JOURNAL OF RESEARCH of the National Bureau of Standards—D. Radio Propagation
Vol. 64D, No. 5, September–October 1960

# On Electromagnetic Radiation in Magneto-Ionic Media [1]

Herwig Kogelnik

(January 12, 1960; revised April 5, 1960)

A method of treating radiation problems in magneto-ionic (anisotropic) media is presented. A "wave matrix" is defined, the zeros of whose determinant are the propagation constants of the ordinary and the extraordinary plane waves. A derivation of the dyadic Green's function for the unbounded medium is given, which is also based on this matrix. A formula is arrived at, which gives the power radiated by any distribution of alternating current in terms of the wave matrix and the spatial Fourier transforms of the currents. The method is illustrated by a discussion of the power radiated by an elementary dipole.

## 1. Introduction

An ionized gas in a permanent magnetic field is an anisotropic dielectric medium. Two well known examples are the ionosphere and the plasma investigated in controlled thermonuclear fusion research. We shall call this medium a "magneto-ionic medium," but other names, like "magneto-plasma," are also used in the literature.

The dielectric properties of a magneto-ionic medium can be described by a dielectric permittivity tensor [1,2,3,4] [2] and the propagation of plane electromagnetic waves in such a medium has been subject to many investigations [3, 5]. The purpose of this paper is to present a general treatment of electromagnetic radiation in magneto-ionic media. (The radiation properties of antennas and moving charged particles are, of course, modified by the anisotropic properties of the surrounding medium.) The method proposed avoids the introduction of vector potentials, Hertz vectors, anisotropic potentials, and the like. The computation of the fields excited by a known distribution of oscillating current will be reduced to elementary matrix operations and the evaluation of integrals. We are particularly interested in the (complex) power radiated by a current distribution and shall illustrate the method by discussing in detail the power radiated by an elementary dipole.

We would also like to direct the reader's attention to other published methods [6, 7, 8, 9, 10, 11, 12], most of which have been applied to special radiation problems, like Cerenkov Radiation, in magneto-ionic media.

## 2. Background

To simplify the analysis, we assume the medium to be homogeneous, of infinite extension, and nonmagnetic ($\mu_{\text{rel}}=1$). All a-c quantities shall be described by their complex amplitude. If the factor exp $(j\omega t)$ is dropped and MKS units are used, Maxwell's equations take the form

$$\nabla\times\mathbf{E}=-j\omega\mu_0\mathbf{H} \tag{1a}$$

$$\nabla\times\mathbf{H}=j\omega\epsilon_0\hat{\epsilon}\,\mathbf{E}+\mathbf{J}, \tag{1b}$$

[1] Contribution from Engineering Laboratory, 19 Parks Road, Oxford, England.

[2] Figures in brackets indicate the literature references at the end of this paper.

where **J** is the external current density—as produced by antennas or moving charged particles—which we assume to be known. The usual matrix formalism is used, and vectors are regarded as column matrices. The permittivity tensor is represented by the matrix $\hat{\epsilon}$, whose components are in general complex numbers, to include conducting (i.e., lossy) media. For most applications a matrix $\hat{\epsilon}$ of the simple form

$$\hat{\epsilon}=\begin{bmatrix} \epsilon_1 & -j\epsilon_2 & 0 \\ j\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \tag{2}$$

can be used (see e.g., [2, 3]). Here the $z$-axis of the coordinate system is orientated in the direction of the applied permanent magnetic field. It can occur, however, that more complicated matrices have to be used, in order to describe the medium's physical behavior correctly [4].

