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Propagation of Microwaves Through a Magneto-Plasma, and a Possible Method for Determining the Electron Velocity Distributions[1]

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Sagdeyev and Shafranov have shown that the absorption of microwaves in a hot plasma in a steady magnetic field can be calculated in simple closed form with the help of the Boltzmann equation, provided that the effect of collision can be ignored.

The present paper is restricted to the special case of propagation of circularly polarized waves parallel to the magnetic field, and the extraordinary ray, in magneto-ionic terminology, is given special attention. It is shown that the formula given by Sagdeyev and Shafranov for this case can be deduced by considering the motions of individual electrons by elementary dynamical methods, using the concepts of Doppler shift and velocity distribution functions to obtain a macroscopic conductivity formula for a high-temperature plasma. From this, the absorption is easily calculated.

It is emphasized that the calculation in no way depends upon the assumption of a Maxwellian velocity distribution function. The absorption can in fact be obtained in closed form for any arbitrary velocity distribution function.

This suggests that a diagnostic technique for the determination of velocity distribution could be based on measurements of absorption of the extraordinary ray, and the potentialities and limitations of this proposal are briefly discussed.

1. Introduction

Sagdeyev and Shafranov [2] have shown that the absorption of microwave energy in a hot plasma can be calculated in simple closed form from the Boltzmann equation provided that the effect of collisions can be ignored.

In the present paper, the same formula is obtained from a simple dynamical-collisional argument, and it is shown that if the collision frequency is small in comparison with the Doppler frequency shift due to thermal motions of the electrons, the value of the collision frequency has only a second-order effect on the absorption coefficient. Furthermore, the results of Sagdeyev and Shafranov are extended to cover the case of an arbitrary, rather than a Maxwellian, velocity distribution function, and a microwave diagnostic technique for determining the distribution functions in low density hot plasmas is proposed.

2. Equations of Motion

In these equations, the effect of the radiofrequency magnetic field will be ignored. The electric field is assumed in the first instance to have the form

$$\left.\begin{aligned} E_x &= E_0 \sin \omega t \\ E_y &= E_0 \cos \omega t \end{aligned}\right\}. \tag{1}$$

It is also assumed that there is a constant magnetic field along the z-axis of value B_0. If the strength of this field is expressed in terms of the cyclotron frequency for electrons $\omega_c = eB_0/m$, the equations of motion can be written

$$\left.\begin{aligned} \frac{dv_x}{dt} &= -\frac{eE_0}{m} \sin \omega t \mp \omega_c v_y \\ \frac{dv_y}{dt} &= -\frac{eE_0}{m} \cos \omega t \pm \omega_c v_x \end{aligned}\right\}. \tag{2}$$

The upper sign corresponds to a magnetic field directed along the positive z-axis. Looking in this direction, the electric vector described by (1) rotates *anticlockwise*. The natural direction of rotation of electrons in this magnetic field is *clockwise*. Thus if (1) described the electric field of a circularly polarized wave traveling along the z-axis in the positive direction, the upper sign will correspond to the *ordinary* ray, and the lower sign to the *extraordinary* ray, in ionosphere theory terminology.

The *extraordinary* ray is the case of interest here, for this wave can be heavily absorbed by synchronous acceleration of the electrons by a cyclotron-type mechanism. In what follows, the lower sign will be taken. To use the equations for the ordinary ray, it is only necessary to change the sign of ω_c.

From eq (2) we find

$$\left.\begin{aligned} \frac{d^2v_x}{dt^2} + \omega_c^2 v_x &= -\frac{eE_0}{m}(\omega+\omega_c)\cos \omega t \\ \frac{d^2v_y}{dt^2} + \omega_c^2 v_y &= +\frac{eE_0}{m}(\omega+\omega_c)\sin \omega t \end{aligned}\right\}. \tag{3}$$

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[2] Sagdeyev and Shafranov, Absorption of high-frequency electromagnetic field energy in the high-temperature plasma. Proceedings of the Second International Conference on Peaceful Uses of Atomic Energy, Geneva, September 1958. Paper P/2215.

3. Collisions and Boundary Conditions

For our present purpose the details of the collision process are not important, since it will emerge that the collision frequency does not enter into the final formula for absorption.

