JOURNAL OF RESEARCH of the National Bureau of Standards—D. Radio Propagation
Vol. 64D, No. 5, September–October 1960

Methods of Predicting the Atmospheric Bending of Radio Rays[1]

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(March 3, 1960; revised March 22, 1960)

Three methods for predicting the bending of radio rays when the refractive index profile above the surface layer is unknown have been developed recently by the authors. These methods are: a statistical technique for refraction at high initial elevation angles, estimation of bending from an exponential model of atmospheric refractive index, and a modification of the exponential model to account for the heavily weighted effects of anomalous initial refractive index gradients at small initial elevation angles. Each model is dependent upon the value of the refractive index at ground level or, in the case of superrefraction, the additional knowledge of the refractive index gradient next to the earth's surface. Each method works best in a particular range of initial elevation angles or meteorological conditions. The height and angular ranges of application of each method are checked by comparison with values obtained from 77 diverse refractive index profiles representative of wide climatic variation. It is found that the use of the best of the three methods will always result in a prediction of the total atmospheric bending within 10 percent for initial elevation angles from zero to 10 milliradians and to within 4 percent for initial elevation angles greater than 17 milliradians (~ 1 deg).

Glossary of Terms

n=the radio refractive index.

n_s=the value of n at the earth's surface.

N=the radio refractivity, $N\equiv (n-1)\times 10^6$.

N_s=value of N at the earth's surface.

ΔN=difference between N_s and the N value at one km above the surface, $-\Delta N\equiv N_s-N_1$.

$(dN/dh)_0$=gradient of N with respect to height, dN/dh, evaluated at the earth's surface.

r=vector radius from the center of the earth.

r_0=radial distance from the center to the earth's surface.

h=height above the surface, $h\equiv r-r_0$.

θ=elevation angle of a radio ray, the (acute) angle between the tangent to the ray path and the local horizontal (i.e., perpendicular to the radius vector.)

θ_0=the value of θ at the ray path origin (transmitting or receiving point).

θ_p=the angle of penetration for a radio duct, i.e., the smallest value of θ_0 for which the radio ray will *not* be trapped, or conversely, the largest θ_0 for which the ray *will* be trapped.

τ=the angular refraction, or bending, of a radio ray.

ϵ=elevation angle error, the angular difference between θ_0 and the true elevation angle to a target at a given point on the ray path.

[1] Contribution from Central Radio Propagation Laboratory, National Bureau of Standards, Boulder, Colo.

1. Introduction

Recent years have seen considerable activity in the evaluation of refraction effects in the troposphere. Schulkin [1][2] outlined a simple method for refraction calculations and applied it to determine the mean refraction expected in arctic, temperate, and tropical climates. Fannin and Jehn [2] made an extensive analysis of elevation angle errors expected in various air masses and geographic locations for initial elevation angles in excess of 3 deg above the horizontal. In a series of papers [3,4,5,6] the present authors have examined atmospheric refractive index structure, the effect of this structure upon radio-ray refraction and have evolved methods of estimating the refraction of radio rays for *all* initial elevation angles. These methods are unique in that they depend only upon a knowledge of the refractive index at the earth's surface or, in the case of superrefraction, the gradient of the refractive index in the earth-boundary layer. Thus distributions of elevation angle error, angular bending, and other refraction effects may be determined for the majority of practical applications by simple reference to distributions of the surface value of the refractive index such as those for the United States [7]. It is assumed of course, that these various methods will be applied only when either details of the actual refractive index profile are unknown or it is impractical to obtain these details. It is under this assumption that this paper tests the relative accuracy of these various prediction methods to arrive at a delineation of conditions under which each method works best.

[2] Figures in brackets indicate the literature references at the end of this paper.