We now rewrite eq (1a)

$$\mathbf{H}=\frac{j}{\omega\mu_0}\nabla\times\mathbf{E} \tag{3}$$

and eliminate the a-c magnetic field **H** from eq (1b) to obtain the wave equation

$$\left(\nabla\nabla-\Delta\hat{1}-\frac{\omega^2}{c^2}\hat{\epsilon}\right)\mathbf{E}=-j\omega\mu_0\mathbf{J}, \tag{4}$$

where $\hat{1}$ is the unit matrix, $\Delta\equiv\nabla^2$ the Laplacian operator, and the dyade $\nabla\nabla$ is standing for a matrix, whose elements are the differential operators $\partial^2/\partial x_i\partial x_k$. We have to solve this wave equation to obtain the amplitudes **E** (**r**) of the a-c electric field in every point of space produced by a given distribution of current. The solutions shall satisfy the condition that at great distances from the sources the fields represent divergent traveling waves.

## 3. Plane Waves

A set of simple solutions of the homogeneous wave equation ($\mathbf{J}=0$) describe plane electromagnetic waves, which we propose to discuss in this chapter. The planes of equal phase are specified by the wave normal

$$\mathbf{n}=(n_1,\ n_2,\ n_3)$$

whose components are the direction cosines

$$n_1=\sin\alpha\cos\beta,$$

$$n_2=\sin\alpha\sin\beta,$$

$$n_3=\cos\alpha.$$

Let $\mathbf{r}=(x,y,z)=(x_1,\ x_2,\ x_3)$ be the radius vector, drawn from the origin to any point in space, then

$$n_1x+n_2y+n_3z=\mathbf{n}\cdot\mathbf{r}=\text{const} \tag{5}$$

is the equation of a plane, and the amplitude of a plane wave with the wave normal **n** will vary as

$$\mathbf{E}(\mathbf{r})=\mathbf{E}_0\,e^{-jk\mathbf{n}\cdot\mathbf{r}}. \tag{6}$$

With $k$ being any complex number, it is convenient to define the vector

$$\mathbf{k}=k\mathbf{n}.$$

We note that

$$\frac{\partial}{\partial x_i} e^{-j\mathbf{k}\cdot\mathbf{r}} = -jk_i e^{-j\mathbf{k}\cdot\mathbf{r}}, \tag{7}$$

and, operating the matrices of the wave equation on the same exponential

$$\left(\nabla\nabla - \Delta\hat{1} - \frac{\omega^2}{c^2}\hat{\epsilon}\right) e^{-j\mathbf{k}\cdot\mathbf{r}} = -\hat{\lambda}\,(\mathbf{k})\, e^{-j\mathbf{k}\cdot\mathbf{r}}, \tag{8}$$

we get the "wave matrix"

$$\hat{\lambda} = \mathbf{kk} - k^2\,\hat{1} + k_0^2\,\hat{\epsilon}., \tag{9}$$

Here is introduced the propagation constant of electromagnetic waves in vacuum

$$k_0 = \frac{\omega}{c}.$$

The electric field of eq (6) has to satisfy the homogeneous wave equation. Using eq (8) and dropping the exponential, this condition can be written

$$\hat{\lambda}\,(\mathbf{k})\,\mathbf{E}_0 = 0. \tag{10}$$

As we look for nonvanishing fields, this condition can only be fulfilled if

$$\det\,\hat{\lambda}\,(\mathbf{k}) = 0. \tag{11}$$

Equation (11) determines the propagation constants of possible plane waves with given wave normal **n**.

If $\hat{\epsilon}$ has the form as in eq (2), an elementary computation shows that

$$\det\,\hat{\lambda}\,(\mathbf{k}) = k_0^2(\epsilon_1 \sin^2\alpha + \epsilon_3 \cos^2\alpha)\,(k^2 - k_I^2)\,(k^2 - k_{II}^2), \tag{12}$$

with

$$k_{I,II}^2/k_0^2 = \frac{(\epsilon_1^2 - \epsilon_2^2)\sin^2\alpha + \epsilon_1\epsilon_3(1 + \cos^2\alpha) \pm \sqrt{(\epsilon_1^2 - \epsilon_2^2 - \epsilon_1\epsilon_3)^2 \sin^4\alpha + 4\epsilon_2^2\epsilon_3^2\cos^2\alpha}}{2(\epsilon_1 \sin^2\alpha + \epsilon_3 \cos^2\alpha)}. \tag{13}$$

