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Amplitude Distribution For Radio Signals Reflected by Meteor Trails. I[1]

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The probability distribution for the envelope of the received signal composed of reflections from many meteor trails is derived theoretically. Both the effects of numerous, small meteors and the residual reflections from infrequent, large meteors are treated simultaneously. For the particular example of exponential decay of initial spikes which are themselves distributed as the inverse square of their amplitudes, we find that the probability that the composite residual signal amplitude exceeds a prescribed level r is given by

$$P(R>r)=\frac{1}{\left[1+\frac{r^2}{(\nu\eta Q)^2}\right]^{1/2}}.$$

This function behaves as a Rayleigh distribution for small amplitude margins r. For the larger, less likely amplitudes it agrees with the result predicted by elementary analysis of isolated meteor reflections. Possible refinements of these results are also discussed. A second paper will discuss time correlation of composite meteor signals at different times.

1. Introduction

Backscattering of radiowaves by meteor trails in the E region of the ionosphere is a valuable direct means for studying meteors. VHF signals are also propagated obliquely to as far as 1,500 km by oblique reflections from the same meteor trails. Signals reflected from the largest meteors are easily recognized as individual spikes in amplitude records. There are also overlapping signal contributions from much smaller meteors which cannot be so distinguished.

The smaller meteors have been suggested as a possible source of the continuous background signal observed on the VHF scatter circuits. To distinguish between the signal due to turbulence and that due to small meteors, the cumulative probability distribution for signal amplitudes has been measured for narrow beams directed both on and off a great circle path. However, a theoretical distribution for overlapping meteors does not seem to have been developed thus far, and this paper is addressed to that problem.

The very small meteors can be analyzed if one considers only the meteor signals which arrive at the precise instant of signal evaluation. A vector combination of many randomly oriented (phased) signal vectors is known to follow a Rayleigh distribution. The corresponding probability that the echo signal lies in the range R to $R+dR$ is:

$$\text{Small:}\qquad W(R)RdR=\frac{RdR}{\sigma^2}e^{-R^2/(2\sigma^2)} \tag{1.1}$$

where

$$\sigma^2=\tfrac{1}{2}\langle R^2\rangle \tag{1.2}$$

is the mean square voltage in the ensemble of meteor echoes. Although this description does recognize a distribution of meteor signals, it is deficient in that it ignores the residual effect of meteor signals created prior to the measuring instant. Even though such signals may have experienced appreciable decay, their combined effect may make a significant contribution to the distribution. This is especially true of the larger meteors, which have a poorer chance of occurring precisely at the instant of measurement, although their residual signal may still be comparatively large.

The very large meteors can be treated as isolated random events. The probability of receiving such an echo signal with an initial pulse height lying between p and $p+dp$ (volts) is experimentally found to follow a distribution of the form

$$D(p)dp=\frac{Q}{p^{2+\epsilon}}dp, \tag{1.3}$$

where the parameter ϵ is commonly taken to be zero for analytical convenience. The residual signal left after t seconds is adequately described by an exponential decay of the initial spike p.

$$R=pe^{-t/\eta}, \tag{1.4}$$

where η is the characteristic (diffusion) decay time of the meteor trail itself. The probability that the residual signal exceeds a prescribed level r is thus an interlocking marginal average over the distribution of observing a signal of exactly strength p and the probability of having received an echo at all. Since the echoes are found to occur at random at an average rate ν,

$$P(R>r)=\int_0^\infty dt\nu\int_{r/e^{-t/\eta}}^\infty dpD(p)=\nu\eta\frac{Q}{r} \tag{1.5}$$

[1] Contribution from Space Technology Laboratories, Inc., P.O. Box 95001, Los Angeles 45, Calif.