We shall assume that an electron is brought to rest at each collision. Thus, for an electron which makes a collision at time t', its motion until the next collision takes place can be found by solving the eq (3) subject to the conditions

$$\left.\begin{aligned} v_x&=v_y=0 \\ \frac{dv_x}{dt}&=-\frac{eE_0}{m}\sin\omega t' \\ \frac{dv_y}{dt}&=-\frac{eE_0}{m}\cos\omega t' \end{aligned}\right\}. \tag{4}$$

The last two formulas follow by substituting the first in (2). The solution of (3) appropriate to the boundary conditions (4) is given by

$$v_x=\frac{eE_0}{m(\omega-\omega_c)}[\cos\omega t-\cos\{\omega_c t+(\omega-\omega_c)t'\}] \tag{5}$$

$$v_y=-\frac{eE_0}{m(\omega-\omega_c)}[\sin\omega t-\sin\{\omega_c t+(\omega-\omega_c)t'\}]. \tag{6}$$

Equations (5) and (6) can be expressed more compactly thus

$$v_x-jv_y=\frac{eE_0}{m(\omega-\omega_c)}[e^{j\omega t}-e^{j\{\omega_c t+(\omega-\omega_c)t'\}}]. \tag{7}$$

Equation (7) gives the x and y components of velocity at time t of an electron which was at rest at time t'.

Assuming that n, the number of electrons per unit volume, is sufficiently large, we can assume that, on the average, at any time t the number of electrons per unit volume whose most recent collision occurred between t' and $t'+dt'$ is given by

$$dn=n\nu e^{-\nu(t-t')}\,dt'. \tag{8}$$

Using this formula, which in fact defines the collision frequency ν, we can calculate the current density at any time t. The contribution to J_x from the electrons whose most recent collision was in the interval t' to $t'+dt'$ is $dJ_x=-nev_xdn$, where v_x is given by (5). Hence, using (7), we find

$$J_x-jJ_y=\frac{-n\nu e^2E_0}{m(\omega-\omega_c)}\int_{-\infty}^{t}[e^{j\omega t}-e^{j\{\omega t+(\omega-\omega_c)t'\}}]e^{-\nu(t-t')}\,dt'. \tag{9}$$

Carrying out the integrations in (9) and separating the real and imaginary parts, we find

$$J_x=\frac{ne^2E_0}{m}\left[\frac{\nu\sin\omega t-(\omega-\omega_c)\cos\omega t}{\nu^2+(\omega-\omega_c)^2}\right] \tag{10}$$

$$J_y=\frac{ne^2E_0}{m}\left[\frac{\nu\cos\omega t+(\omega-\omega_c)\sin\omega t}{\nu^2+(\omega-\omega_c)^2}\right]. \tag{11}$$

4. Effect of Velocity Distribution

So far, the formulas we have derived are familiar in magneto-ionic theory. We now consider the effect of thermal velocities of the electrons. If the electric field we have been discussing is associated with a wave of frequency ω_0 traveling in the direction of increasing z, with phase constant β, then an electron traveling in the same direction with velocity v_z will be subjected to a field of angular frequency $\omega_0-\beta v_z=\omega$ say.

If dn_0 electrons/unit volume have a z-component of velocity between v_z and v_z+dv_z, and if the velocity distribution is Maxwellian, we have

$$dn_0=n_0\left(\frac{m}{2\pi kT}\right)^{1/2}e^{-mv_z^2/2kT}\,dv_z. \tag{12}$$

Equations (10) and (11) can be used to calculate the contribution to the current density components J_x and J_y in a frame of reference traveling with the electrons, i.e., with velocity v_z along the original z-axis, if we substitute dn_0 for n, and interpret ω as $\omega_0-\beta v_z$. Thus

$$dJ_x=\frac{n_0e^2E_0}{m}\left(\frac{m}{2\pi kT}\right)^{1/2}e^{-mv_z^2/2kT}\left[\frac{\nu\sin\omega t-(\omega-\omega_c)\cos\omega t}{\nu^2+(\omega-\omega_c)^2}\right]dv_z. \tag{13}$$

