (including some authors of senior and graduate level texts) begin their discussion directly with eq (4.20). The only correct procedure is to formulate a problem which is soluble in principle and to introduce judicious approximations as needed during the course of solution.

Iterative solutions have the disadvantage that they become tedious in second and third order even if approximations are made. High-order solutions with approximate kernels are not even desirable as they may have nothing to do with the original problem. When all is considered, one needs to know over what range of antenna parameters such solutions can be used with confidence. Obviously, the kernels used in Hallén and King-Middleton theory approximate $g(z-\zeta)$ over a range comparable to antenna length only if h/a is large. The point at which these theories

[3] A choice of $f(z)$ to serve in constructing $W(z, \zeta)$ and $\psi(z)$ need not influence the choice of $f_0(z)$ which is used to start the interative process.

break down is somewhat arbitrary since it must depend upon an arbitrarily selected amount of tolerable error.

5. Fourier Series Solutions of Storm and Zuhrt

One disadvantage of the iterative solutions discussed in section 1 is that they become extremely tedious beyond second or third order. Storm attempted to invent a theory which could be extended to higher order [8]. Unfortunately, he introduced the kernel distance $R_1=[a^2+(z-\zeta)^2]^{1/2}$ and studied

$$\int_{-h}^{h} f(\zeta)\frac{e^{-jkR_1}}{R_1}d\zeta=C\cos kz+\frac{1}{2}\sin k|z| \qquad (5.1)$$

which has, in fact, no solution at all. "Solutions" to (5.1) are physically meaningful only if they apply to sufficiently thin antennas and if they are restricted to low order. In addition, Storm's theory contains fundamental errors which invalidate his solution no matter what kernel distance is used.

Storm expands the unknown function in the form

$$f(\zeta)=B\sin k(h-|\zeta|)+\sum_{n=0}^{n=N} F_n\cos(2n+1)\pi\zeta/2h. \qquad (5.2)$$

Since $f(\zeta)$ is an even function the expansion also represents $f(\zeta)$ in the range $-h\leq\zeta\leq 0$.

Following Storm, we seek to determine the coefficients B and F_n of the above expansion. The result of substituting (5.2) into (5.1) is

$$M_0(z)B+\sum_{n=0}^{n=N} S_n(z)F_n=C\cos kz+\frac{1}{2}\sin k|z|, \qquad (5.3)$$

where

$$M_0(z)=\int_{-h}^{h}\sin k(h-|\zeta|)\frac{e^{-jkR_1}}{R_1}d\zeta, \qquad (5.4)$$

$$S_n(z)=\int_{-h}^{h}\cos\frac{(2n+1)\pi\zeta}{2h}\frac{e^{-jkR_1}}{R_1}d\zeta. \qquad (5.5)$$

The integrals in (5.4) and (5.5) are somewhat difficult to evaluate unless approximations are made. Storm replaces the kernel with $|z-\zeta|^{-1}\exp(-jk|z-\zeta|)$ outside the range $z-5a<\zeta<z+5a$. Inside this range he replaces the kernel by $[a^2+(z-\zeta)^2]^{-1/2}$. These ranges are ambiguous if z is within $5a$ of the ends of the antenna since ζ must also be restricted to the range $-h\leq\zeta\leq h$. Presumably we are not to consider values of z too close to the ends of the antenna in what follows. Once the approximations are made, the integrations required in (5.4) and (5.5) can be performed in terms of elementary functions. We shall omit the details and return to consideration of (5.3).

Storm explicitly satisfies (5.3) at $N+2$ points and obtains $N+2$ equations in $N+2$ unknowns which

Table 1

kh	Storm: 5 point calculation	King-Middleton
$\pi/2$	$Z_s=81.5+j\,44.5$	$Z_{KM}=81.5+j\,43.4$
π	$Z_s=1162-j\,1354$	$Z_{KM}=1000-j\,1350$

are B, C, and N of the F_n. He has performed calculations with $N=0$, 1, 2, and 3 for both full and half wavelength antennas with $h/a=904$. The agreement with King-Middleton iterative theory is remarkable. A comparison between the impedances from the latter theory with those from Storm's five-point calculation is made in table 1. King-Middleton data used in this comparison are second order except for the resistance in the half wavelength case which is third order.

In spite of the success of Storm's calculation, his theory breaks down in higher order or for smaller h/a ratios. The difficulties are made more evident if we specify the unknown function by the expansion

$$f(\zeta)=B\sum_{n=0}^{n=N}A_n\cos\frac{(2n+1)\pi\zeta}{2h}+\sum_{n=0}^{n=N}F_n\cos\frac{(2n+1)\pi\zeta}{2h}, \qquad (5.6)$$

where the A_n are the first N terms of an expansion of $\sin k(h-|\zeta|)$. As such the A_n are known explicitly. It is clear that (5.6) approaches (5.2) as N becomes large. If (5.6) is used for $f(\zeta)$, (5.4) must be changed to

$$M_0'(z)=B\int_{-h}^{h}\frac{e^{-jkR_1}}{R_1}\left\{\sum_{n=0}^{n=N}A_n\cos\frac{(2n+1)\pi\zeta}{2h}\right\}d\zeta. \qquad (5.7)$$

The definition of $S_n(z)$ does not change. The new equation to be satisfied is

$$M_0'(z)B+\sum_{n=0}^{n=N}S_n(z)F_n=C\cos kz+\frac{1}{2}\sin k|z|. \qquad (5.8)$$

Set aside, for the moment, the question of how C is to be determined and consider the solution in terms of C. A set of linear simultaneous equations for the unknowns B, F_0, F_1, . . . F_N is obtained by satisfying (5.8) at $(N+1)$ values of z. The matrix of coefficients is

$$\left\|\begin{matrix} M_0'(z_1) & S_0(z_1) & S_1(z_1) & \cdots & S_N(z_1)\\ M_0'(z_2) & S_0(z_2) & S_1(z_2) & \cdots & S_N(z_2)\\ \cdot & \cdot & \cdot & & \cdot\\ \cdot & \cdot & \cdot & & \cdot\\ \cdot & \cdot & \cdot & & \cdot\\ M_0'(z_N) & S_0(z_N) & S_1(z_N) & \cdots & S_N(z_N)\end{matrix}\right\|.$$

By comparing (5.5) and (5.7), one sees that the $M_0'(z_m)$ are linear combinations of the $S_n(z_m)$. The determinant of the above matrix is identically zero for any N. If the left hand column is replaced by $M_0(z_m)$ so that all of $B \sin k(h-|\zeta|)$ is used, the determinant of the matrix must approach zero as N becomes large. No high order solution can be obtained unless one of the unknowns is assigned an arbitrary value. A likely candidate for the assignment is B which can be selected from the elementary induced emf theory according to $B=(jZ_0/4\pi)I_m$.