The distribution $W(R)$ for the signal produced by large isolated meteor echoes is obtained from this result by differentiation.

$$\text{Large:} \qquad W(R)dR = \nu\eta Q \frac{dR}{R^2} \tag{1.6}$$

This form is evidently quite different in nature from the Rayleigh distribution (1.1) ascribed to the smaller meteor contributions. However, these two results will emerge as asymptotic behaviors of a distribution which accounts for the effect of both the large and small meteors simultaneously. This distribution is derived in section 3, after the basic probabilistic expressions are developed in section 2. The bivariate probability density function for observing two meteor echo signals within prescribed ranges at different times will be discussed in a second paper on the subject.

2. General Amplitude Distribution Expressions

To derive the statistical distribution of the fading signal amplitude produced by a variety of meteor signals, one must recognize a spectrum of echo signal strengths in various stages of decay. It is convenient to tabulate the random occurrence of each meteor echo according to the envelope amplitude p with which the echo first appears. A typical sequence of meteor echoes is so separated in figure 1. The individual signals are randomly phased as they arrive, but figure 1 plots only the envelope magnitudes, independent of phase. The larger, less frequent signals are plotted on the top line as they might occur in time; with the smaller, more frequent echoes plotted on the lower scales. Actually, we shall wish to deal with a continuum of initial echo amplitudes p, and one should really show an infinite number of traces to handle each signal size interval p to $p+dp$.

At any given time, the total measured signal is the vector summation of the individual residual signals produced by each meteor in all size classes. Of course, the pulses which occur closest to the measuring instant produce the greatest remnant signals. On the other hand, there are an infinite number of very small signal remnants in the receiver from *all* previous meteors which may well contribute significantly to the composite total signal. To calculate the precise distribution in which both effects take their balanced roles, we use the Markoff method. The application of this method to the meteor echo problem follows closely Chandrasekhar's derivation of the Holtzmark distribution [1] for stellar attractive forces.

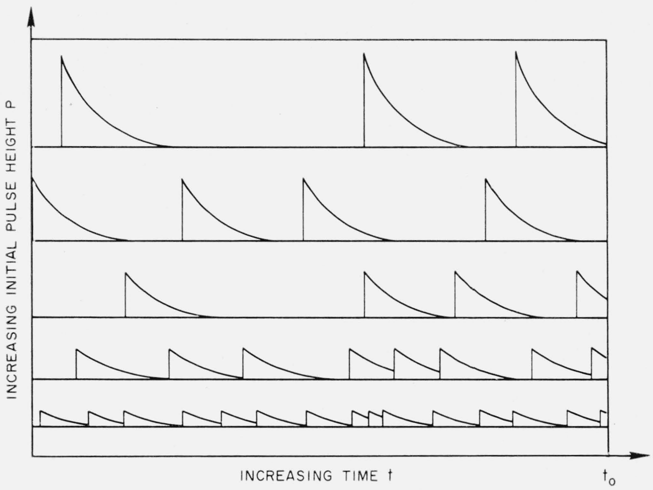

FIGURE 1. *Typical occurrence history of individual meteor echoes arranged according to increasing initial pulse height.*

Consider first a finite time interval T prior to the time of measurement. The number of meteors which are likely to have occurred during this fixed interval is, of course, a random variable. Let us suppose, however, that exactly N meteor echoes occur in this interval. Since the meteor echoes form a Markoff process of small probability, one can argue that the probability of observing exactly N echoes in a fixed interval T should follow a Poisson distribution.

$$P(N|T) = \frac{(\nu T)^T}{N!} e^{-\nu T}, \tag{2.1}$$

The average number of meteors to be expected in an interval T is νT, and this estimate becomes sharper as this interval is lengthened. Let us assume that N is fixed for the moment, and label the individual meteor echoes by a subscript i. The residual vector signal S_i remaining at the measuring instant t_0 produced by an initial pulse $\vec{p}_i$ at time t_i becomes:

$$\vec{S}_i = \vec{p}_i F(t_0 - t_i), \qquad 1 < i < N, \tag{2.2}$$

where $F(\tau)$ is the form factor which describes the temporal decay of the initial pulse. The composite signal at t_0 is the vector sum of all N residual signals.

