JOURNAL OF RESEARCH of the National Bureau of Standards—D. Radio Propagation
Vol. 64D, No. 5, September–October 1960

Guiding of Whistlers in a Homogeneous Medium

R. L. Smith

(May 4, 1960)

The velocity of energy flow of whistlers in a homogeneous medium is computed as a function of wave-normal angles. The maximum allowable cone of ray angles approaches 19°29′ at very low frequencies, decreases with frequency to a minimum of 11° at a wave frequency of one-fifth the gyrofrequency, then increases to 90° at the gyrofrequency. The velocity of energy flow departs markedly from the longitudinal value except at very low frequencies or very small wave-normal angles.

1. Introduction

Part of the energy from a lightning stroke can penetrate the lower ionosphere and be guided approximately along the earth's magnetic field in the outer ionosphere. The energy is dispersed as it travels in this region, causing the original signal, which can be considered an impulse, to be stretched or dispersed into a gliding tone, typically lasting about a second. This gliding tone is called a whistler. The dispersion is a measure of the electron content along the path.

Storey [1] [2] has analyzed the properties of whistler propagation for the case of wave frequency very low compared to the local electron plasma frequency and gyrofrequency. Briefly, his results show that (a) the direction of energy flow lies within a limiting cone of 19°29′ around the magnetic field direction; (b) the longitudinal expression for group velocity can be used to describe the velocity of the energy flow with an error less than 8 percent for wave normal angles up to 70°; (c) if the initial wave normal angles for the different frequency components are identical, the path of energy flow is independent of frequency; and (d) the time delay T from the causative lightning stroke is given by $T=D\,f^{-1/2}$, where f is the frequency and D is a constant for a particular whistler. This relation is sometimes called "Eckersley's law". Storey found good agreement between this relation and data he obtained at Cambridge, England.

From recordings made at high latitudes, a new phenomenon, called the nose whistler, was discovered by Helliwell et al. [2] in 1955. These whistlers exhibited simultaneous rising and falling tones starting at the frequency of minimum time delay called the "nose". The nose whistler occurs when the wave frequency of the whistler becomes comparable to the local gyrofrequency at the top of a field line. Even at the lowest frequencies of most nose whistlers the departures from Eckersley's law are significant. Other whistlers, though they may not exhibit a nose, often show similar departures from Eckersley's law. These deviations are important in determining the location of the path of propagation [3]. Without the high-frequency information, it is usually necessary to assume that whistler energy follows a field line terminating at the receiver. The diameter of the effective area of whistlers has been shown by Storey [1] and Crary et al. [4] to be approximately 1,000 km. Consider two possible whistler paths terminating at 55° and 60° geomagnetic latitude, well within the effective area of a whistler. The calculated average electron density for a whistler of given dispersion will differ by 2:1 for these two paths. Clearly the path needs to be defined more precisely. Before we use the high-frequency information however, we must reexamine the equations used to determine the dispersion and path.

The present paper is an extension of Storey's work, removing the restriction that the wave frequency be small compared to the gyrofrequency. As frequency increases, the analysis shows that the limiting cone of rays first decreases to 11° at about one-fifth the gyrofrequency, then increases to 90° at the gyrofrequency. The error in the quasi-longitudinal approximation for group velocity increases with frequency and wave normal angle.

2. Refractive Index for Whistlers

Experimental evidence shows that the attenuation of whistlers in the outer ionosphere is very low. There are estimates of attenuation of the order of 10 db and lower over paths which are roughly 10,000 km long. The attenuation will therefore be neglected. The square of the wave refractive index for the ordinary mode, as defined by Booker, is then given from the Appleton-Hartree equation as:

$$\mu^2=1-\frac{X}{1-\dfrac{Y_H^2\sin^2\theta}{2(1-X)}-Y_H\cos\theta\left[1+\dfrac{Y_H^2\sin^4\theta}{4\cos^2\theta\,(1-X)^2}\right]^{\frac{1}{2}}} \tag{1}$$

[1] Contribution from Radioscience Laboratory, Stanford University, Stanford, Calif. The main results of this paper were presented at the Symposium on VLF Radio Waves held in Boulder, Colo., January 1957.