2. Theory and Background

A basic measure of atmospheric refraction effects on high frequency radio propagation is the bending, or angular refraction, of individual rays. By assuming the refractive index to be a function only of height above the surface of a smooth, spherical earth rays can be traced using Snell's law in the following form [1]:

$$nr \cos \theta = n_s r_0 \cos \theta_0. \tag{1}$$

The geometry is shown in figure 1. The equation for the ray bending, τ, can be obtained from eq (1) as [1, 9]

$$\tau_{s,1} = -\int_{n_s}^{n_1} \frac{dn}{n} \cot \theta \tag{2}$$

where $\theta = \cos^{-1}\left\{\frac{n_s r_0 \cos \theta_0}{n_1 r},\right\},$

and $\cot \theta$ is given to a very high degree of approximation by:

$$\cot \theta = \frac{\cos \theta_0}{\sqrt{\sin^2 \theta_0 + \frac{2h}{r_0} - 2(N_s - N) \times 10^{-6} \cos^2 \theta_0}}. \tag{3}$$

It can be seen from inspection of eq (2) and (3) that in order to evaluate τ directly it is necessary to know the refractive index of the atmosphere, n as a function of height, h.

How one evaluates eqs (2) and (3) depends upon the availability of data and computing facilities. If details of the refractive index profile are available

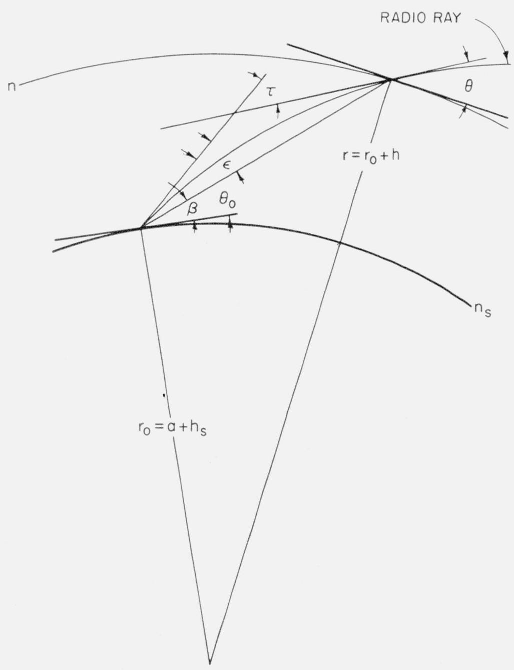

FIGURE 1. *Geometry of radio-ray refraction.*

from either refractometer or radiosonde ascents, then one could evaluate refraction effects with the aid of of an electronic computer or simplified graphical techniques [9]. However, frequently the desired refractive index profile data are simply not available and one must fall back upon less exact methods of estimating τ. One may use average values of τ such as given by Schulkin or the distributions of refraction effects given by Fannin and Jehn. However, simple, standard measurements of pressure, temperature, and humidity at the earth's surface are almost always available and may be used to estimate tropospheric refraction effects. In the sections that follow, the evaluation of the integral for τ will be examined from three different viewpoints:

1. Simplification of eqs (2) and (3), to permit evaluation of τ without knowledge of details of the actual refractive index profile.
2. Evaluation of τ for actual observed refractive index profiles and the statistical reduction of the data so derived into a function of some observable parameter (e.g., N_s).
3. Construction of an analytic model of $n(r)$ for normal conditions, thus yielding expected values of τ by direct integration of the model.

3. τ as a Function of N_s

As a first approximation towards evaluation of τ without detailed knowledge of $n(r)$, consider the integration by parts of eq (2):

$$\tau_{s,1} = -\int_{n_s}^{n_1} \frac{dn}{n} \cot \theta = -\ln \{n\} \cot \theta \Big]_{n_s}^{n_1} + \int_{\cot \theta_0}^{\cot \theta_1} \ln \{n\} d(\cot \theta)$$

or:

$$\tau_{s,1} = \ln \{n_s\} \cot \theta_0 - \ln \{n_1\} \cot \theta_1 - \int_{\theta_0}^{\theta_1} \ln \{n\}\, csc^2 \theta d\theta$$