$$\vec{R} = \sum_{i=1}^{N} \vec{S}_i, \tag{2.3}$$

According to Markoff's method, the probability distribution for the total measured vector $\vec{R}$ at time t_0 is the two-dimensional Fourier transform of a finite product taken over the set of initial echoes.

$$W(\vec{R}) = \frac{1}{(2\pi)^2} \int d^2k e^{i\vec{k}\cdot\vec{R}} A(k), \tag{2.4}$$

where

$$A(k) = \left\langle \prod_{i=1}^{N} \exp i\vec{k}\cdot\vec{p}_i F(t_0 - t_i) \right\rangle. \tag{2.5}$$

In the definition of $A(k)$, the averaging brackets must sum over all possible: (1) times of echo occurrence t_i, (2) initial echo vector pulse amplitude $\vec{p}_i$, and (3) total number of echoes N in the interval

[1] S. Chandrasekhar, Stochastic problems in physics and astronomy, Rev. Mod. Phys. **15**, 1 (1943).

T. The initial pulses $\vec{p}_i$ are independent of one another, since multiple (i.e., trail-to-trail) scattering is apparently unimportant, and there is insignificant gravitational interaction between the meteors. The infinite product thus becomes:

$$A(k)=\left\langle\left[\int d^2 p \int_0^T dt\, \gamma\, (\vec{p},t)e^{i\vec{k}\cdot\vec{p}F(t_0-t)}\right]^N\right\rangle_N, \quad (2.6)$$

where $\gamma(\vec{p},t)$ is the probability that a *single* meteor echo occurs at time t and produces a vector signal p in the receiver.

The average over N can be performed by multiplying with the probability (2.1) of observing exactly N echoes in the interval T and summing over all N.

$$A(k)=\sum_{N=0}^{\infty}\frac{(\nu T)^N}{N!}\, e^{-\nu T}$$

$$\left[\int d^2p \int_0^T dt\, \gamma\, (p,t)e^{i\vec{k}\cdot\vec{p}F(t_0-t)}\right]^N$$

$$= \text{xp}-\nu T\left[1-\int d^2p\int_0^T dt\, \gamma\, (p,t)e^{i\vec{k}\cdot\vec{p}F(t_0-t)}\right]. \quad (2.7)$$

To proceed further, one must examine the probability density function $\gamma(\vec{p},t)$ for a single echo pulse. If we were to examine the interval T in an a priori fashion, we could estimate that $N=\nu T$ echoes would most probably occur somewhere in the interval. However, their actual time of occurrence could not be predicted at all accurately, and one could only say that an individual meteor is equally likely to occur anywhere in the interval, viz,

$$\gamma\, (\vec{p},t)=\frac{1}{T}\gamma\, (\vec{p}). \quad (2.8)$$

One can exploit this form in equation (2.7) by noting that

$$\frac{1}{T}\int d^2\vec{p} \int_0^T dt\, \gamma\, (\vec{p})=1,$$

since $\gamma(\vec{p})$ itself must be normalized to unity. Substituting this expression for the one in the exponent of (2.7) allows one to cancel off the arbitrary finite time interval T.

$$A(k)=\exp-\nu \int d^2p \int_0^T dt\, \gamma\, (\vec{p})\, [1-e^{i\vec{k}\cdot\vec{p}F(t-t_0)}] \quad (2.9)$$

At this stage one can safely take the limit of infinite sample length, $T\rightarrow\infty$, since the exponential term's unit value for large time displacements (i.e., F small) is now cancelled in the integrand.