These relations show that in a magneto-ionic medium, as is well known, two types of plane waves are possible for a given wave normal. They are called the "ordinary" and the "extraordinary" wave, and their (complex) propagation constants are $k_I$ and $k_{II}$ respectively, whose values depend on the angle $\alpha$ between wave normal and permanent magnetic field. The quantities $k_{I,II}/k_0$ are known as the "refractive indices" of the corresponding waves.

## 4. Dyadic Green's Function

In this chapter we propose to present a solution of the wave eq (4) for any known current-distribution $\mathbf{J}(\mathbf{r})$ (confined to a finite region of space). Because of the linearity of Maxwell's equations there must be a linear relation between the components of a current *element* and the components of the electric field produced by the latter at a point **r**. We can, therefore write for the electric field $\mathbf{E}(\mathbf{r})$ produced by the *entire* distribution $\mathbf{J}(\mathbf{r}')$

$$\mathbf{E}(\mathbf{r}) = \int d\mathbf{r}'\,\hat{G}(\mathbf{r},\mathbf{r}')\,\mathbf{J}(\mathbf{r}'), \tag{14}$$

where $\int d\mathbf{r}'$ stands for $\iiint_{-\infty}^{+\infty} dx'\,dy'\,dz'$. The matrix $\hat{G}$ is called "dyadic Green's function." It has proved a powerful tool in treating other problems of electrodynamics and is well known for an unbounded isotropic medium [13, 14]. In the following we shall derive $\hat{G}$ for an unbounded anistropic medium. An alternative derivation has been described by Bunkin [6].

We use the following identities holding for Dirac's $\delta$-function

$$\int_{-\infty}^{+\infty} f(x')\delta(x-x')dx'=f(x) \tag{15}$$

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-jk_1x}dk_1, \tag{16}$$

and the abbreviation

$$\delta(\mathbf{r})=\delta(x)\delta(y)\delta(z).$$

Because of eq (15) the electric field $\mathbf{E}$ of eq (14) satisfies the wave equation if $\hat{G}$ satisfies

$$(\nabla\nabla-\Delta\hat{1}-k_0^2\hat{\epsilon})\hat{G}(\mathbf{r},\mathbf{r}')=-j\omega\mu_0\hat{1}\delta(\mathbf{r}-\mathbf{r}'). \tag{17}$$

We note that $\nabla\nabla$ and $\Delta$ operate on the variables $\mathbf{r}$ only and not on $\mathbf{r}'$, and we have to assume that the interchange of these two operators with the integration $\int d\mathbf{r}'$ is permitted.

To find a suitable matrix $\hat{G}$ we multiply eq (8) by the inverse of the wave matrix from the right to get

$$(\nabla\nabla-\Delta\hat{1}-k_0^2\hat{\epsilon})\hat{\lambda}^{-1}e^{-j\mathbf{k}\cdot\mathbf{r}}=-\hat{1}e^{-j\mathbf{k}\cdot\mathbf{r}}. \tag{18}$$

We can do this for all real $k_1$, $k_2$, and $k_3$, if we assume the medium to be at least slightly lossy. Then, the zeros of det $\hat{\lambda}$ ($\mathbf{k}$), $k_I^2$, and $k_{II}^2$, will have imaginary parts (i.e., plane waves are attenuated). But as shown later the results are also valid for lossless media.