Evidently, Storm did not become aware of these difficulties because he included C among the unknowns. If this is done and the matrix is augmented by appropriate bordering elements, a system of equations with an unique solution for every N results. However, the value of C which is obtained may not be the correct one. Storm's expansion for $f(z)$ is identically zero at the boundaries. If the coefficients, F_n, of that expansion are calculated in terms of a spurious C it may happen that the expansion (for large N) does not approach zero at the boundaries even though it is identically zero at $z=\pm h$. The only way out of this dilemma is to expand the current in functions which are not identically zero at $z=\pm h$ and then use the boundary condition $f(\pm h)=0$ to determine C.

An actual calculation to support these criticisms is worthwhile. In performing the calculation we used the theory of the next section which is equivalent to a corrected version of Storm's theory even though several of the technical details are quite different. Storm was followed to the extent that $f(\zeta)$ was expanded in the set $\{\cos(2n+1)\pi\zeta/2h\}$, and C was treated as an independent unknown. A structure with $h/a=60$ was selected for study. The results are shown in figure 2 which displays the real and

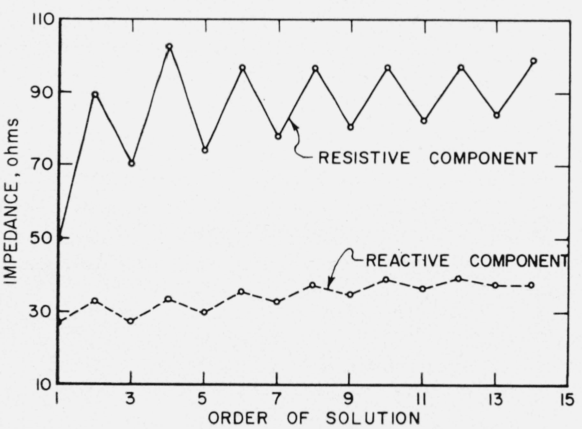

FIGURE 2. *Impedance versus order of solution (Storm's procedure).*

imaginary parts of antenna impedance as a function of the order of solution. Corresponding values of C are shown in figure 3. The result of mistreating the parameter C is easily observed from these curves. We have conjectured that the amplitude of oscillation in the results may decrease with increasing h/a.

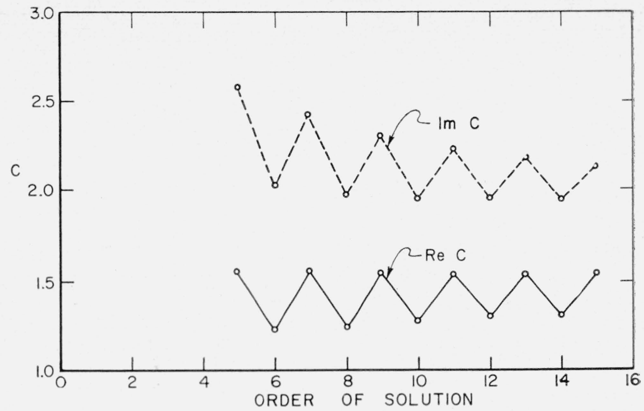

FIGURE 3. *The parameter C versus order of solution (Storm's procedure).*

If this is so it explains why Storm obtained good results with a defective theory.

Zuhrt considered the problem of developing $f(z)$ in a Fourier series from a somewhat different point of view [9]. Instead of considering a single isolated antenna, he formulated a boundary value problem for an infinite collinear array of such antennas spaced along the z-axis with centers at $z=\pm nd$, where $d>h$. Each unit is center-driven by a potential $V_n=(-1)^n V_0$ impressed across a gap of finite width. Ultimately he allows d to approach infinity and the gap width to become small. In this limit only the center unit remains and his theory represents the simple tubular model. This approach is unnecessarily intricate. Zuhrt's final equation can be derived directly from the Hallén integral equation. Equation (3.3) may be written

$$g(z-\zeta)=-j\int_0^\infty J_0(\beta a)H_0^{(2)}(\beta a)\cos\alpha(z-\zeta)d\alpha. \tag{5.9}$$

The term $\cos\alpha(z-\zeta)$ in the integrand can be expanded and the term $\sin\alpha z \sin\alpha\zeta$ omitted since $f(\zeta)$ is even for a center-fed antenna. Therefore, we can consider

$$\int_{-h}^{h} f(\zeta)K(z,\zeta)d\zeta=C\cos kz+\frac{1}{2}\sin k|z|, \tag{5.10}$$

where

$$K(z,\zeta)=-j\int_0^\infty J_0(\beta a)H_0^{(2)}(\beta a)\cos\alpha z\cos\alpha\zeta d\zeta. \tag{5.11}$$

Applying the operator $L_z=\partial^2/\partial z^2+k^2$ to both sides of (5.10), one obtains

$$\int_{-h}^{h} f(\zeta)L_zK(z,\zeta)d\zeta=k\delta(z). \tag{5.12}$$

Now expand $f(\zeta)$ as

$$f(\zeta)=\sum_{n=0}^{n=N} F_n \cos H n\zeta, \tag{5.13}$$

where

$$\frac{(2n+1)\pi}{2h}=H_n. \tag{5.14}$$

Substituting the assumed current expansion into (5.12), one obtains

$$\sum_{n=0}^{n=N}\left\{\int_{-h}^{h} \cos H_n\zeta L_z K(z,\zeta)d\zeta\right\} F_n=k\delta(z). \tag{5.15}$$