Now since $n = 1 + N \times 10^{-6}$ and $N \times 10^{-6} < 5 \times 10^{-4}$

then

$$\ln \{n\} = N \times 10^{-6} \{1 - \tfrac{1}{2}(N \times 10^{-6}) + \ldots\} \cong N \times 10^{-6}$$

with this approximation (i.e., a maximum error of less than 0.2%) the above equation becomes

$$\tau_{s,1} \cong N_s \times 10^{-6} \cot \theta_0 - N_1 \times 10^{-6} \cot \theta_1 - \int_{\theta_0}^{\theta_1} N \times 10^{-6}\, csc^2 \theta d\theta \text{ for } \theta_0 \pm 0. \tag{4}$$

The integral in eq (4) has been found to contribute no more than 3 percent to the value of $\tau_{s,\infty}$ for an initial elevation angle of 10 deg or greater [1] while the second term is zero due to $N_\infty \equiv 0$. Thus the first term of (4) forms an approximation to $\tau_{s,\infty}$ which is asymptotic to the true value of $\tau_{s,\infty}$ as θ_0

approaches 90 deg. It may be shown that for normal conditions and all heights the integral in (4) is essentially independent of N_s for $\theta_0 > 17$ mr (~ 1 deg); the term $N_1 \cot \theta_1$ tends to be constant; thus (4) reduces to a linear equation

$$\tau_{s,1} \cong b_1 N_s + a_1. \tag{5}$$

The form of eq (5) is very attractive, since it implies two things:

1. $\tau_{s,1}$ may be predicted with some accuracy as a function only of N_s (h_1 and θ_0 constant), a parameter which may be observed from simple surface measurements of the common meteorological elements of temperature, pressure, and humidity.
2. The simple linear form of the equation indicates that, given a large mass of observed $\tau_{s,1}$ versus N_s for many values of h and θ_0, the expected (or best estimate) values of b and a can be obtained by the standard method of statistical linear regression.

The method of attack indicated by implication number two has, in fact, been carried out by the authors [3]. The results show that for $h_1 = \infty$ the method is accurate to within ± 3 percent of the true value (as an rms error) for initial elevation angles as small as 1 deg. The accuracy of this method and the following methods, will be examined for 0.1 km$\leq h \leq \infty$ in the following sections.

4. Exponential Model

The development of a model of $N(h)$ to describe the normal behavior of atmospheric N as a function of N_s and height has received a considerable amount of treatment in the past. One of the simplest, and, at the same time, most accurate models which has emerged from these studies is that in which $N(h)$ has an exponential decrease with height [4, 5, 10],

$$N(h) = N_s \exp\{-c_e h\}. \tag{6}$$

One of the earliest applications of this particular model has been attributed by Garfinkel to Sir Isaac Newton, who used the exponential form in a study of astronomical refraction [11].

If it is assumed that $N(h)$ is indeed an exponential function of height, then the gradient of $N(h)$ would also be an exponential function of height. The most extensive source of data with which to evaluate the coefficients in the exponential is that of ΔN (the value of N at 1 km minus the surface value, N_s) which has received wide application in radio propagation problems [12]. Thus we would expect

$$\frac{\Delta N}{\Delta h} = k_1 \exp\{-k_2 h\} \tag{7}$$

to take the form

$$\Delta N = k_1 \exp\{-k_2\}$$

for our special case of $\Delta h = h = 1$ km. Examination of the ΔN data soon revealed that k_2 was dependent upon N_s i.e., the higher the surface value of N the greater the expected drop in N over 1 km. Examination of the data indicated that

$$k_2 = k_3 N_s$$

and the resultant equation,

$$\Delta N = k_1 \exp\{-k_3 N_s\} \tag{8}$$

was solved by least squares. The least squares determination was facilitated by converting (8) to the form

$$\ln|\Delta N| = -k_3 N_s + \ln k_1 \tag{9}$$

or, in words, expressing the natural logarithm of ΔN as a linear function of N_s. The values of k_1 and k_3 were established from some 888 sets of 8-year means of ΔN and N_s from 45 U.S. weather stations. The results of this study are shown graphically in figure 2 where the least squares exponential fit of $\overline{\Delta N}$ and $\overline{N}_s$ is given by:

$$-\overline{\Delta N} = 7.32 \exp\{0.005577 \overline{N}_s\}. \tag{10}$$

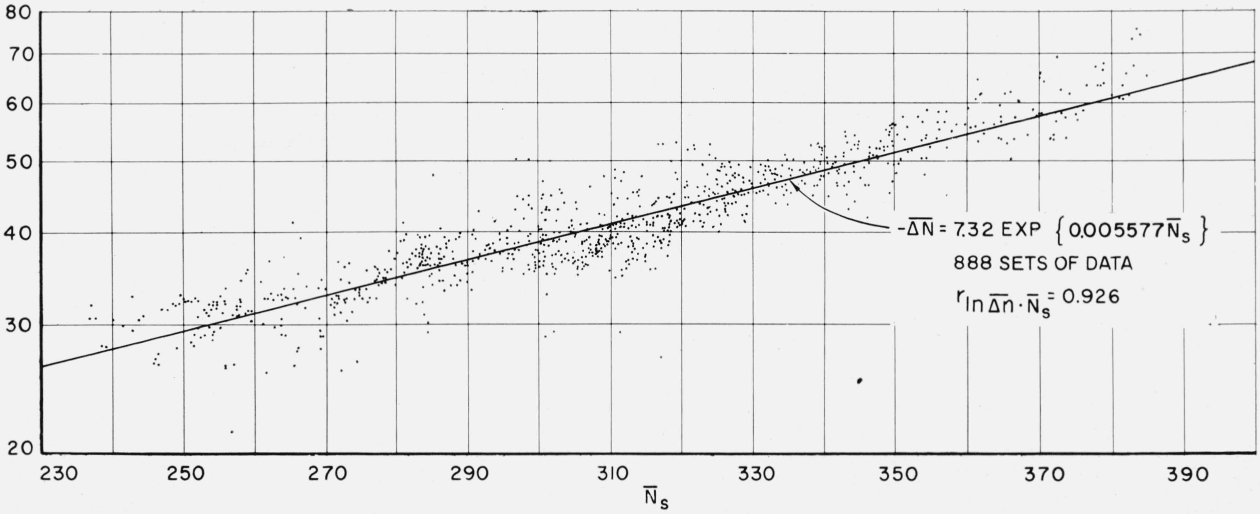

FIGURE 2. *Regression of ln $|\overline{\Delta N}|$ upon Ns.*

From this equation the CRPL Exponential Reference Atmosphere [5] was determined, the profiles being defined by the following equations:

$$N = N_s \exp \{-c_e h\}$$

$$c_e = ln \left\{ \frac{N_s}{N_s + \Delta N} \right\}. \qquad (11)$$

Ray tracings have been computed for this model covering more than the normal range of N_s, and the results may be used, either in tabular or graphical form, to predict τ for any normal combination of N_s, θ_0, and height [5].

5. Initial Gradient Correction Method

The importance of the initial gradient in radio propagation where the initial elevation angle of a ray path is near zero has long been recognized. For example, if $dN/dh = 1/r_0$ then $\Delta\tau = \infty$, an expression of the fact that the ray path will travel at a constant height above the earth's surface. This is called ducting, or trapping of the radio ray. The effect of anomalous initial N-gradients on ray propagation at elevation angles near zero, and for gradients less than ducting, ($|dN/dh| < 157$/km, or $dn/dh > -157$/km), may also be quite large. A method has been presented [4] for correcting the predicted refraction (from the exponential reference atmosphere) to account for anomalous initial N-gradients, assuming that the actual value of the initial gradient is known. The result is,

$$\tau_h = \tau_h(N_s, \theta_0) + [\tau_{100}(N_s^*, \theta_0) - \tau_{100}(N_s, \theta_0)] \qquad (12)$$

where $\tau_h(N_s) = \tau$ at height h, for the exponential reference atmosphere corresponding to N_s and N_s^* is the N_s for the exponential reference atmosphere having the same initial gradient as that observed; τ_{100} is τ at 100 meters height.

This procedure has the effect of correcting the predicted bending by assuming that the observed initial gradient exists throughout a surface layer 100 meters thick, calculating the bending at the top of the 100-meter-thick layer, and assuming that the atmosphere behaves according to the exponential reference profile corresponding to the observed value of N_s for all heights above 100 meters. This approach has proven quite successful in predicting τ for initial elevation angles under 10 mr, and will, of course, predict ducting when it occurs.