We finally multiply eq (18) by $e^{j\mathbf{k}\cdot\mathbf{r}'}$ and perform the integrations $\int d\mathbf{k}=\iiint_{-\infty}^{+\infty} dk_1\, dk_2\, dk_3$ to find that the matrix

$$\hat{G}=\frac{j\omega\mu_0}{8\pi^3}\int d\mathbf{k}\hat{\lambda}^{-1}e^{-j\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \tag{19}$$

satisfies eq (17). We have thus found the dyadic Green's function for the unbounded anisotropic medium. To this solution, of course, we can add any solution of the homogeneous wave equation, if required by the behavior of the fields "at infinity." It can be shown, however, that the result given in eq (18) satisfies the required condition that no incoming waves shall occur.

The inverse of $\hat{\lambda}$ can be computed by using Cramer's rule

$$\hat{\lambda}^{-1}=\frac{\hat{\Lambda}}{\det\hat{\lambda}}, \tag{20}$$

where the matrix $\hat{\Lambda}$ is the "adjoint" of $\hat{\lambda}$. For a magneto-ionic medium with a dielectric tensor, as in eq (2), we find

$$\hat{\Lambda}(\mathbf{k})=k^4\mathbf{n}\,\mathbf{n}-k^2k_0^2\hat{L}+k_0^4\hat{\mathrm{E}}, \tag{21}$$

where $\hat{\mathrm{E}}$ is the adjoint of $\hat{\epsilon}$

$$\hat{\mathrm{E}}=\begin{bmatrix} \epsilon_1\epsilon_3, & j\epsilon_2\epsilon_3, & 0 \\ -j\epsilon_2\epsilon_3, & \epsilon_1\epsilon_3, & 0 \\ 0 & 0 & \epsilon_1^2-\epsilon_2^2 \end{bmatrix} \tag{22}$$

and

$$\hat{L}=\begin{bmatrix} \epsilon_1(n_1^2+n_2^2)+\epsilon_3(n_1^2+n_3^2); & j\epsilon_2(n_1^2+n_2^2)+\epsilon_3 n_1 n_2; & \epsilon_1 n_1 n_3+j\epsilon_2 n_2 n_3 \\ -j\epsilon_2(n_1^2+n_2^2)+\epsilon_3 n_1 n_2; & \epsilon_1(n_1^2+n_2^2)+\epsilon_3(n_2^2+n_3^2); & \epsilon_1 n_2 n_3-j\epsilon_2 n_1 n_3 \\ \epsilon_1 n_1 n_3-j\epsilon_2 n_2 n_3; & \epsilon_1 n_2 n_3+j\epsilon_2 n_1 n_3; & \epsilon_1(1+n_3^2) \end{bmatrix}. \tag{23}$$

The equivalence of $\hat{G}$ and Bunkin's Green's function can be shown by using eq (7) to replace each $k_i$ in $\hat{\Lambda}$ by the operation $j\partial/\partial x_i$ and interchanging the latter with the integration $\int d\mathbf{k}$.

In this chapter the problem of finding the fields, produced by a known distribution of current, has been reduced to the problem of evaluating integrals. The methods of integration will have to be adapted to the particular problem. For example, the method of steepest descents has been employed to find the dipole fields at great distances [6].

## 5. Power Radiated by a Distribution of Current

The mean complex power radiated by the current distribution $\mathbf{J}(\mathbf{r})$ is given by

$$P=-\tfrac{1}{2}\int d\,\mathbf{r}\;\mathbf{J}^{+}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r}), \tag{24}$$

where the row matrix $\mathbf{J}^+$ is the Hermite conjugate of $\mathbf{J}$. For some purposes it is convenient to rewrite this expression in terms of the wave matrix and the spatial Fourier transform $\mathbf{J}_\mathbf{k}$ of the current density

$$\mathbf{J}(\mathbf{r})=\int d\mathbf{k}\;\mathbf{J}_{\mathbf{k}}\,e^{-j\mathbf{k}\cdot\mathbf{r}}. \tag{25}$$