One can further reduce expression (2.9) by recalling that the initial echo pulses are randomly phased, since the distance from the transmitter to each meteor (and back) is a random variable, when expressed in wavelength units.

$$\int d^2p\gamma\, (p)=\int_0^\infty dpp \int_0^{2\pi} d\phi\, \frac{D(p)}{2\pi p}, \quad (2.10)$$

Here $D(p)$ is the distribution of initial pulse heights, and ϕ is the angle between p and a convenient reference, which we choose as the transform vector k. One can now use the integral definition of the zero-order Bessel function to carry out the angular ϕ integration in (2.9).

$$A\,(k)=\exp-\nu \int_0^\infty dt \int_0^\infty dpD(p)\ \{1-J_0\,[kpF\,(t_0-t)]\} \quad (2.11)$$

This expression, in conjunction with the Fourier transform (2.4), represent the formal solution to the problem at hand. To proceed further with the calculation of the probability density, one must assume explicit forms for the temporal decay function $F(\tau)$ and the pulse height distribution $D(p)$.

3. Meteors Which Decay Exponentially

Most of the smaller, underdense meteor echo signals are found to decay exponentially, viz,

$$F\,(t)=e^{-t/\eta}, \quad (3.1)$$

corresponding to molecular diffusion of reflecting electrons in the ionized column of the meteor trail itself. The decay time constant η is related to the diffusion constant D at the height of reflection and the wavelength λ of the radiation employed.

$$(\eta)^{-1}=\frac{16\pi^2 D}{\lambda^2}. \quad (3.2)$$

There are, of course, overdense meteor echoes [2] which do not obey the simple decay law (3.1), and one must treat them separately.

In evaluating $A(k)$ from eq (2.11), it is convenient to take the reference or measuring time t_0 to be zero and to run the time backward in a positive sense.

$$A\,(k)=\exp-\nu \int_0^\infty dpD\,(p) \int_0^\infty dt\,[1-J_0\,(kpe^{-t/\eta})]. \quad (3.3)$$

One can simplify the calculation by settling $u=kp$ $\exp-t/\eta$ and reversing the order of integration.

$$A\,(k)=\exp-\nu\eta \int_0^\infty \frac{du}{u}\,[1-J_0\,(u)] \int_{u/k}^\infty dpD\,(p). \quad (3.4)$$

[2] L. A. Manning and V. R. Eshelman, Meteors in the ionosphere, Proc. IRE **47,** 186 (1959).

The cumulative integral of $D(p)$ expresses the probability that the initial pulse height equals or exceeds the lower limit. As noted earlier, measurements of individual echo pulse heights show that

$$\int_r^\infty dpD(p)=\frac{Q}{r^{1+\epsilon}}, \tag{3.5}$$

and it is presumed that this same law extends down to the smaller meteors which cannot be distinguished as individual echoes. The fractional exponent ϵ has been variously reported to lie between 0 and 0.3.

The case $\epsilon=0$ is analytically important, since all of the required integrations can be performed for this case and it serves as a good working example. Combining (3.4) and (3.5), we find for this special case,

$$A(k)=\exp-\nu\eta\int_0^\infty\frac{du}{u}\,[1-J_0(u)]\left(\frac{Qk}{u}\right),$$

or since the definite integral has unit value.

$$A(k)=\exp-\nu\eta Qk. \tag{3.6}$$

One may now compute the probability distribution for the resultant signal by introducing (3.6) into expression (2.4).

$$W(R,\phi)=\frac{R}{4\pi^2}\int_0^\infty dkk\int_0^{2\pi}d\omega e^{ikR\cos(\omega-\phi)}e^{-\nu\eta Qk}$$

$$=\frac{R}{2\pi}\int_0^\infty dkkJ_0(kR)e^{-\nu\eta Qk}$$

or

$$W(R,\phi)=\frac{R}{2\pi}\,\frac{\nu\eta Q}{[R^2+(\nu\eta Q)^2]^{3/2}} \tag{3.7}$$

This distribution is independent of the phase angle ϕ, expressing the fact that the vector sum of a large number of randomly phased vectors is itself randomly oriented. The probability density for R alone is obtained by integrating over ϕ.