[2] Figures in brackets indicate the literature references at the end of this paper.

where

$X = \frac{f_0^2}{f^2}$,

$Y_H = \frac{f_H}{f}$,

θ = angle between the wave normal and magnetic field,

f = wave frequency,

$f_0 = \sqrt{\frac{Ne^2}{4\pi^2 m\epsilon_0}}$ = plasma frequency,

$f_H = \frac{\mu_0 eH}{2\pi m}$ = gyrofrequency,

N = density of electrons,

e = charge of the electron,

m = mass of the electron,

H = magnetic field strength,

ϵ_0 = permittivity of free space.

If we assume that $\tan\theta \sin\theta << 2\,(X-1)/Y_H$, i.e., if we assume quasi-longitudinal propagation, then eq (1) is simplified to:

$$\mu^2 = 1 - \frac{X}{1 - Y_H \cos\theta}. \qquad (2)$$

For most cases of interest in whistler propagation it can be assumed that the square of the refractive index is large compared to unity. Equation (2) can then be further simplified to:

$$\mu^2 = \frac{X}{Y_H \cos\theta - 1} = \frac{f_0^2}{f(f_H \cos\theta - f)}. \qquad (3)$$

The phase velocity is then

$$v = \frac{c}{\mu} = \frac{c(Y_H \cos\theta - 1)^{1/2}}{X^{1/2}} = \frac{c(\cos\theta - \lambda)^{1/2}}{\lambda^{1/2} X^{1/2}}, \qquad (4)$$

where $\lambda = 1/Y_H = f/f_H$ = normalized wave frequency. This relation defines the wave surface.

Equation (4) shows that for a given value of λ, the angle θ between the wave normal and the field direction must be less than that given by $\cos\theta_{max} = \lambda$ for a propagating wave. The condition of validity of the quasi-longitudinal assumption at the maximum angle can easily be shown to be:

$$f_H^2 \ll (2f_0^2 - f^2). \qquad (5)$$

Assuming the wave refractive index is given by eq (3), then the group refractive index, from which we determine the group velocity, is:

$$\mu' = \frac{d}{df}(\mu f) = \frac{d}{df}\frac{f_0 f^{1/2}}{(f_H \cos\theta - f)^{1/2}} = \frac{f_0 f_H \cos\theta}{2f^{1/2}(f_H\cos\theta - f)^{3/2}} \qquad (6)$$

The error in the group velocity as derived from eq (3) instead of eq (2) is negligible at $\lambda = 0$, 0.75, and 1.0. The greatest error occurs at $\lambda = 0.283$ and is approximately

$$0.11\,\frac{f_H^2 \cos^2\theta}{f_0^2}.$$

3. Behavior of the Ray

The direction of the ray is the direction of constructive interference of phase fronts for infinitesimal changes in the wave normal direction. If the direction of the ray with respect to the wave normal is α, and α is positive when the wave normal lies between the ray and the field direction, then α is given by

$$\tan\alpha = -\frac{1}{\mu}\frac{\partial\mu}{\partial\theta} = -\frac{\sin\theta}{2(\cos\theta - \lambda)}. \qquad (7)$$

This relation has been shown by Bremmer [5], Storey [1], and others.

The total angle between the ray direction and the field is $\theta+\alpha$. A sketch of $(\theta+\alpha)$ as a function of θ and parametric in λ is shown in figure 1.

The diagonal line represents the ray direction when the wave normal is at its maximum value. The greatest negative value of $(\theta+\alpha)$ is given by