Using eqs (14), (19), and (25) we get

$$P=-\frac{j\omega\mu_0}{16\pi^3}\int d\mathbf{k}\;d\mathbf{k}'\;d\mathbf{k}''\;d\mathbf{r}\;d\mathbf{r}'\;\mathbf{J}_{\mathbf{k}'}^{+}\,\hat{\lambda}^{-1}(\mathbf{k})\;\mathbf{J}_{\mathbf{k}''}\,e^{-j[\mathbf{r}\cdot(\mathbf{k}-\mathbf{k}')-\mathbf{r}'(\mathbf{k}-\mathbf{k}'')]} \tag{26}$$

With the help of the relations (15) and (16) these 15 integrals can be reduced to 3, and we obtain the relatively useful formula

$$P=-4j\pi^3\omega\mu_0\int d\mathbf{k}\;\mathbf{J}_{\mathbf{k}}^{+}\,\hat{\lambda}^{-1}\mathbf{J}_{\mathbf{k}}. \tag{27}$$

One can, of course, rewrite eq (24) in this form almost immediately, if Parseval's equation is used.

An application of formula (27) is given in the next section.

## 6. Power Radiated by an Elementary Dipole

To simplify the problem, we assume here that the medium is lossless. Let an elementary electric dipole with moment $\mathbf{p}$ be placed at the origin of our coordinate system. The spatial distribution of current is then

$$\mathbf{J}(\mathbf{r})=j\omega\mathbf{p}\delta(\mathbf{r}). \tag{28}$$

For the calculation of many physical quantities, like the fields at great distances or the real power radiated, many current configurations $\mathbf{J}'(\mathbf{r})$, which are concentrated in an electrically small region, can be considered equivalent to an elementary dipole (see e.g. [15]). The equivalent moment is given by

$$j\omega\mathbf{p}=\int d\mathbf{r}\,\mathbf{J}'(\mathbf{r}). \tag{29}$$

Certain quantities, however, like the fields in the immediate neighborhood or the reactive power, depend very strongly on the dimensions of this region. We are, therefore, mainly interested in the real power, radiated by the dipole. In an anisotropic medium the power is expected to be different for different orientations and polarizations of the dipole.

As we want to apply formula (27), we need the Fourier transform of $\mathbf{J}(\mathbf{r})$, which is

$$\mathbf{J}_{\mathbf{k}}=\frac{j\omega\mathbf{p}}{8\pi^3}. \tag{30}$$

Putting this into the formula, we see that the complex power can be written as a bilinear form

$$P=\frac{\omega^2}{2}\mathbf{p}^{+}\hat{z}\mathbf{p}, \tag{31}$$

where the components of the matrix

$$\hat{z}=-\frac{j\omega\mu_0}{8\pi^3}\int d\mathbf{k}\hat{\lambda}^{-1} \tag{32}$$

are measured in $\Omega m^{-2}$ (impedance units per unit area). This complex matrix $\hat{z}$ can be split into a Hermitian and an anti-Hermitian part

$$\hat{z}=\hat{r}+j\hat{x}, \tag{33}$$

that is to say, the matrices $\hat{r}$ and $\hat{x}$ are both Hermitian ($\hat{r}=\hat{r}^{+}$, $\hat{x}=\hat{x}^{+}$). With the help of these two matrices the bilinear form of eq (31) splits up into two Hermitian forms (which are real numbers) thus separating real and reactive power. The real power $P_r$ is therefore given by

$$P_r=\frac{\omega^2}{2}\mathbf{p}^{+}\hat{r}\mathbf{p}. \tag{34}$$

The reactive power produced by a region filled with current increases without limit as the region contracts. It has, therefore, no physical meaning for an elementary dipole.

In the following we would like to sketch the steps of the computation of $\hat{r}$ for an $\hat{\epsilon}$ as in eq (2). That means, we have to pick the Hermitian part of $\hat{z}$, given by eq (32), which is possible after performing two steps of the integration of $\int d\mathbf{k}\hat{\lambda}^{-1}$.

