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Use of Logarithmic Frequency Spacing in Ionogram Analysis[1]

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The use of logarithmic frequency spacing brings several advantages to the reduction of ionograms to electron density profiles. Among them is the fact that, when computing factors for the analysis, one need not determine the group refractive index. Formulas involving only the phase refractive index are presented; for the ordinary component one exact and one approximate formula are given, while for the extraordinary component there is an approximate formula valid over a wide range of geomagnetic latitudes. There is a brief discussion of quasi-longitudinal approximations to the extraordinary phase refractive index.

1. Introduction

In his important paper on obtaining electron density profiles, Budden [1][3] used for illustration a linear sampling of the frequency scale of the ionogram. He also mentioned the possibility of other samplings, including logarithmic. At about the same time, King [2] presented an illustration based on logarithmic sampling, and later [3] showed that this spacing simplifies inclusion of the earth's magnetic field in the analysis, by the use of an accurate approximation.

The methods of real height analysis using summation of spaced ordinates due to Kelso [4], Shinn (described by Thomas [5]), and Schmerling [6], are much easier to apply if the ionograms have a logarithmic frequency scale, for then overlays can be used. This derives from the fact that the properties of the propagation equation depend mainly on f_N/f, the ratio of the plasma frequency to the exploring frequency, and to a much less extent on the absolute frequencies (determined by f_H/f, where f_H is the gyrofrequency. As is usual in treating this subject, the effects of electron collisions are neglected.)

The purpose of this note is to inquire more closely into the advantages of logarithmic spacing and to present an approximation using it for analysis of the extraordinary ionogram trace.

2. General Equations

The basic equation relating the virtual height, h', and the real height, h_r, at the reflection point (assuming geometrical optics) is, [2]

$$h'=\int_0^{h_r}\mu' dh, \tag{1}$$

where μ' is the group refractive index. This can be rewritten

$$h'=\int_{\Phi_0}^{\Phi_r}\mu'\frac{dh}{d\Phi}d\Phi, \tag{2}$$

where Φ is any single valued function of the electron density. The integration can now be divided into a series of steps within which $dh/d\Phi$ is varying sufficiently slowly to be taken out of the integral sign.

$$h'_m=\sum_m\left(\frac{\Delta h}{\Delta\phi}\right)_m\int_{\Delta\phi m}\mu' d\phi \tag{3}$$

The integral $\int_{\Delta\Phi m}\mu' d\Phi$, which we shall denote by F_m, defines constant factors used in the analysis of the ionograms to obtain h. Once the table of factors is prepared, the reduction to real heights is simply a matter of solving a set of simultaneous equations in $(\Delta h/\Delta\Phi)_m$.

If now Φ is identified with the logarithm of f_N, a simplification results. (The natural logarithm, ln, will be discussed here to simplify the presentation, although common logarithms may be used in practice.)

For, at fixed f_N,

$$\mu'=\left[\frac{\partial(\mu f)}{\partial f}\right]_{f_N}$$

$$=\mu+\left[\frac{\partial\mu}{\partial\ln f}\right]_{f_N}, \tag{4}$$

where μ is the phase refractive index; it is a function of both f and f_N.

Using the theorem

$$\left[\frac{\partial\mu}{\partial\ln f}\right]_{f_N/f}=\left[\frac{\partial\mu}{\partial\ln f}\right]_{f_N}+\left[\frac{\partial\mu}{\partial\ln f}\right]_f \tag{5}$$

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[3] Figures in brackets indicate the literature references at the end of this paper.

relating the partial differential at fixed f_N/f to partial differentials at fixed f_N and at fixed f, we get

$$\mu' = \mu - \left[\frac{\partial \mu}{\partial \ln f_N}\right]_f + \left[\frac{\partial \mu}{\partial \ln f}\right]_{f_N/f}. \tag{6}$$

The factors used in the ionogram analysis then become

$$\int_{\Delta\Phi} \mu' d\Phi = \int_{\Delta \ln f_N} \mu' d\ln f_N$$

$$= \bar{\mu} \Delta \ln f_N - [\Delta\mu]_f + \overline{\left[\frac{\partial \mu}{\partial \ln f}\right]}_{f_N/f} \Delta \ln f_N, \tag{7}$$

where the "bars" denote mean values of the quantities over the interval.

All terms of eq (7) can be evaluated from a table of $\mu(\ln f, \ln f_N)$, so that one can avoid the tedious calculation of the group refractive index. In a recent paper, Titheridge [7] has given an expression with the same advantage.