6. Analysis of the Accuracy of the Prediction Methods

A test sample of ray bendings for the range of N-profiles likely to be encountered was prepared from 77 refractivity profiles derived from radiosonde observations. These 77 profiles represent both normal and extreme refractivity profiles for 13 climatically diverse locations in the United States and are actually representative of a nearly worldwide range of conditions [13]. Values of τ were calculated by numerical integration for values of θ_0 from 0 to 900 mr. (For a thorough discussion of the ray-tracing techniques employed see reference [5].) The results of this general refraction study provided a large mass of data for checking the accuracy of each of the prediction methods.

The relative accuracy of the exponential model and the initial gradient correction method were tested by predicting the bending at particular heights and initial elevation angles for the 77 sample profiles and finding the rms error of prediction for each case. Since 13 of the 77 profiles had surface ducts they could be used only for elevation angles greater than the angle of penetration for each case, given by:

$$\theta_p = \sqrt{0.2[156.9 - (dN/dh)_0]} \times 10^6 \text{ radians} \qquad (13)$$

where $(dN/dh)_0$ is the observed initial gradient of N per km, assumed to extend over 100 m. Thus there were only 64 profiles analyzed at $\theta_0 = 0$, and 77 at $\theta_0 = 10$ mr. These same data were also used to derive the regression of τ upon N_s for various heights and initial elevation angles. The scatter of points about the regression line was then used as an estimate of the minimum rms error that would be expected from any of the three prediction methods. Note carefully, however, that the regression lines are a "best fit" to the test data. Although a future sample of data would presumably yield similar rms deviations, the possibility exists that the present sample data are systematically biased, in this case there would be an additional error encountered in practice due to this bias.

The primary purpose of the present analysis is to determine over which regions of height and θ_0 each method will give the best results. It was found that the statistical correlation method is more accurate from 1 deg to vertical incidence and that for all altitudes in this range of θ_0 it is the most accurate of the three methods. At θ_0 smaller than 1 deg, and especially at $\theta_0 < 10$ mr, the methods based on the exponential reference atmosphere, particularly the corrected exponential reference atmosphere, are more accurate. Figure 3 illustrates these conclusions by

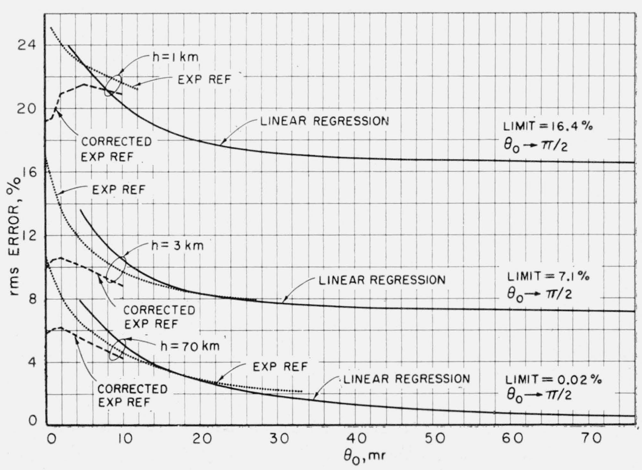

FIGURE 3. *RMS error of predicting τ at various heights as a percent of mean τ, excluding superrefractive profiles.*

comparing the rms error of prediction as a percent of mean τ for all three methods at three different heights and over a large range of θ_0, excluding all superrefractive profiles. A superrefractive profile is here defined as one with an initial gradient of N in excess of 100 N-units per km, i.e., $(dN/dh)_0 < -100$/km.

It is evident from figure 3 that the percentage error of predicting τ decreases with increasing thickness of the atmosphere through which the ray passes. This is as one would expect since the value of the refractive index becomes less variable with increasing height and, one might say, the upper limit of integration of (2) becomes more a function of the lower limit, N_s. Further, the sensitivity of refraction effects to low-level profile anomalies for small values of θ_0 is reflected by the relatively small percentage error of the corrected exponential model for $\theta_0 < 10$ mr.