For this purpose we introduce polar coordinates in $\mathbf{k}$-space, with the volume element

$$d\mathbf{k}=k^2\sin\alpha\, dk\, d\alpha\, d\beta. \tag{34}$$

We choose the intervals of integration from $-\infty$ to $+\infty$ for $k$, from 0 to $\pi/2$ for $\alpha$, and from 0 to $2\pi$ for $\beta$, to cover all $\mathbf{k}$-space. Equations (12) and (20) are used to substitute for $\hat{\lambda}^{-1}$, and eq (21) to rewrite

$$\frac{k^2\hat{\Lambda}(k)}{(k^2-k_I^2)(k^2-k_{II}^2)}=k^2\mathbf{n}\,\mathbf{n}+(k_I^2+k_{II}^2)\mathbf{n}\,\mathbf{n}-k_0^2\hat{L}+\frac{1}{k_I^2-k_{II}^2}\left[\frac{k_I^2}{k^2-k_I^2}\hat{\Lambda}(k_I)-\frac{k_{II}^2}{k^2-k_{II}^2}\hat{\Lambda}(k_{II})\right]. \tag{35}$$

As $k_I$ and $k_{II}$ are independent of $\beta$, the integration with respect to this variable can be performed. As a result of this, we introduce two new matrices, $\hat{N}$ and $\hat{M}$,

$$\hat{N}=\frac{1}{\pi}\int_0^{2\pi}\mathbf{nn}\,d\beta=\begin{bmatrix}\sin^2\alpha & 0 & 0\\ 0 & \sin^2\alpha & 0\\ 0 & 0 & 2\cos^2\alpha\end{bmatrix} \tag{36}$$

$$\hat{M}=\frac{1}{\pi}\int_0^{2\pi}\hat{L}d\beta=\begin{bmatrix}2\epsilon_1\sin^2\alpha+\epsilon_3(1+\cos^2\alpha); & 2j\epsilon_2\sin^2\alpha; & 0\\ -2j\epsilon_2\sin^2\alpha; & 2\epsilon_1\sin^2\alpha+\epsilon_3(1+\cos^2\alpha); & 0\\ 0; & 0; & 2\epsilon_1(1+\cos^2\alpha)\end{bmatrix}. \tag{37}$$

We have postulated a lossless medium. Therefore, $\epsilon_1$, $\epsilon_2$, and $\epsilon_3$ have to be real numbers and the matrices $\hat{M}$ and $\hat{N}$ are Hermitian. $k_I^2$ and $k_{II}^2$ are also real but $k_I$ and/or $k_{II}$ can be real or imaginary ("cut off" plane wave) because negative values of $\epsilon_1$ and/or $\epsilon_3$ can occur in magneto-ionic media.

The next step is to perform the integration $\int_{-\infty}^{+\infty} dk$. From (35) we notice that the two last terms of the integrands have poles if $k_I$ and $k_{II}$ are real. As a consequence of this, the values of the corresponding integrals are not uniquely determined. And it is here that we have to remember the medium is regarded as at least slightly lossy (see sec. 4). The poles are then removed from the path of integration, and, we obtain with the help of Cauchy's formula and the residue concept

$$\int_{-\infty}^{+\infty} \frac{dk}{k^2-k_I^2} = -\frac{\pi j}{k_I}, \tag{38}$$

where the sign of the root of $k_I^2$ has been chosen such that Im $k_I \leq 0$ and Re $k_I > 0$, and similarly for $k_{II}$. The result of the integration is now uniquely determined and can also be used for the lossless case.

The first three terms of the integrand contribute to the anti-Hermitian part of $\hat{z}$ only, and can be dropped. Their integral does not exist (see remarks on reactive power).