3. Ordinary Component

While one would normally use the complete eq (7), it is worth noting that, for routine analysis of the ordinary component, the last term can be dropped [3]. Experience shows that the heights deduced from the ionograms are not seriously affected

The assumption here is

$$\left[\frac{\partial \mu}{\partial \ln f}\right]_{f_N/f} \doteq 0. \tag{8}$$

4. Extraordinary Component

While eq (7) applies formally to either the ordinary or the extraordinary component its use with the latter is made difficult by the rapid change of the last term near the gyrofrequency. By making the transformation

$$\xi^2 = f^2(1-y), \tag{9}$$

where $y = f_H/f$, one can adopt an assumption similar to eq (8),

$$\left[\frac{\partial \mu}{\partial \ln \xi}\right]_{f_N/\xi} \doteq 0, \tag{10}$$

and so simplify the computations.

The approximation of eq (10) will be considered in the next section where it will be shown to hold extremely well down to geomagnetic latitudes as low as 30° (where the propagation angle, θ, between the earth's magnetic field and the vertical is as large as 50°).

When eq (7) is put in terms of ξ and eq (10) is applied, one obtains

$$F_{xm} = \int_{\Delta \ln f_N} \mu' d \ln f_N = \bar{\mu} \Delta \ln f_N - \frac{d \ln \xi}{d \ln f} [\Delta \mu]_\xi, \tag{11}$$

where the subscript x in F_{xm} denotes the extraordinary component.

As constant ξ implies constant f, and

$$\frac{d\ln \xi}{d\ln f} = \frac{1-y/2}{1-y}, \tag{12}$$

the factors for analysis of the extraordinary component become

$$F_{xm} = \bar{\mu}\, \Delta \ln f_N - \frac{1-y/2}{1-y} [\Delta\mu]_f. \tag{13}$$

5. Phase Refractive Index for the Extraordinary Mode

In order to justify use of the approximation (10), a few remarks on quasi-longitudinal approximations to the phase refractive index of the extraordinary mode are appropriate.

Common approximations are [8],

$$\frac{1}{1-\mu^2} \doteq \frac{f^2}{f_N^2}(1-y\cos\theta) \tag{14}$$

and [9],

$$\frac{1}{1-\mu^2} \doteq \frac{f^2}{f_N^2}(1-y) = \xi^2/f_N^2. \tag{15}$$

The exact expression is

$$\frac{1}{1-\mu^2} = \frac{f^2}{f_N^2}\left(1 - y\cos\theta\cot\frac{\psi}{2}\right), \tag{16}$$

where

$$\tan\psi = \frac{2(f^2-f_N^2)}{f\, f_H}\, \frac{\cos\theta}{\sin^2\theta}. \tag{17}$$

Equations (14), (15), and (16) can be compared as follows: For the extraordinary trace, the *least* value of $\tan\psi$ occurs at the reflection point, where it is $2\cos\theta/\sin^2\theta$, so that the greatest value of $\cot \psi/2$ is $1/\cos\theta$.

That is to say,

$$1 < \cot\frac{\psi}{2} < \frac{1}{\cos\theta} \tag{18}$$

Therefore the exact value for $1/1\text{-}\mu^2$ (16) always lies between those given by the two approximate expressions (14) and (15), except at the reflection point, where (15) is exact.

Now if we let A represent the ratio of (16) to (14), and B the ratio of (16) to (15), i.e.,

$$A = \frac{1 - y\cos\theta\cot\frac{\psi}{2}}{1-y\cos\theta}, \tag{19}$$

and

$$B = \frac{1 - y\cos\theta\cot\frac{\psi}{2}}{1-y}, \tag{20}$$

the following table compares the ratios for $y=\frac{1}{2}$ and various values of $(f_N/f)^2$, at different geomagnetic latitudes.

Latitude	$\left(\frac{fN^2}{f}\right)=$	0.5	0.4	0.3	0.2	0.1	0.0
14°	A	0.644	0.714	0.762	0.797	0.823*	0.844*
	B	1.000	1.109	1.184	1.238	1.278	1.310
23½°	A	0.743	0.792	0.825	0.850	0.868*	0.883*
	B	1.000	1.065	1.110	1.143	1.168	1.187
30°	A	0.804	0.840	0.864	0.883	0.897*	0.908*
	B	1.000	1.045	1.075	1.098	1.116	1.129
45°	A	0.904	0.921	0.933	0.941	0.948	0.954*
	B	1.000	1.018	1.031	1.041	1.048	1.054
60°	A	0.962	0.969	0.973	0.977	0.979	0.982*
	B	1.000	1.007	1.011	1.015	1.018	1.020

Only in those positions of the table marked by an asterisk is A closer to unity than B. Clearly, then, the approximation (15) is the better, especially near the reflection point $[(f_N/f)^2=1-y]$ where the need for accuracy is greatest. Accepting eq (15), eq (10) follows.