Transforming this contribution to the current back to the original frame of reference does not affect its amplitude but restores the frequency to the original value ω_0. Thus, summing all such contributions

$$J_x=\frac{n_0e^2E_0}{m}\left(\frac{m}{2\pi kT}\right)^{1/2}\int_{-\infty}^{+\infty}e^{-mv_z^2/2kT}\left[\frac{\nu\sin\omega_0 t-(\omega-\omega_c)\cos\omega_0 t}{\nu^2+(\omega-\omega_c)^2}\right]dv_z. \tag{14}$$

Similarly,

$$J_y=\frac{n_0e^2E_0}{m}\left(\frac{m}{2\pi kT}\right)^{1/2}\int_{-\infty}^{+\infty}e^{-mv_z^2/2kT}\left[\frac{\nu\cos\omega_0 t+(\omega-\omega_c)\sin\omega_0 t}{\nu^2+(\omega-\omega_c)^2}\right]dv_z. \tag{15}$$

Note that parts of J_x and J_y which are in phase with the electric field components E_x and E_y are equal, and correspond to a transverse conductivity σ_1 given by

$$\sigma_\perp=\frac{n_0e^2}{m}\left(\frac{m}{2\pi kT}\right)^{1/2}\int_{-\infty}^{+\infty}e^{-mv_z^2/2kT}\left[\frac{\nu}{\nu^2+(\omega_0-\omega_c-\beta v_z)^2}\right]dv_z. \tag{16}$$

Similarly, the transverse permittivity ϵ_1 is

$$\epsilon_{\perp}=\epsilon_0-\frac{n_0e^2}{\omega_0 m}\left(\frac{m}{2\pi kT}\right)^{1/2}\int_{-\infty}^{+\infty}e^{-mv_z^2/2kT}\left[\frac{(\omega_0-\omega_c-\beta v_z)}{\nu^2+(\omega_0-\omega_c-\beta v_z)^2}\right]dv_z. \quad (17)$$

The effect of collisions on the v_z component of velocity has not been considered, since the absorption phenomenon is not affected by this process. We do not need to have detailed knowledge of v_z for each electron, we only need to know the number of electrons in any given small range of velocities.

5. Wave Propagation in a Hot Plasma

We assume now that the plasma temperature T is large [3] and the collision frequency small, so that

$$\nu\ll\beta\sqrt{\frac{kT}{m}}. \quad (18)$$

In this case the bracketed factor in the integrand of (16) is small unless v_z is such that

$$\omega_0-\omega_c-\beta v_z<\nu. \quad (19)$$

Within this small range of values of v_z the exponential factor can be given the value corresponding to $v_z=(\omega_0-\omega_c)/\beta$, and taken outside the integral sign. It is then a simple matter to evaluate the integral, and the resulting formula for the conductivity is

$$\sigma=\frac{\pi n_0e^2}{\beta m}\left(\frac{m}{2\pi kT}\right)^{1/2}e^{-m/2kT\,[(\omega_0-\omega_c)/\beta]^2}. \quad (20)$$

If the conductivity is small, the absorption coefficient κ can be calculated very simply from the formula

$$\kappa=\frac{\alpha}{\beta}=\frac{\sigma_\perp}{2\omega\epsilon_\perp}=\frac{\sigma_\perp}{2\omega\epsilon_0p^2}; \quad (21)$$

where α and β are the attenuation and phase constants respectively, and p is the real part of the refractive index. If, in accordance with Sagdeyev and Shafranov, we define $v_{||}$ by the equation

$$v_{||}=\sqrt{\frac{2kT}{m}} \quad (22)$$

we find

$$\kappa=\frac{\sqrt{\pi}}{2}\cdot\frac{c}{v_{||}}\frac{\omega_p^2}{\omega_0^2p^3}\exp\left\{-\frac{c^2}{v_{||}^2p^2}\left(\frac{\omega_0-\omega_c}{\omega_0}\right)^2\right\}. \quad (23)$$

Here ω_p is the plasma frequency ($\omega_p^2=ne^2/m\epsilon_0$), and ω_0 and ω_c are the frequency of the wave and the electron cyclotron frequency respectively.

This formula agrees exactly with the result of Sagdeyev and Shafranov, as stated in eq (13) of their paper (see footnote 2).