With $u = \cos\alpha$, the vacuum characteristic impedance $Z_0 = \sqrt{\mu_0/\epsilon_0}$, and the vacuum wave length $\lambda_0 = 2\pi/k_0$ we obtain finally

$$\hat{r} = -\frac{\pi}{2}\frac{Z_0}{\lambda_0^2}\int_0^1 \frac{du}{k_0^2[\epsilon_1+(\epsilon_3-\epsilon_1)u^2](k_I^2-k_{II}^2)}\hat{F}, \tag{39}$$

where

$$\hat{F} = \text{Herm}\{(k_I^5-k_{II}^5)\hat{N} - k_0^2(k_I^3-k_{II}^3)\hat{M} + 2(k_I-k_{II})k_0^4\hat{\text{E}}\}.$$

The symbol "Herm" stands for "Hermitian part of." As $\hat{N}$, $\hat{M}$, and $\hat{\text{E}}$ are Hermitian, this simply means that all terms with an imaginary $k_I$ or $k_{II}$ have to be dropped. $k_I$ and $k_{II}$ vary with $u(=\cos\alpha)$ according to eq (13) and can be real for some regions of integration and imaginary for others. The latter is the case for angles $\alpha$ between wave normal and permanent magnetic field where the corresponding plane wave is "cut-off" and cannot transmit power.

Because of the particular structure of $\hat{N}$, $\hat{M}$, and $\hat{\text{E}}$ the matrix $\hat{r}$ has the form

$$\hat{r} = \begin{bmatrix} r_1 & -jr_2 & 0 \\ jr_2 & r_1 & 0 \\ 0 & 0 & r_3 \end{bmatrix}, \tag{40}$$

from which the power $P_r$, radiated by dipoles of any orientation and polarization, can be computed.

To discuss two special cases of polarization let us consider:

(a) A linearly polarized dipole moment $\mathbf{p}$, as produced by an oscillating charged particle or an electrically small linear antenna. In this case the terms with $r_2$ cancel in the bilinear form and the dependence of $P_r$ on the orientation of the dipole can be computed from

$$P_r = \frac{\omega^2}{2}\mathbf{p}^+ \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_1 & 0 \\ 0 & 0 & r_3 \end{pmatrix} \mathbf{p}; \tag{41}$$

which gives, if plotted in a polar diagram, an ellipsoid of revolution for $1/\sqrt{P_r}$. For the special orientation perpendicular to the magnetic field the power radiated is

$$P_{r\perp} = \frac{\omega^2}{2} r_1 |p_\perp|^2$$

and for parallel orientation we find

$$P_{r_{\|}}=\frac{\omega^2}{2}r_3|p_{\|}|^2;$$

(b) A circularly polarized moment, as produced by a charged particle on an electrically small circular orbit or two crossed linear antennas with a difference of $\pi/2$ in phase. We propose to discuss two particular orientations only. The first is such that $p_z=0$, and $p_y=\pm jp_x$ (for left- or right-hand polarization respectively) where we obtain

$$P_r=\omega^2|p_x|^2(r_1\pm r_2)$$

which is a characteristic result for magneto-ionic media. It shows that the two crossed antennas are "coupled" by the medium. The power they radiate is not the sum of the powers that each individual antenna would radiate, if the other one were not excited.

The second orientation is one where $p_y=0$ and $p_z=\pm jp_x$. Here the total power

$$P_r=\frac{\omega^2}{2}|p_x|^2(r_1+r_3)$$

is equal to the sum of the powers that would be radiated by the individual antennas.