Now multiply both sides of (5.15) by $\cos H_p z$, and integrate on z over the range $-h\leq z\leq h$ to obtain

$$\sum_{n=0}^{n=N}\left\{\frac{1}{k}\int_{-h}^{h} \cos H_p z dz \int_{-h}^{h} \cos H_n \zeta L_z K(z,\zeta)d\zeta\right\} F_n=1. \tag{5.16}$$

Equation (5.16) generates an infinite set of linear simultaneous equations as the indexing parameter, p, is allowed to range from zero to infinity. The coefficients are given by

$$Z_{pn}=\frac{1}{k}\int_{-h}^{h} \cos H_p z dz\int_{-h}^{h} \cos H_n \zeta L_z K(z,\zeta)d\zeta. \tag{5.17}$$

Further reduction is accomplished by substituting (5.11) for $K(z, \zeta)$ and carrying out the indicated operations. When this is done,

$$Z_{pn}=-4j(-1)^{p+n}H_pH_n \int_0^{\infty} \frac{J_0(\beta a)H_0^{(2)}(\beta a)(k^2-\alpha^2) \cos^2 \alpha k}{(H_p^2-\alpha^2)(H_n^2-\alpha^2)} d\alpha. \tag{5.18}$$

This is the same as Zuhrt's formula except for trivial differences in notation.

Zuhrt obtains an Nth order theory by truncating the infinite scheme at Nth order. The integral which defines the matrix elements is difficult to evaluate. Zuhrt resolves this difficulty by resorting to graphical integration. Each coefficient has a real and an imaginary part, so that an Nth order theory requires $2N^2$ graphical integrations, a formidable amount of labor even for small N.

6. Further Development of Fourier Series Solutions

The kernel of Hallén's integral equation,

$$g(z-\zeta)=\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-jkR}}{R} d\phi, \tag{6.1}$$

where

$$R=[4a^2 \sin^2 \phi/2+(z-\zeta)^2]^{1/2}, \tag{6.2}$$

represents the radiation at any value of z on a cylinder of radius a from a ring source of radius a located at ζ. As such it is meaningful in the entire domain $-\infty< z<+\infty$. However, the integral equation is valid only on the range $-h\leq z\leq +h$ and operations on the source coordinate, ζ, are restricted to the same range. Hence, a special expansion of $g(z-\zeta)$ for $-h\leq z,\zeta\leq +h$ is desirable. A frontal attack on the problem of obtaining such an expansion has been made by Bohn in an investigation of a theoretical model which is quite different from the one used here [10]. The point of interest at present is his method of handling $g(z-\zeta)$ which is the kernel of an integral equation occurring in his theory.

In our notation Bohn's expansion is

$$g(z-\zeta)=\sum_{n=-\infty}^{n=\infty}\ \sum_{m=-\infty}^{m=\infty} G_{nm}e^{j(n\pi z/h-m\pi\zeta/h)} \tag{6.3}$$

for $-h\leq z,\zeta\leq +h$, where the coefficients of the Fourier series are given by

$$G_{nm}=\frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h} g(z-\zeta)e^{-j(n\pi z/h-m\pi\zeta/h)}dzd\zeta. \tag{6.4}$$

Substituting (5.9) for $g(z-\zeta)$, (6.4) becomes

$$G_{nm}=-2jh^2\int_{-\infty}^{+\infty} \frac{J_0(\beta a)H_0^{(2)}(\beta a) \sin (n\pi-\alpha h) \sin (m\pi-\alpha h)d\alpha.}{(n\pi-\alpha h)(m\pi-\alpha h)} \tag{6.5}$$

This expression exhibits the difficulties involved in a direct attack. The reader will appreciate that the integration is not trivial. Bohn evaluates (6.5) approximately by means of ingenious distortions of the contour of integration. Details will be omitted here, it being sufficient for our purpose to note that no investigator to date has been able to obtain the exact coefficients of a double Fourier series representation of $g(z-\zeta)$.

Fortunately, it is possible to avoid the double Fourier series representation entirely. It is only necessary to make use of the fact that z and ζ enter only as the square of their difference. Thus, $g(z-\zeta)$ is not a general function of (z, ζ). To proceed, let

$$\xi=(z-\zeta). \tag{6.6}$$

If z and ζ are separately in the range $-h\leq z,\zeta\leq h$, then ξ is in the range $-2h\leq\xi\leq 2h$. Since $g(\xi)$ contains only ξ^2, a cosine series in the range 0 to $2h$ will suffice. Thus, we seek an expansion of the form

$$g(\xi)=\frac{D_0}{2}+\sum_{m=1}^{m=\infty} D_m \cos m\pi\xi/2h, \text{ for } 0\leq\xi\leq 2h. \tag{6.7}$$

The fundamental formula for any coefficient is

$$D_m=\frac{1}{2h}\int_0^{2h} g(\xi) \cos m\pi\xi/2hd\xi. \tag{6.8}$$

Direct integration of (6.8) is difficult. An indirect method can be constructed by writing

$$g(\xi)=g_1(\xi)+g_2(\xi), \tag{6.9}$$

where $g_1(\xi)=g(\xi)$ in the range $0\leq\xi\leq 2h$, and is zero outside this range, $g_2(\xi)$ is equal to zero in the range $0\leq\xi\leq 2h$, and is identical to $g(\xi)$ in the range $2h\leq\xi\leq\infty$. Transposition of (6.9) gives

$$g_1(\xi)=g(\xi)-g_2(\xi). \tag{6.10}$$

The symbolic cosine transform of (6.10) is

$$G_1(\alpha)=G(\alpha)-G_2(\alpha). \tag{6.11}$$

The transform of $g_1(\xi)$ and its inverse are given by

$$G_1(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^{2h} g_1(\xi)\cos\alpha\xi d\xi, \tag{6.12}$$

and

$$g_1(\xi)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} G_1(\alpha)\cos\alpha\xi d\alpha. \tag{6.13}$$