The calculation of $\epsilon_\perp$ from (17) is carried out in the appendix. When this is done, we find

$$p^2=1-\frac{2c}{v_{||}}\cdot\frac{\omega_p^2}{\omega_0^2p}e^{-c^2/v_{||}^2p^2[(\omega_0-\omega_c)\omega_0]^2}\left\{\frac{c}{v_{||}p}\left(\frac{\omega_0-\omega_c}{\omega_0}\right)-\frac{1}{3}\frac{c^3}{v_{||}^3p^3}\left(\frac{\omega_0-\omega_c}{\omega_0}\right)^3+\cdots\right\}. \quad (24)$$

This also agrees with the form given by Sagdeyev and Shafranov, although their result is expressed in terms of an error integral. (The lower limit of this integral, which is missing, is presumably zero.) There is also an error in the sign of the second term in their formula.

6. A Microwave Diagnostic Technique for Determining the Electron Velocity Distribution in a Hot Plasma

It is easy to see that the method employed for integrating (16) does not depend in any way on the velocity distribution. This need not be Maxwellian, but can be quite arbitrary.

Let the velocity distribution parallel to the magnetic field be defined by formulas

$$dn_0=n_0f(v_z)dv_z;\ \int_{-\infty}^{+\infty}f(v_z)dv_z=1 \quad (25)$$

in place of the Maxwellian distribution formula eq (12).

We find for the conductivity of the plasma the very simple expression

$$\sigma_\perp=\frac{\pi n_0e^2}{\beta m}f\left(\frac{\omega_0-\omega_c}{\beta}\right).$$

The attenuation coefficient can be written

$$\alpha=\frac{\pi e^2\mu_0}{2m}\cdot\frac{\omega_0}{\beta^2}n_0f\left(\frac{\omega_0-\omega_c}{\beta}\right).$$

Thus the distribution function can be expressed in terms of experimentally observable quantities as follows

$$n_0f\left(\frac{\omega_0-\omega_c}{\beta}\right)=\frac{2m}{\pi e^2\mu_0}\cdot\frac{\alpha\beta^2}{\omega_0}. \quad (26)$$

If the right-hand side of (26) is plotted as a function of $(\omega_0-\omega_c)/\beta$, the electron velocity distribution parallel to the magnetic field is obtained. The area under this curve will give the electron density. In practical applications, the distribution function may depend on z. If this is the case, the total attenuation measured is approximately $\int\alpha dz$ along the transmission path, provided that the rate of change of attenuation per wavelength of path is not too great. Thus from the total attenuation we can determine, using (26), the *average* distribution function, the average being taken along the microwave path.

[3] Since this is the only temperature involved, we use T for $T_{||}$, the effective temperature associated with thermal motion of electrons parallel to the magnetic field lines.

7. Wave Propagation in a Cold Plasma

Formulas for $\sigma_\perp$ and $\epsilon_\perp$ in a cold plasma, in which $\nu \gg \beta\sqrt{kT/m}$, are easily obtained from (16) and (17) if the collision frequency can be assumed independent of v_z. The familiar magneto-ionic theory formulas are obtained:

$$\left.\begin{aligned} \sigma_\perp &= \frac{ne^2}{m}\cdot\frac{\nu}{\nu^2+(\omega_0-\omega_c)^2} \\ \frac{\epsilon_1}{\epsilon_0} &= 1-\frac{ne^2}{\epsilon_0 m}\cdot\frac{(\omega_0-\omega_c)}{\nu^2+(\omega_0-\omega_c)^2} \end{aligned}\right\}. \tag{27}$$

Note that if $\nu \gg (\omega_0-\omega_c)$, $\sigma_\perp$ is proportional to $1/\nu$, whilst if $\nu \ll (\omega_0-\omega_c)$, $\sigma_\perp$ is proportional to ν.

8. Discussion

The purpose of this section is to contrast the behavior of the conductivity of hot and cold plasmas with respect to variation of collision frequency.