The values of $r_1$, $r_2$, and $r_3$ are determined by the corresponding integrals of eq (39), which have yet to be evaluated. For the special case of an isotropic medium ($\epsilon_2=0$, $\epsilon_1=\epsilon_3$)

$$r_2=0; \qquad r_1=r_3=\frac{2\pi}{3}\frac{Z_0}{\lambda_0^2}\sqrt{\epsilon_0}\equiv r_0$$

if $\epsilon_1=\epsilon_3\geq 0$, and $r_1=r_2=r_3=0$ if $\epsilon_1=\epsilon_3<0$, which is the case in an isotropic plasma at frequencies below the plasma frequency. If $\epsilon_2=0$ and $\epsilon_1$, $\epsilon_3>0$, which occurs in magneto-ionic media with a very high gyrofrequency or in an uniaxial crystal, the integration yields

$$r_1/r_0=\tfrac{1}{4}(3+\epsilon_3/\epsilon_1);$$

$$r_3/r_0=1; \qquad r_2=0.$$

With the exception of these and a few other special cases, the integrals have to be evaluated numerically. Some plots of an evaluation of $r_1/r_0$ and $r_3/r_0$ by means of an electronic computer can be found in [16].

If the power radiated and the current distribution of an antenna is known, the radiation resistance (which is defined by them) can be easily computed. Take for example a short linear antenna of length $l$ and with a constant distribution of current $I$, which is orientated perpendicular to the permanent magnetic field. It produces a moment with the components $p_x=Il/j\omega$, $p_y=p_z=0$, and its radiation resistance is $R=l^2r_1$.

---

The author thanks H. Motz for encouragement and valuable discussion.

## 7. References

[1] H. W. Nichols, J. C. Schelleng, B.S.T.J. **4,** 215 (1925).
[2] W. P. Allis; Motions of electrons and ions, Handb. Phys. **21,**, 383 (1956).
[3] J. A. Ratcliffe, The magneto-ionic theory and its application to the ionosphere (Cambridge Univ. Press, Cambridge, England, 1959) (with further references).
[4] J. E. Drummond, Basic microwave properties of hot magneto-plasmas, Phys. Rev. **110,** 293 (1958) (with further references).
[5] M. Born, Optik, pp. 413–420 (Berlin, Germany, 1933).
[6] F. V. Bunkin; On radiation in anisotropic media, J. Exptl. Theoret. Phys. (USSR) **32,** 338 (1957).
[7] A. A. Kolomenskii, Radiation from a plasma electron in uniform motion in a magnetic field, Doklady Akad. Nauk S.S.S.R. **106,** 982 (1956) (with further references).

[8] A. G. Sitenko, A. A. Kolomenskii, Motion of a charged particle in an optically active anisotropic medium, J. Exptl. Theoret. Phys. (USSR) **30,** 511 (1956).
[9] K. A. Barsukov, On the Doppler effect in an anisotropic and gyrotropic medium, J. Exptl. Theoret. Phys. (USSR) **36,** 1485 (1959).
[10] G. A. Begiashvile, E. V. Gedalin, Cerenkov radiation of a magnetic dipole in an anisotropic medium, J. Exptl. Theoret. Phys. (USSR) **36,** 1939 (1959).
[11] V. L. Ginzburg, V. Ya. Eidman, The radiation reaction in the motion of a charge in a medium, J. Exptl. Theoret. Phys. (USSR) **36,** 1823 (1959).
[12] A. D. Bresler, The far fields exited by a point source in a passive dissipationless anisotropic uniform waveguide; Trans. I.R.E. **MTT-7,** 282 (1959).
[13] F. E. Borgnis, Ch. H. Papas, Randwertprobleme der Mikrowellenphysik, p. 251–60 (Springer, Berlin, Germany, 1955) (with further references).
[14] P. M. Morse, H. Feshbach, Methods of theoretical physics, Pt. II, Ch. 13 (McGraw-Hill Book Co., Inc. New York, N.Y., 1953).
[15] J. A. Stratton, Electromagnetic theory, p. 432 (McGraw-Hill Book Co., Inc., New York, N.Y., 1941).
[16] H. Kogelnik, The radiation resistance of an elementary dipole in anisotropic plasmas, Fourth Intern. Conf. on Ionization Phenomena in Gases; (Uppsala, Sweden (North-Holland Publishing Co., Amsterdam, Aug. 1960).

(Paper 64D5–90)