Comparison of (6.12) and (6.8) shows that the coefficients of the cosine series expansion for $g_1(\xi)$ are simply proportional to sample values of its cosine transform, $G_1(\alpha)$. Thus,

$$D_m=\frac{1}{h}\sqrt{\frac{\pi}{2}}\left[G\left(\frac{m\pi}{2h}\right)-G_2\left(\frac{m\pi}{2h}\right)\right]. \tag{6.14}$$

$G(\alpha)$ can be immediately identified from (5.9) as

$$G(\alpha)=-j\sqrt{\frac{\pi}{2}}J_0(\beta a)H_0^{(2)}(\beta a). \tag{6.15}$$

Incidentally, an asymptotic expansion of $G(\alpha)$ shows that, for small ξ, $g(\xi)\sim\ln\xi$. Since the singularity in $g(\xi)$ is no worse than logarithmic, the integrals defining the D_m exist.

Determination of $G_2(\alpha)$ is tedious but not difficult. The definition of $G_2(\alpha)$ is

$$G_2(\alpha)=\sqrt{\frac{2}{\pi}}\int_{2h}^{\infty} g(\xi)\cos\alpha\xi d\xi, \tag{6.16}$$

which can be expanded in a highly convergent infinite series. If, for convenience, we set

$$y^2=4a^2\sin^2\phi/2 \tag{6.17}$$

and

$$u^2=y^2/\xi^2, \tag{6.18}$$

(6.1) becomes

$$g(\xi^2>y^2)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-jk\xi\sqrt{1+u^2}}}{\xi}(1+u^2)^{-1/2}d\phi. \tag{6.19}$$

When appropriate expansions of the integrand are made and the ϕ-integration is performed, there results

$$g(\xi^2>y^2)=\frac{e^{-jk\xi}}{\xi}\left[1-\frac{jka^2}{\xi}-\frac{a^2}{\xi^2}-\ldots\right]. \tag{6.20}$$

The above expression is to be used when $\xi\geq 2h$. The parameter k is of the same order as $1/h$. Therefore the terms retained in the square bracket are of order $(a/h)^2$. The first omitted term is of order $(a/h)^4$. We shall now omit all of (6.20) except the first term. It is not difficult to restore the small correction terms later.

With this omission, (6.16) becomes

$$G_2(\alpha)\simeq\sqrt{\frac{2}{\pi}}\int_{2h}^{\infty}\frac{e^{-jk\xi}\cos\alpha\xi}{\xi}d\xi. \tag{6.21}$$

Sine and cosine integral functions are defined by

$$\begin{aligned} Si(x)&=\int_0^x\frac{\sin x}{x}dx=\frac{\pi}{2}-\int_x^{\infty}\frac{\sin x}{x}dx;\\ Ci(x)&=-\int_x^{\infty}\frac{\cos x}{x}dx. \end{aligned} \tag{6.22}$$

With the above, a few trigonometric identities, and a few elementary changes of variable, the integration of (6.21) follows almost by definition, it being only necessary to exercise a little care depending on whether $\alpha<k$ or $\alpha>k$. The case of $\alpha=k$ will require special attention. We have

$$\begin{aligned} G_2(\alpha<k)\simeq-\sqrt{\frac{1}{2\pi}}\{&Ci[(k+)\alpha 2h]+Ci[(k-\alpha)2h]\\ &+j\pi-jSi[(k+\alpha)2h]-jSi[(k-\alpha)2h]\}. \end{aligned} \tag{6.23}$$

$$\begin{aligned} G_2(\alpha>k)\simeq-\sqrt{\frac{1}{2\pi}}\{&Ci[(\alpha+k)2h]+Ci[(\alpha-k)2h]\\ &-jSi[(\alpha+k)2h]+jSi[(\alpha-k)2h]\}. \end{aligned} \tag{6.24}$$

It is convenient to rewrite $G(\alpha)$ from (6.15) separately for the cases $\alpha<k$ and $\alpha>k$. Let $|\beta|=b$. The phase of β has been defined so that $\beta=b$ if $\alpha<k$ and $\beta=-jb$ if $\alpha>k$. Then

$$G(\alpha<k)=-j\sqrt{\frac{\pi}{2}}J_0(ba)H_0^{(2)}(ba), \tag{6.25}$$

and

$$G(\alpha>k)=\sqrt{\frac{2}{\pi}}I_0(ba)K_0(ba). \tag{6.26}$$

Now $G_2(\alpha<k)$, $G_2(\alpha>k)$, $G(\alpha<k)$, $G(\alpha>k)$ are all singular at $\alpha=k$. In the first two functions the singularity comes from the cosine integral function, in the last two $H_0^{(2)}(ba)$ and $K_0(ba)$ become singular. Since these functions are to be sampled at the points

$m\pi/2h$, the singularities apparently give trouble if $h=m\lambda/4$. Actually we are concerned only with the difference $G(\alpha)-G_2(\alpha)$ and this turns out to be finite and independent of whether α approaches k from above or below. The special formula required for the $\alpha=k$ case is found from combining (6.23), (6.25) and making use of small argument formulas for the various functions involved. The latter are tabulated for the reader's convenience.

For small x,

$$J_0(x)\simeq 1, \quad Si(x)\simeq 0,$$

$$H_0^{(2)}(x)\simeq 1+j\frac{2}{\pi}\ln\frac{2}{\gamma x}, \quad Ci\,(x)\simeq \ln\gamma\,x. \qquad (6.27)$$

The logarithmic singularities subtract off in $G(\alpha)-G_2(\alpha)$ and one obtains

$$G_1(\alpha=k)\simeq\frac{1}{\sqrt{2\pi}}\left\{\ln\frac{4h}{\gamma ka^2}+Ci\,(4kh)-jSi(4kh)\right\}. \qquad (6.28)$$

The same result can be obtained by using (6.24), (6.26), and the small argument formulas for $I_0(ba)$ and $K_0(ba)$.