$$W(R)dR=\frac{(\nu\eta Q)RdR}{[R^2+(\nu\eta Q)^2]^{3/2}}. \tag{3.8}$$

It is important to note that this distribution does not possess finite moments of any order, although it is properly normalized to unity. This means that one cannot define an RMS signal level for describing the cumulative probability as suggested in eqs (1.1) and (1.2). The root of the problem, of course, lies in the initial pulse height distribution assumption of eq (3.5). The integrals of $W(R)$ diverge for large amplitudes, which, in turn, are produced by the very large individual echoes. The assumed distribution (3.5) does not suppress these large echoes rapidly enough to insure convergence, although most workers agree that the form (3.5) must eventually change its rate of decrease with r so as to properly represent the rarity of really large meteors.

The function which is commonly measured experimentally is the cumulative probability that the total signal amplitude R exceeds a prescribed level r.

$$P(|R|>r)=\int_r^\infty dRW(R)=\frac{1}{\left[1+\left(\frac{r}{\nu\eta Q}\right)^2\right]^{1/2}} \tag{3.9}$$

This result is plotted on Rayleigh graph paper versus the ratio $r/\nu\eta Q$ in figure 2. The Rayleigh cumulative distribution $P=\exp-(r^2/2\sigma^2)$ plots as a straight line with slope minus one on this paper. The probability of observing very small signals r is seen to follow the straight line Rayleigh behavior with slope minus one. This is because the small argument expansion of eq (3.9)

$$\lim_{r\to0}(P)=1-\frac{1}{2}\,\frac{r^2}{(\nu\eta Q)^2} \tag{3.10}$$

is essentially identical to that for the Rayleigh distribution (1.1) with $\sigma=(\nu\eta Q)$. On the other hand, we have already noted that the meteor distribution (3.8) does not possess a finite variance, so that $\nu\eta Q$ cannot be identified with an RMS signal level. Note, however, that the curve in figure 2 is displaced upward from the normal Rayleigh curve by a factor of $\sqrt{2}=0.707$, since P (r) is plotted versus $\nu\eta Q$, not $\sqrt{2}\nu\eta Q$ which would be the root mean square signal level of a Rayleigh distribution with the same small amplitude asymptotic behavior.

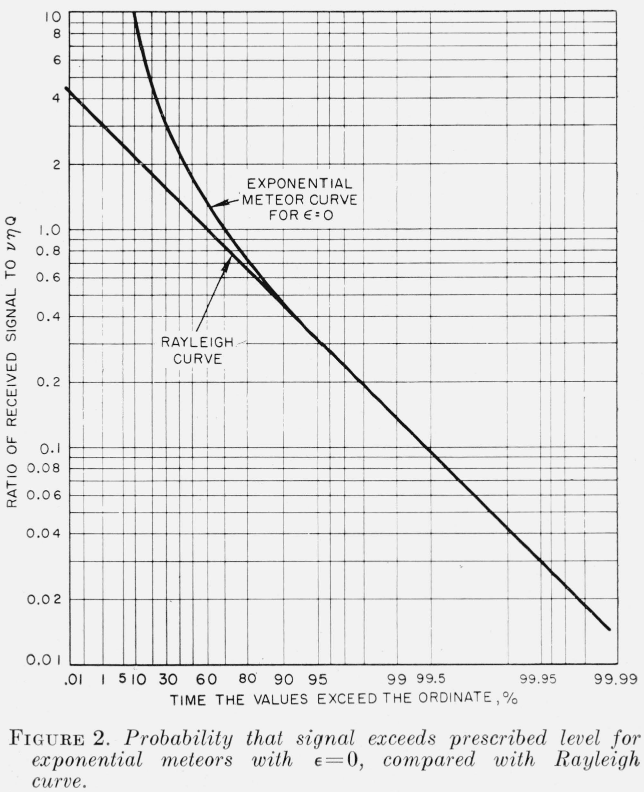

FIGURE 2. *Probability that signal exceeds prescribed level for exponential meteors with $\epsilon=0$, compared with Rayleigh curve.*