Consider first the cold plasma conductivity as described by (27). For a large collision frequency, the conductivity is proportional to $1/\nu$. The physical meaning is clear. It is easy to show from (7) that an electron starting from rest at time $t=0$ acquires in time t a kinetic energy given by

$$W=\frac{1}{2}\frac{e^2E_0}{m}\left\{\frac{\sin\left[\left(\frac{\omega-\omega_c}{2}\right)t\right]}{\frac{\omega-\omega_c}{2}}\right\}^2 \tag{28}$$

assuming that no collisions take place. Differentiating, we find

$$\frac{dW}{dt}=\frac{e^2E_0^2}{m}\frac{\sin(\omega-\omega_c)t}{(\omega-\omega_c)}. \tag{28a}$$

If the collision frequency ν is large, in comparison with $(\omega_0-\omega_c)$, the time interval t, during which the gain in kinetic energy described by (28a) can take place before the process is interrupted by another collision, is small, and $(\omega_0-\omega_c)\, t \ll 1$. Hence, (28a) can be replaced by

$$\frac{dW}{dt}=\frac{e^2E_0^2}{m}\cdot t. \tag{29}$$

The average rate of absorption per electron is given by

$$\frac{\overline{dW}}{dt}=\frac{e^2E_0^2}{m}\cdot\frac{1}{\nu}. \tag{30}$$

Thus the physical significance of the inverse variation of conductivity with collision frequency, when the collision frequency is large in comparison with the frequency difference, $\omega_0-\omega_c$ becomes clear.

When the collision frequency is small in comparison with $(\omega_0-\omega_c)$, the amplitude of oscillation, and hence the mean kinetic energy acquired by the electrons, is not limited by collisions, but by the lack of synchronism between the frequency of the alternating force by the wave and the natural frequency of gyration of the electrons. If $\omega_0 > \omega_c$, the amplitude of oscillation is limited mainly by electron inertia, while if $\omega_0 < \omega_c$, it is limited by the obstructing effect of the magnetic field. If (28) is averaged over the long time which elapsed between collision, we find

$$\overline{W}=\frac{e^2E_0^2}{m(\omega_0-\omega_c)^2}.$$

If it is assumed that, on the average, this energy, or a definite fraction of it, is given up by an electron at each collision, the average rate of absorption of energy per electron is given by

$$\nu\overline{W}=\frac{e^2E_0^2}{m(\omega_0-\omega_c)^2}\cdot\nu, \tag{31}$$

and is clearly directly proportional to ν, the collision frequency. Thus the reason for the variation of conductivity with collision frequency predicted by (27) is clear, and the physical mechanism by which the energy is absorbed has been exhibited. Energy extracted from the wave appears first as kinetic energy of the electron, and is then transferred to molecules or ions of the gas by collisions. This absorption process is called *collisional absorption*, and as we have seen, depends strongly on the collision frequency.

In direct contrast to this situation, the absorption by a "hot" plasma, in which $\nu \ll \beta\sqrt{kT/m}$, is independent of ν to first order, as eq (20) shows. The reason for this is that in a hot plasma, the effective frequency seen by an electron depends on its velocity in the direction of the propagation of the wave, due to the Doppler shift. Since the electron velocities vary widely in a hot plasma, a wide range of effective frequencies exists, and the hot conductivity is essentially an average of the cold conductivity over all frequencies. That this is independent of ν can be easily seen using (27)

$$\begin{aligned}\int_{-\infty}^{+\infty}\sigma_\perp d\omega &= \frac{ne^2}{m}\int_{-\infty}^{+\infty}\frac{\nu}{\nu^2+(\omega-\omega_c)^2}\,d\omega \\ &= \frac{ne^2}{m}\int_{-\infty}^{+\infty}\frac{\nu}{\nu^2+\omega_1^2}\,d\omega_1=\frac{\pi ne^2}{m}. \end{aligned} \tag{32}$$

Provided that ν is small enough, the integrand is very small except when $(\omega-\omega_c)\sim\nu$, so that a very small range of effective frequencies (and electron velocities) contributes significantly to the absorption. Provided that the plasma temperature is high enough, the variation of the distribution function over this small range of significant electron velocities may be neglected, and the distribution function can be regarded as constant as was done in obtaining eq (20).

Further, if ν depends on v_z, the analysis is still valid, provided that the dependence is not strong in the small range of significant velocities.

The case of no collisions at all can be regarded as the limiting case $\nu \rightarrow 0$; the "cold conductivity" term in the integrand of (16) then has the character of a delta function, and eq (10) is exact.

Thus, the agreement of our analysis with that of Sagdeyev and Shafranov is to be expected, in spite of the fact that in their treatment it was assumed ab initio that no collisions occurred, whereas the present analysis necessarily involves the consideration of collisions.