The degree of approximation in the above formulas may be improved by calculating the cosine transforms of the correction terms in (6.20). The next term to be included is

$$T_1(\alpha)=-jka^2\sqrt{\frac{2}{\pi}}\int_{2h}^{\infty}\frac{e^{-jk\xi}}{\xi^2}\cos\alpha\xi d\xi, \qquad (6.29)$$

which reduces to trigonometric functions, sine integral functions and cosine integral functions. All higher order correction terms may be similarly treated.

Thus, an expansion of $g(z-\zeta)$ in the form (6.7) can be achieved and the coefficients can be calculated with any desired degree of accuracy. It will be convenient in what follows to re-define D_0 so that the leading term can be included under the summation sign. If this is done

$$g(z-\zeta)=\sum_{m=0}^{m=\infty}D_m[\cos\,m\pi z/2h\,\cos\,m\pi\zeta/2h$$
$$+\sin\,m\pi z/2h\,\sin\,m\pi\zeta/2h]. \qquad (6.30)$$

The sine terms are not needed in the treatment of a center-fed antenna. We are then led to consider

$$\int_{-h}^{h}f(\zeta)\,K(z,\zeta)\,d\zeta=C\cos kz+\frac{1}{2}\sin k|z|, \qquad (6.31)$$

where

$$K(z,\zeta)=\sum_{m=0}^{m=\infty}D_m\cos\,m\,\pi\,z/2h\,\cos\,m\pi\,\zeta/2h, \qquad (6.32)$$

subject to $f(\pm h)=0$.

We now expand $f(\zeta)$ as

$$f(\zeta)=\sum_{n=0}^{n=\infty}F_n\cos\,n\,\pi\,\zeta/h \qquad (6.33)$$

and substitute into (6.31). After the ζ integration,

$$\sum_{n=0}^{n=\infty}\sum_{m=0}^{m=\infty}F_nD_m\gamma_{nm}$$
$$\cos\,m\pi z/2h=C\cos kz+\frac{1}{2}\sin k|z|, \qquad (6.34)$$

$$\gamma_{nm}=2\int_{o}^{h}\cos\,n\,\pi\,\zeta/h\cos\,m\,\pi\,\zeta/2\,h\,d\zeta. \qquad (6.35)$$

An infinite set of linear simultaneous equations is obtained by multiplying both sides of (6.34) by $\{\cos\,(2p+1)\pi z/2h\}$ and integrating on z from $-h$ to $+h$. The result is

$$\sum_{n=0}^{n=\infty}\sum_{m=0}^{m=\infty}F_nD_m\gamma_{nm}\beta_{pm}=Cr_p+v_p, \qquad (6.36)$$

where

$$\beta_{pm}=2\int_0^h\cos\,[(2p+1)\pi z/2h]\,\cos\,[m\pi z/2h]dz, \qquad (6.37)$$

$$r_p=2\int_0^h\cos kz\,\cos\,[(2p+1)\pi z/2h]dz, \qquad (6.38)$$

and

$$v_p=2\int_0^h\sin kz\,\cos\,[(2p+1)\pi z/2h]dz. \qquad (6.39)$$

Equation (6.36) can be written

$$\sum_{n=0}^{n=\infty}\Gamma_{pn}F_n=Cr_p+v_p, \qquad (6.40)$$

where

$$\Gamma_{pn}=\sum_{m=0}^{m=\infty}D_m\gamma_{nm}\beta_{pm}. \qquad (6.41)$$

If (6.41) were actually an infinite sum, many terms would be required to satisfactorily approximate each Γ_{pn} and the theory would be laborious except in low order. However, (6.41) contains only two nonzero terms! To appreciate this fact consider the set of functions $\{\cos(m\pi z/2h)\}$ which appear in the expansion of $K(z,\ \zeta)$. These functions are complete on $0\leq z\leq 2h$. They appeared in the theory because we expanded a function of $(z-\zeta)$. Clearly since m is either even or odd, this basic set of functions can be divided into two subsets $\{\cos\,m\pi z/h\}$ and $\{\cos(2m+1)\pi z/2h\}$. Both of the subsets are complete and orthogonal on the range where they are actually used. In a sense we can refer to (6.32) as an overcomplete expansion of $K(z,\zeta)$. Therefore, the summation in (6.41) can be broken into two parts, one for m even, the other for m odd, and reduced to

$$\Gamma_{pn}=h[\beta_{p,2n}D_{2n}+\gamma_{n,2p+1}D_{2p+1}]. \qquad (6.42)$$

The set of equations generated by (6.40) as p ranges from zero to infinity can be cast in matrix form as

$$\Gamma F = Cr + v, \tag{6.43}$$

where Γ is a matrix of the Γ_{pn}; F, r, and v are column vectors.

The method of obtaining simultaneous equations from (6.34) used here is formally equivalent to any other method which might be used. Consider multiplying (6.34) by $X_p(z)$ and integrating on z from $-h$ to h, where $X_p(z)$ is arbitrary. Now $X_p(z)$ can be expanded in the set $\{\cos(2p+1)\pi z/2h\}$ so that the set of equations obtained by using $X_p(z)$ is a linear combination of the equations represented by (6.43). An Nth order colocation scheme is equivalent to choosing $X_p(z)$ from a set of N delta functions, $\delta(z-z_p)$ with $p=1, 2, 3, \ldots, N$.