$$|\theta+\alpha|_{max} = \sin^{-1}\lambda \qquad (8)$$

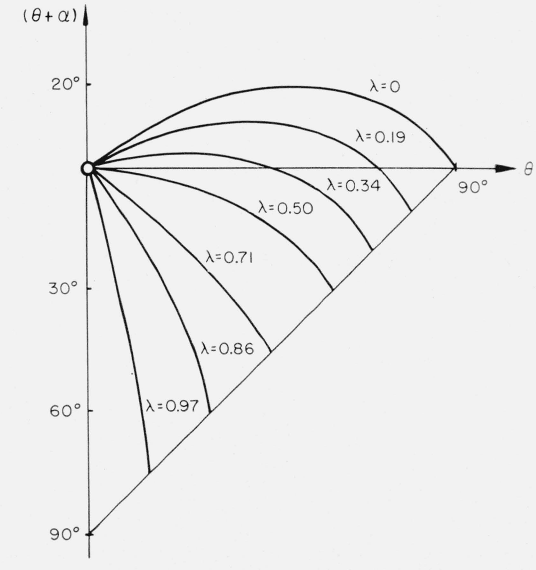

FIGURE 1. *The ray direction as a function of the wave direction.*

The greatest positive value of $(\theta+\alpha)$ is determined by setting

$$\frac{\partial(\theta+\alpha)}{\partial\theta}=0$$

$$\frac{\partial(\theta+\alpha)}{\partial\theta}=1+\frac{\partial}{\partial\theta}\tan^{-1}\left[-\frac{\sin\theta}{2\ (\cos\theta-\lambda)}\right]$$

$$=\frac{3\cos^2\theta-6\lambda\cos\theta+4\lambda^2-1}{2\ (\cos\theta-\lambda)^2+\sin^2\theta}=0. \tag{9}$$

Solving eq (7) for cos θ, we find

$$\cos\theta=\lambda+\frac{(1-\lambda^2)^{1/2}}{3^{1/2}} \tag{10}$$

The maximum positive value is then easily shown to be

$$\tan\ (\theta+\alpha)_{\max}=\frac{[(1-\lambda^2)^{1/2}-\sqrt{3}\lambda]^{3/2}}{2^{3/2}(1-\lambda^2)^{3/4}}. \tag{11}$$

The maximum ray direction for any given normalized frequency is given by the larger of the values determined from eqs (8) and (11). Figure 2 shows the results graphically.

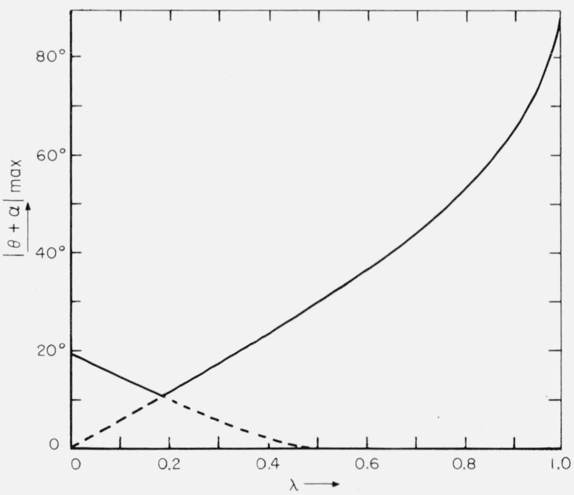

FIGURE 2. *Maximum possible ray direction as a function of* λ.

The minimum ray cone of 11° occurs at $\lambda=0.189$. The value at $\lambda=0$ is found to be 19°29′ in accordance with Storey. The half beamwidth lies between these two values for all λ between 0 and 0.33.

The group ray refractive index is given by $M'=\mu'\cos\alpha$. The group ray velocity is the velocity of propagation of a point on a wave packet limited in both length and width where the phase is stationary with respect to independent variations of both frequency and wave normal direction. From eq (6), we can obtain

$$\mu'=\frac{X^{1/2}\lambda^{1/2}}{2\left(1-\frac{\lambda}{\cos\theta}\right)^{3/2}(\cos\theta)^{1/2}} \tag{12}$$

and from eq (7) we can obtain

$$\cos\alpha=\frac{2\left(1-\frac{\lambda}{\cos\theta}\right)}{\left[\tan^2\theta+4\left(1-\frac{\lambda}{\cos\theta}\right)^2\right]^{1/2}}. \tag{13}$$

Combining (12) and (13), we obtain

$$M'=\frac{X^{1/2}\lambda^{1/2}}{2(1-\lambda)^{3/2}}\Phi(\theta,\lambda)=\mu_L{}'\Phi(\theta,\lambda) \tag{14}$$

where

$$\Phi(\theta,\lambda)=\frac{(1-\lambda)^{3/2}}{\left[\frac{1}{4}\tan^2\theta(\cos\theta-\lambda)+\frac{(\cos\theta-\lambda)^3}{\cos^2\theta}\right]^{1/2}} \tag{15}$$

and $\mu_L{}'$=group ray refractive index of a longitudinal wave. The factor Φ is plotted in figure 3 as a function of θ, with λ as a parameter. The curves show to what extent the group ray refractive index is independent of wave normal direction.