The cumulative probability distribution for the very large, unusual signals is markedly different than a Rayleigh distribution, reflecting the unique character of meteor echoes. One may expand (3.9) for large r to find

$$P(R>r)\simeq\frac{\nu\eta Q}{r}$$

in agreement with the qualitative result (1.5). This also agrees in form with the basic assumption (3.5) for the cumulative probability that an individual meteor echo amplitude exceeds the level r. The important difference is that eq (3.5) assumes that a meteor signal has just been received, whereas eq (3.11) calculates the residual large meteor signal at any time. The additional factor $\nu\eta$ in (3.11) is the average rate of occurrence times the half life of individual meteors, and is a measure of the fraction of time that the large, isolated meteor signals are greater than e^{-1} of their initial value.

The simplified theoretical result of eq (3.9) was compared with experimental data gathered by Bowles [3] on the Havana, Ill., to Boulder, Colo., VHF scatter link operated by the National Bureau of Standards. Totalizer outputs obtained with rhombic antennas directed off path were employed, so as to accentuate the meteoric signal contribution. The experimental points follow a Rayleigh distribution above the 50 percent level, but indicate a higher probability of observing the very large signals produced by combinations of strong echoes than is predicted by the Rayleigh distribution. This is in qualitative agreement with the theoretical result plotted in figure 2, although the quantitative agreement is not as precise as one would like. It is believed that the residual discrepancy can be traced to the three basic assumptions used in deriving eq (3.9):

1. The large meteors do not decay exponentially, as assumed in eq (3.1), especially if they are strong enough to produce overdense echoes.
2. The assumption $\epsilon=0$ in applying eq (3.5) is not consonant with some meteor radar experiments, which suggest small fractional values.
3. The initial pulse height cumulative distribution (3.5) is almost certainly not correct for the very large meteor end of the spectrum.

The second possibility was checked numerically by rederiving the transform function $A(k)$ for arbitrary ϵ.

$$A_\epsilon(k)=\exp-\nu\eta\int_0^\infty\frac{du}{u}\,[1-J_0(u)]\cdot\frac{Qk^{1+\epsilon}}{u^{1+\epsilon}}$$

$$=\exp-\nu\eta Qk^{1+\epsilon}\cdot\xi \qquad (3.12)$$

where

$$\xi=\frac{1}{1+\epsilon}\,\frac{1}{2^{1+\epsilon}}\cdot\frac{\Gamma\left(\frac{1-\epsilon}{2}\right)}{\Gamma\left(\frac{3+\epsilon}{2}\right)}.$$

[3] K. L. Bowles, private communication.

However, the coefficient of $k^{1+\epsilon}$ which appears in the polar integration for computing $W(R)$ may be removed by renormalizing k itself.

$$W(R)RdR=RdR\int_0^\infty dkkJ_0(kR)e^{-\gamma k^{1+\epsilon}}$$

$$=d\rho\left[\rho\int_0^\infty dzzJ_0(z\rho)e^{-z^{1+\epsilon}}\right], \qquad (3.13)$$

with

$$\rho=\frac{R}{\gamma^{1/1+\epsilon}}\text{ and }z=k\gamma^{1/1+\epsilon}.$$

The function given by the bracketed integral in (3.13) was tabulated numerically on a digital computing machine for the following values: $\epsilon=0.1$, 0.2, 0.25, 0.3, and the results are plotted in figure 3.