The formal solution of (6.43) in terms of Γ^{-1} is simply

$$F = C\Gamma^{-1}r + \Gamma^{-1}v. \tag{6.44}$$

We have not been able to discover a general form for Γ^{-1}. Consequently, it has been necessary to truncate the system of equations to finite order and invert finite matrices using digital computer methods.[4] This procedure is a cause of some concern in that there are apparently no mathematical theorems which justify the assumption that a sequence of such inverses converges to the inverse of the infinite matrix. However, there is evidence that the procedure being used here produces the correct solution of Hallén's integral equation. First of all, we have examined sequences of finite inverses up to 25th order for $h/a=60$ and $kh=\pi/2$. These are well-behaved and stable. For a rough definition of stability, we shall say that a stable Nth order inverse has been found if the elements of an Nth order matrix formed by truncating the inverse matrix of an $(N+M)$th order solution do not change appreciably as M is increased. M will be referred to as the stability margin. For the problem at hand our work indicates that $M \sim 3$.

The results obtained for finite antennas are thoroughly reasonable when compared to results obtained by other methods. From a pragmatic point of view there seems to be sufficient evidence that the matrix inversion procedure does indeed produce a finite number of terms of the correct solution to the original integral equation.

If the elements of the inverse of an Nth order truncation of Γ are designated as H^N_{np} the numerical solution is

$$F_n(C) = \sum_{p=0}^{p=N} H^N_{np}(Cr_p + v_p). \tag{6.45}$$

[4] Numerical inversion of the required matrices was accomplished on the IBM 650, using a library program furnished by IBM Corporation.

The Nth approximation to $f(z)$ given by the computational program is

$$f_N(z) = \sum_{n=0}^{n=N} F_n(C)\cos(n\pi z/h). \tag{6.46}$$

The leading terms of (6.46) are not a good approximation to antenna current. Consequently, this representation is slowly convergent. To improve convergence a good low order approximation to $f(z)$ was chosen, and expanded in a series

$$x(z) = \sum_{n=0}^{n=N} X_n \cos(n\pi z/h). \tag{6.47}$$

Candidates for the role of $x(z)$ are the classical sinusoidal distribution, the zeroth order approximation from an iterative solution, the King-Middleton modified zeroth order approximation, or a new low order approximation by R. W. P. King referred to by its author as the "quasi-zeroth order approximation" [11]. We have used the latter. Once the choice has been made and X_n have been calculated, (6.46) is modified to read

$$f_N(z) = x(z) + \sum_{n=0}^{n=N} [F_n(C) - X_n]\cos(n\pi z/h). \tag{6.48}$$

When the boundary condition $f(h)=0$ is imposed, one obtains an Nth order approximation to C from

$$\sum_{n=0}^{n=N} [F_n(C) - X_n](-1)^n = 0. \tag{6.49}$$

By letting $N=0, 1, 2, \ldots$, N, (6.49) can be used to generate a sequence,

$$C_0, C_1, C_2, \ldots, C_N \tag{6.50}$$

Now an Nth order solution should be regarded as the first N terms of a solution of infinite order, only the latter solution involves the Fourier series for $f(z)$. The Fourier series is unique and logically it must be in terms of C_∞. The most direct method of determining C_∞ is to plot the above sequence versus $1/n$ and extrapolate. Unfortunately, the sequence of C values obtained from the boundary condition oscillates and the extrapolation is subject to large error. This difficulty can be overcome by applying a Ceasaro transformation to the sequence of C values to form a new sequence which converges to the same limit. The transformed sequence plots a smooth curve against $1/n$, the extrapolated limit of which is taken to be C_∞. A graph of a typical treatment of C is shown in figure 4.

Our final expression for $f(z)$ is

$$f_N(z) = x(z) + \sum_{n=0}^{n=N} [F_n(C_\infty) - X_n]\cos(n\pi z/h). \tag{6.51}$$

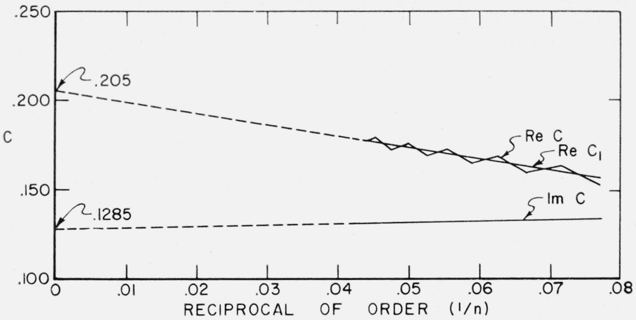

FIGURE 4. *Determination of parameter* C.

This expression does not quite satisfy the boundary condition $f(+h)=0$. However, except for truncation errors which affect only the last two or three of the F_n appreciably, it represents an estimate of the first N terms of a solution of infinite order. Readers who may prefer an expression which satisfies the boundary condition exactly in Nth order should be reminded once again that a finite number of terms of a Fourier series provides a least squares best fit over the entire range of the function being represented. This type of fit is to be preferred over one which is identically equal to the function at one point.

The admittance of the antenna is now simply

$$Y_N=\frac{4\pi}{jZ_0}\left\{x(0)+\sum_{n=0}^{n=N}[F_n(C_\infty)-X_n]\right\}\cdot \tag{6.52}$$

The function $f(\zeta)$ is proportional to the sum of the currents on both the inner and outer surfaces of the tubular conductor. It was pointed out earlier that only the current on the outer surface and not the tube current is to be associated with the experimental admittance of the antenna.