9. Conclusions

Summarizing, the present analysis has shown:

(a) That in the absorption of microwaves by a hot plasma in a magnetic field, collisions have a second-order effect only;

(b) That the results of Sagdeyev and Shafranov, obtained by solving the Boltzmann equations for the distribution functions by a perturbation technique, neglecting collisions at the outset, can be obtained by a direct ballistic analysis of the motion of individual electrons;

(c) That the assumption of a Maxwellian velocity distribution function made by Sagdeyev and Shafranov is not necessary to the analysis, and that a closed-form solution for conductivity can be obtained for any arbitrary distribution function;

(d) That a study of the variation with frequency of the attenuation and phase constant of a circularly polarized electromagnetic wave in an ionized plasma in a magnetic field can in principle yield the velocity distribution function for the thermal motion of electrons parallel to the magnetic field lines. There may be considerable difficulty in applying this technique in some cases. For example, the extraordinary ray only must be employed, and a high degree of discrimination against the *ordinary* ray in the launching and/or receiving antennas will be necessary if errors due to the ordinary ray, which is attenuated much less strongly, are to be negligible. Also diffraction effects will be serious except at microwave frequencies, so that the method can only be applied in the simple form suggested here if the magnetic field strength is of the order of 3,000 gauss or more, so that the cyclotron frequency falls in the microwave-frequency band.

As an alternative to attempting to eliminate the ordinary ray entirely, one could start with a linearly polarized wave and measure the ellipticity of the received signal. This would have some practical advantages, and the necessary theory could be developed very simply from the equations given here.

A more serious limitation arises when the order of magnitude of the attenuation at resonance is estimated. For $n=10^{11}$ electrons/cm^3 $T=10^6$ °K, and $f_0=10{,}000$ Mc/s, the attenuation coefficient at cyclotron resonance has a value of about 100 db/cm or the attenuation length is about 1 mm. For densities of 10^8 electrons/cm^3 under the same conditions, however, the attenuation coefficient is about 0.1 db/cm, and for such a low density plasma the method might be feasible.

It should be noted that the effect of the radio-frequency magnetic field has not been considered in the present analysis. While this seems unlikely to lead to serious errors at electron temperatures of a million degrees or so, these effects might be appreciable for temperatures of the order of a hundred million degrees, when the electron velocity is only one order of magnitude less than the velocity of light.

No comparison has been made with the more elaborate theory of Drummond [4], which starts from the Boltzmann equation but retains the collision term, for which a suitable approximation is later introduced. It would be very desirable to make such a comparison, which would probably shed further light on the significance of collisions in a practical situation.

10. Appendix

We have to evaluate

$$I=\int_{-\infty}^{+\infty} e^{-mv_z^2/2kT}\frac{(\omega_0-\omega_c-\beta v_z)}{\nu^2+(\omega_0-\omega_c-\beta b_z)^2}\,dv_z. \tag{1}$$

Let $t=\sqrt{m/2kT}\,v_z : t_0=\sqrt{m/2kT}\,[(\omega_0-\omega_c)/\beta]$. Let us also put $\nu=0$. We get

$$I=\frac{1}{\beta}\int_{-\infty}^{+\infty}\frac{e^{-t^2}dt}{t_0-t}. \tag{2}$$

Now put $z=t-t_0$.

$$I=-\frac{1}{\beta}\int_{-\infty}^{+\infty}\frac{e^{-(z+t_0)^2}}{z}\,dz. \tag{3}$$

Because of the singularity at $z=0$, we interpret I in the following way

$$-\beta I=e^{-t_0^2}\lim_{\epsilon\rightarrow 0}\left\{\int_{-\infty}^{-\epsilon} e^{-z^2-2t_0z}\frac{dz}{z}+\int_{+\epsilon}^{+\infty} e^{-z^2-2t_0z}\frac{dz}{z}\right\}.$$

Replacing z by $-z$ in the first integral, and combining the two integrals gives

$$+\frac{\beta I}{2}=e^{-t_0^2}\int_0^{\infty}\frac{e^{-z^2}}{z}\sinh 2t_0z\,dz. \tag{4}$$

This integral is easily evaluated by expanding $\sinh 2t_0z$ and integrating term by term. This leads to the result given in eq (24).

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[4] J. E. Drummond, Basic microwave properties of hot magnetoplasma, Phys. Rev. **110** (April 15, 1958).

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