Writing the exact expression, using eqs (16), (20), and (9), as

$$\frac{1}{1-\mu^2}=\frac{\xi^2}{f_N{}^2}B, \tag{21}$$

it can be shown that, because of the slow rate of change of B with $(f_N/f)^2$ (and hence with $\xi^2/f_N{}^2$), eq (10) is applicable to ionogram analysis over a wider range of latitudes than eq (15). Rydbeck [9] used a relation equivalent to eq (15) in a method of ionogram analysis for $\theta<20°$ (latitudes greater than 54°), where it certainly holds within the accuracies to which the ionograms can be read. Use of eq (10) extends the range of application down to geomagnetic latitude 30°.

6. A Numerical Example

The following tables illustrate the computation of the factors F_{xm} for analyzing the extraordinary ionogram trace.

The first table gives μ as a function of $\log_{10} f_N$ and $\log_{10}\xi$. This was calculated using eq (16).

Table of $\mu\times 10^4$		$f_H=1.4753$ Mc/s				$\theta=21°53'$				
f Mc/s	$\log\xi$	$\log f_N$								
		0.32	0.28	0.24	0.20	0.16	0.12	0.08	0.04	0.00
2.95	0.32	0000	4209	5658	6610	7302	7827	8236	8560	8821
2.78	0.28	------	0000	4210	5660	6613	7305	7830	8239	8562
2.63	0.24	------	------	0000	4211	5662	6616	7308	7833	8242
2.49	0.20	------	------	------	0000	4212	5664	6619	7311	7836
2.36	0.16	------	------	------	------	0000	4214	5667	6622	7314
2.25	0.12	------	------	------	------	------	0000	4215	5669	6625
2.15	0.08	------	------	------	------	------	------	0000	4217	5672
2.06	0.04	------	------	------	------	------	------	------	0000	4219
1.98	0.00	------	------	------	------	------	------	------	------	0000

Lines parallel to the diagonal in the table define values of μ for constant values of the ratio f_N/ξ. Differences between the values along such a line, therefore, give $[\partial\mu/\partial\log\xi]_{f_N/\xi}$; and as this is very small compared with $[\partial\mu/\partial\log\xi]_{f_N}$ and $[\partial\mu/\partial\log f_N]_\xi$, the assumption made in section 4 (eq (10)) is quite satisfactory.

From this table one can obtain the components $\bar{\mu}\Delta\log f_N$ and $M[(1-y/2)/(1-y)]\,[\Delta\mu]_f$, of the factors F_{xm} (eq (13)). Here, $M=0.4343$ is the conversion factor to change from natural logarithms to common logarithms. Thus for $\log\xi=0.24$, we obtain the factor for the interval $\Delta\log f_N$ (0.16—0.12) by adding $\bar{\mu}\log f_N$ (0.0246) and $-M\dfrac{1-y/2}{1-y}[\Delta\mu]_f$ (0.0680) giving a value of 0.0926.

The table of $F_{xm}\times 10^4$ is then,

$\log\xi$	Interval of $\log f_N$										
	0.28 to 0.32	0.24 to 0.28	0.20 to 0.24	0.16 to 0.20	0.12 to 0.16	0.08 to 0.12	0.04 to 0.08	0.00 to 0.04	−0.04 to 0.00	−0.08 to −0.04	−0.12 to −0.08
0.32	2855	1142	866	730	645	587	574	518	494	476	462
.28	------	2976	1184	894	749	660	599	556	525	500	481
.24	------	------	3117	1233	926	772	677	613	566	534	506
.20	------	------	------	3280	1290	964	799	697	628	579	544
.16	------	------	------	------	3470	1356	1006	830	721	646	593
.12	------	------	------	------	------	3692	1434	1058	867	749	668
.08	------	------	------	------	------	------	3954	1525	1117	910	780
.04	------	------	------	------	------	------	------	4262	1631	1187	960
.00	------	------	------	------	------	------	------	------	4625	1757	1269

This example is taken from part of a table prepared for testing methods of ionogram analysis which use both the ordinary and extraordinary traces to give information on the unobserved parts of the ionosphere [10], [11], [12].

7. Conclusions

The use of logarithmic frequency spacing allows easy computation of factors for the real height analysis of ionograms, using tables of phase refractive index; the group refractive index need not be calculated.

For the ordinary wave component, there is an exact formula for the factors eq (7), and an approximate formula good enough for most work. The formula for the extraordinary wave component eq (13) contains an approximation, but it can be used with negligible error for analysis of ionograms from any but the lowest latitudes.

In the justification of the extraordinary wave approximation it was shown that the simplest quasi-longitudinal approximation to the phase refractive index eq (15) is better than one in common use eq (14).

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