The generic expression for the vector potential at a field point (ρ, ϕ, z) when $\rho_0=a$ is

$$A_z=\frac{\mu}{4\pi}\int_{-h}^{h}I(\zeta)g(\rho,z;a,\zeta)d\zeta, \tag{6.53}$$

where $g(\rho, z; a, \zeta)$ is given by either

$$g(\rho,z;a,\zeta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-jkR}}{R}\,d\phi \tag{6.54}$$

with

$$R=[\rho^2+a^2-2\rho a\cos\phi+(z-\zeta)^2]^{1/2} \tag{6.55}$$

or

$$g(\rho,z;a,\zeta)=-j\int_0^\infty J_0(\beta a)H_0^{(2)}(\beta\rho)\cos\alpha(z-\zeta)d\alpha\cdot \tag{6.56}$$

As a consequence of the ϕ-symmetry, only the ϕ-component of $\overline{H}$ is different from zero, and is given by

$$H_\phi=-\frac{1}{\mu}\,\partial A_z/\partial\rho\cdot \tag{6.57}$$

The surface current density on the outer surface of the tube is obtained from the boundary condition $\overline{n}\times\overline{H}=\overline{K}$, and the current distribution along the outer surface is

$$I^0(z)\,\frac{1}{4\pi}\int_{-h}^{h}I(\zeta)g^0(\rho,z;a,\zeta)d\zeta \tag{6.58}$$

where

$$g^0(\rho,z;a,\zeta)=-2\pi a\left[\frac{\partial}{\partial\rho}\,g(\rho,z;a,\zeta)\right]_{\rho=a+}\cdot \tag{6.59}$$

If $g(\rho,z;a,\zeta)$ is expanded in the Fourier series

$$g^0(\rho,z;a,\zeta)=\sum_{m=0}^{m=\infty}D_m^0\cos\frac{m\pi(z-\zeta)}{2h},\ 0\leq|z-\zeta|\leq h, \tag{6.60}$$

the coefficients D_m^0 may be determined by the same procedure used in expanding the kernel of Hallén's equation. Omitting details, the D_m^0 are given by sample values of

$$G_1^0(\alpha)=-j\beta aJ_0(\beta a)H_1^{(2)}(\beta a)+\left\{\frac{\partial}{\partial\rho}\int_{2h}^{\infty}\int_{-\pi}^{\pi}\frac{e^{-jkR}}{R}\cos\alpha(z-\zeta)d\phi d(z-\zeta)\right\}_{\rho=a+}, \tag{6.61}$$

where R is given by (6.55).

This leads to the following expression for Fourier coefficients of the exterior current:

$$I_m^0=\frac{4\pi}{jZ_0}D_m^0\left\{\int_{-h}^{h}x(\zeta)\cos(m\pi\zeta/2h)d\zeta+\sum_{n=0}^{n=N}[F_n-X_n]\int_{-h}^{h}\cos(n\pi\zeta/h)\cos(m\pi\zeta/2h)d\zeta\right\}\cdot \tag{6.62}$$

7. Results

We have applied the theory to half and full wavelength antennas with $h/a=60$ and 500π, respectively. In each case the calculations were performed at 25th order. Correction terms from (6.30) were included in the calculation so that each matrix element is accurate to one part in 10^5. Graphs of total current and plots of total admittance versus order are shown (figs. 5 to 12). In the latter graphs, an nth order admittance is obtained by using the first n terms of a 25th order solution.

Comparison of these results with those of the King-Middleton theory in terms of impedance are shown in table 2.

King-Middleton impedances in table 2 were obtained by graphical interpolation of tables given in reference [2]. Resistances of the half wavelength structures are from third order solutions. Other quantities from King-Middleton theory are second order. Our own Z_T for the full wavelength antenna with $h/a=60$ is obtained by extrapolating the current graph to $z=0$. The current distribution on a full wavelength structure does not have zero slope at the origin as do the cosine terms used to describe it. Consequently, many cosine terms are required for high accuracy, especially in the treatment of a thick structure. The inner current correction is entirely negligible for structures with $h/a=500\ \pi$; it is still small for $h/a=60$.

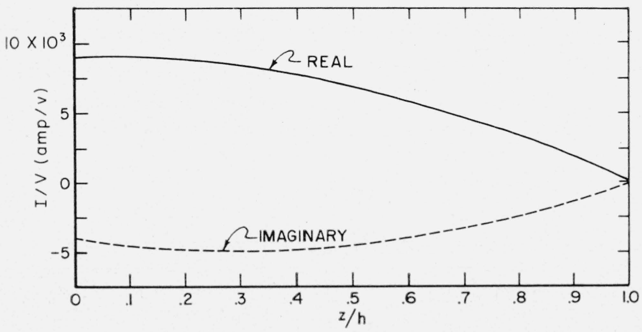

FIGURE 5. *Current distribution:* kh$=\pi/2$, $h/a=60$.

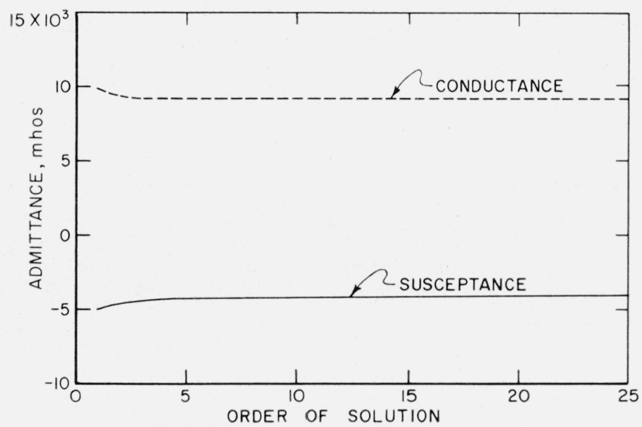

FIGURE 6. *Admittance versus order of solution:* kh$=\pi/2$, h/a$=60$.

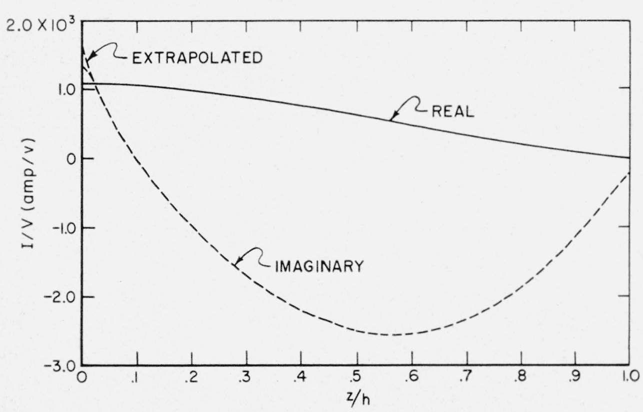

FIGURE 7. *Current distribution:* kh$=\pi$, h/a$=60$.