It must be remembered that the statistical regression technique is only *apparently* superior to methods using the exponential reference atmosphere since, by definition, it must have a minimum rms error provided that τ is a linear function of N_s. Although the difference in the percentage rms error between these two methods appears quite large, the actual rms error of the exponential reference atmosphere is either less than or within 0.1 mr of that of the statistical correlation technique for $\theta_0 \geq 7$ mr, thus, for practical considerations, indicating no clear-cut superiority of one technique over the other; this is illustrated in figure 4.

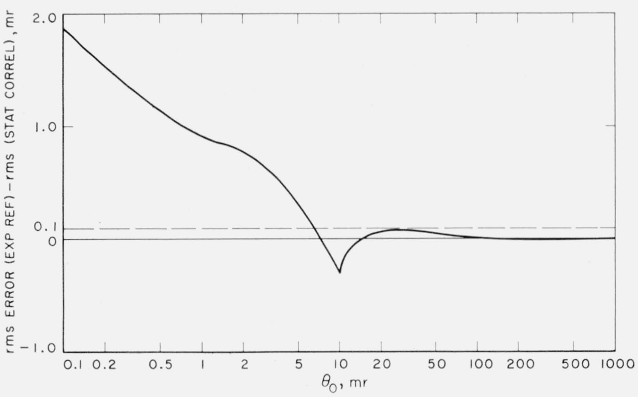

Figure 4. *Difference between rms error of predicting τ at 70 km for the exponential reference atmosphere and the statistical correlation method.*

One may evaluate the minimum rms error expected in predicting τ by these three methods by defining a composite prediction method that utilizes the best of the three methods in each range of θ_0 and height to yield a minimum overall error.

The numerical value of the rms deviations for the optimized composite of the three methods is shown on figure 5 for heights of 1, 3, and 70 km and all profiles. It is seen that the maximum error for any case is about 2 mr and decreases to 1 mr or less for $\theta_0 \geq 10$ mr for all height increments. The error for the total bending case, h=70 km, drops below an rms value of 1 mr for $\theta_0 \geq 5$ mr.

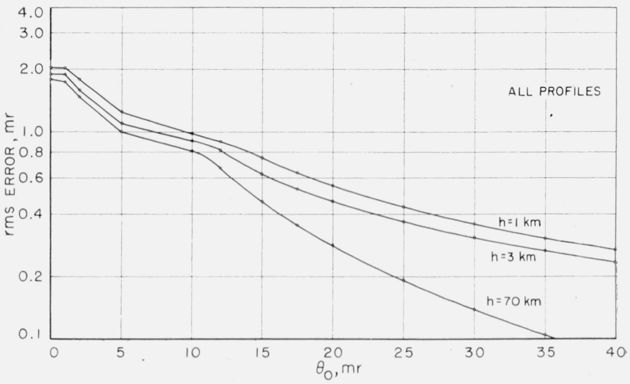

Figure 5. *RMS error of predicting τ, composite prediction method.*

7. Conclusions

The present study appears to indicate that:

1. The statistical regression technique is an adequate solution to the bending problem for all θ_0 larger than about 10 mr, and all heights from 1 km up.

2. The exponential reference atmosphere is equally as good as the statistical technique over the same range of θ_0 and height.

3. The initial gradient correction method is very useful for $\theta_0 < 10$ mr, and can be used to extend predictions of τ down to a θ_0 of 0 for any model which yields N as a function of height.

These conclusions indicate that the reader who desires the quick evaluation of some refraction effect should consult the rather extensive tables of the CRPL Exponential Reference Atmosphere [5]. These tables would allow, for instance, the determination of the elevation angle error as a function of θ_0, N at the earth's surface, and the radar range. The reader desiring the convenience of the statistical regression technique for estimating either τ or the elevation angle error may obtain the necessary statistical parameters from the literature [14].

The authors express their gratitude to Mrs. B. J. Weddle, Mrs. M. A. Fischer, and Mrs. G. M. Richmond for their aid in the calculations of this study.

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