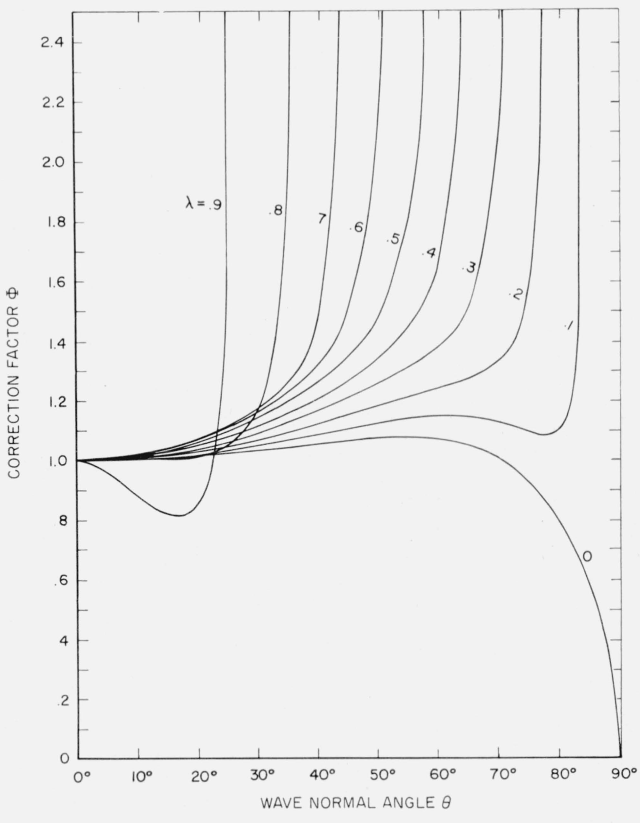

FIGURE 3. *Correction factor* Φ *as a function of wave normal direction, with* λ *as a parameter.*

4. Conclusion

The spread of allowable ray angles approaches 19°29′ as the normalized frequency $\lambda = f/f_H$ tends toward zero. The limiting cone reaches a minimum of 11° at about one-fifth of the gyrofrequency, then rapidly increases with increasing frequency. The energy near and above the nose frequency of whistlers ($\lambda \approx 0.25$) would then be expected to diverge considerably in the outer ionosphere unless an additional confining mechanism is postulated.

Furthermore, the group ray refractive index may depart markedly from the longitudinal value. Even for wave-normal angles less than one-half the limiting angle the correction factor may exceed unity by more than 10 percent. Evidence for the spread in group ray refractive index should be found in the time delay spreading of received whistlers. Since this spread is observed to be less than 1 percent in most nose whistler traces, a mechanism for confining the wave normal angles to small values is indicated. Such a mechanism might be enhanced columns of ionization alined along the magnetic field as discussed by Helliwell [3] and Smith et al. [6].

5. References

[1] L. R. O. Storey, An investigation of whistling atmospherics, Phil. Trans. Roy. Soc. [A], **246,** 113 to 141 (July 9, 1953).

[2] R. A. Helliwell, J. H. Crary, J. H. Pope, R. L. Smith, The "nose" whistler—a new high latitude phenomenon. J. Geophys. Res. **61,** 139 to 142 (Mar. 1956).

[3] R. A. Helliwell, Low frequency propagation studies, part I: Whistlers and related phenomena. Final report, Contract AF 19 (604) 795. June 15, 1953 to Sept. 30, 1956. (Revised May 28, 1958.) AFCRC–TR–56–189, ASTIA Document No. AD110184.

[4] J. H. Crary, R. A. Helliwell, R. F. Chase, Stanford-Seattle whistler observations, J. Geophys. Res., **61,** No. 1 (Mar. 1956).

[5] H. Bremmer, Terrestrial radio waves (Amsterdam: Elsevier, 1949).

[6] R. L. Smith, R. A. Helliwell, I. W. Yabroff, A theory of trapping of whistlers in field-aligned columns of enhanced ionization," J. Geophys. Res., **65,** No. 3 (Mar. 1960).

(Paper 64D5–88)