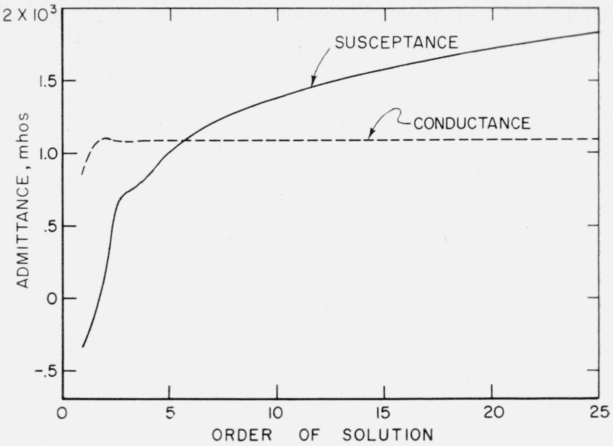

FIGURE 8. *Admittance versus order of solution:* kh$=\pi$, h/a $=60$.

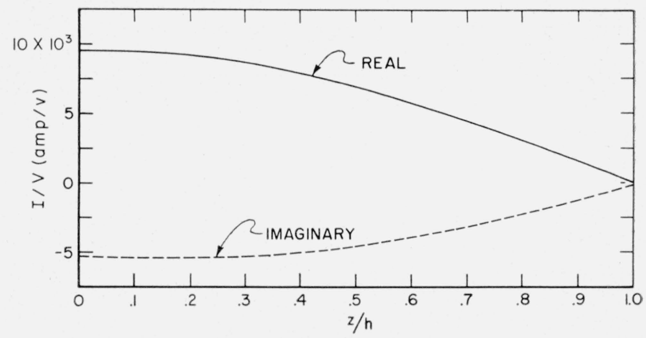

FIGURE 9. *Current distribution:* kh$=\pi/2$, h/a$=500\ \pi$.

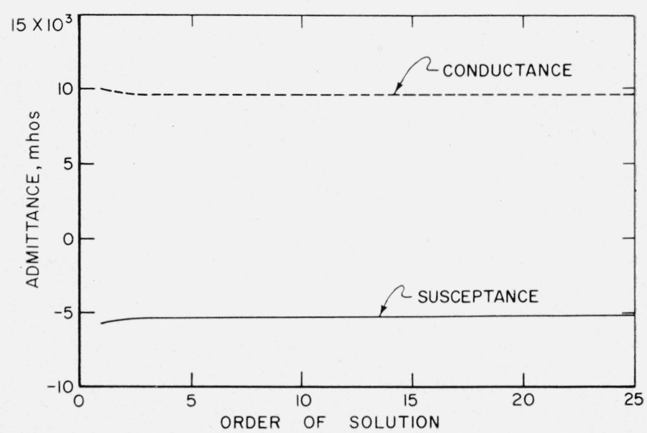

FIGURE 10. *Admittance versus order of solution:* kh$=\pi/2$, h/a$=500\ \pi$.

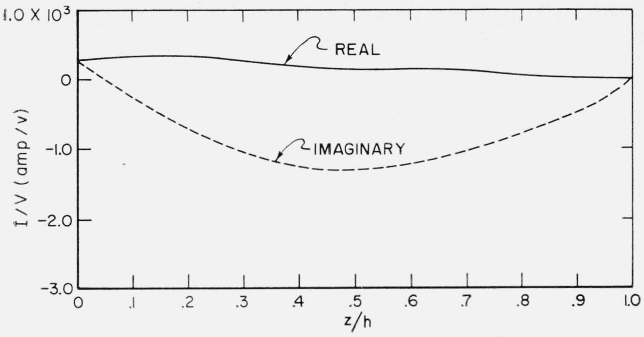

FIGURE 11. *Current distribution:* kh$=\pi$, h/a$=500\ \pi$.

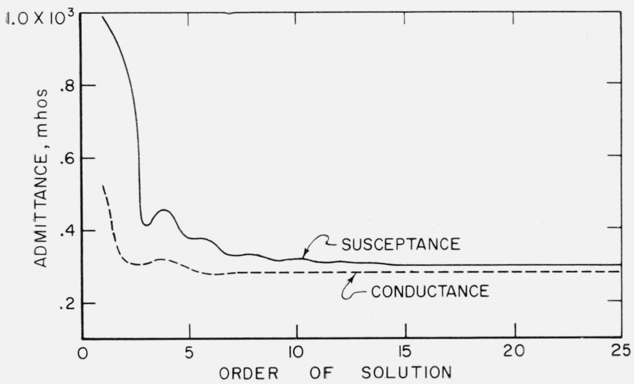

FIGURE 12. *Admittance versus order of solution:* kh$=\pi$, h/a$=500\ \pi$.

It can be seen from table 2 that the amount of disagreement between this theory and the King-Middleton theory is only about two percent. One may expect disagreement of this order or less over the entire range of $h/a>60$.

TABLE 2

kh	h/a	Z_T	Z^0	Z_{KM}
$\pi/2$	60	$91.4+j\ 38.6$	$92.5+j\ 40.6$	$91.4+j\ 41.5$
π	60	$205\ -j\ 382$	$205\ -j\ 380$	$206\ -j\ 380$
$\pi/2$	$500\ \pi$	$79.7-j\ 42.9$	----------	$80.3-j\ 43.4$
π	$500\ \pi$	$1646\ -j\ 1768$	----------	$1625\ -j\ 1744$

It can be seen from the graphs of admittance versus order that, except for thick full wavelength structures, 25 terms are excessive for $h/a>60$. Thus, a great deal of margin for the study of thicker antennas is inherent in 25th order solutions. Studies of full wavelength antennas with $h/a<60$ will require more than 25 terms or further modification of some of the technical details of the theory.

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