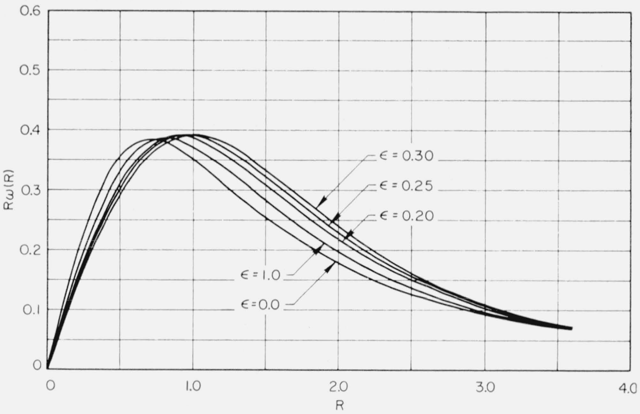

FIGURE 3. *Plots of the probability that the instantaneous signal lies in the range* R *to* R+dR *for various values of the parameter* ϵ *which determines the distribution of initial pulse heights for individual meteors.*

The cumulative probability corresponding to the various ϵ-fractional distributions (3.13) was also computed numerically using the analytical equivalence

$$P(\rho)=\int_\rho^\infty d\rho\rho\int_0^\infty dzzJ_0(\rho z)e^{-z^{1+\epsilon}},$$

$$=1-\rho\int_0^\infty dzJ_1(\rho z)e^{-z^{1+\epsilon}}, \qquad (3.14)$$

which follows by reversing the order of integration, and treating the limits cautiously. The second form is plotted in figure 4 for various values of ϵ on Rayleigh paper. The various curves in this figure do not have the same asymptotic behavior because of different normalizations of the vertical scale for each ϵ. However, one can imagine the signal levels adjusted for each case so that all approach the same Rayleigh limit. This would show that the P (r) curves all fall between the $\epsilon=0$ curve shown in figure 2 and the Rayleigh distribution straight lines. Insofar as the present data of Bowles suggests

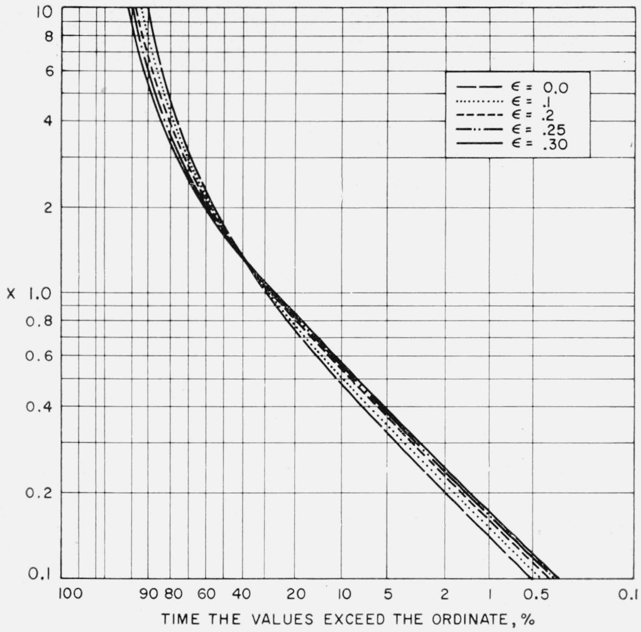

FIGURE 4. *Plots of the cumulative probability that the instantaneous signal exceeds a prescribed level for various values of the parameter ε which determines the distribution of initial pulse heights for individual meteors.*

that the departure from Rayleigh is not as marked as that predicted in eq (3.9), this would seem to indicate that values of ϵ near 0.2 may give better agreement. On the other hand, the data sample now available is certainly too limited to pronounce final judgment.

The first two objections raised above also deserve further attention in a careful comparison of the theory with experiment. The approximate descriptions developed by Manning and Eshelman (see footnote 3) for overdense echoes were examined briefly, but unfortunately the split (p) integrations were not found to be tractable analytically.

Valuable discussions of the problem with V. R. Eshelman and T. A. Magness are acknowledged. K. L. Bowles kindly made his experimental data available prior to publication. B. A. Troesch and L. Stohler computed the numerical results displayed in figures 3 and 4.

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