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Mode Theory and the Propagation of ELF Radio Waves[1]

James R. Wait

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The mode theory of propagation of electromagnetic waves at extremely low frequencies (ELF) (1.0 to 3,000 cycles per second) is treated in this paper. Starting with the representation of the field as a sum of modes, approximate formulas are presented for the attenuation and phase constants. Certain alternate representations of the individual modes are mentioned. These are used as a basis for describing the physical behavior of the field at large distances from the source, particularly near the antipode of the source. At the shorter distances, where the range is comparable to the wavelength, the spherical-earth mode series is best transformed to a series involving cylindrical wave functions. This latter form is used to evaluate the near field behavior of the various field components.

The effect of the earth's magnetic field is also evaluated using a quasi-longitudinal approximation. In general it is indicated that if the gyrofrequency is less than the effective value of the collision frequency, the presence of the earth's magnetic field may be neglected for ELF. When this condition is not met the attenuation may be increased somewhat. The influence of an inhomogeneous ionosphere is also briefly considered and, finally, the propagation of ELF pulses is treated. It is suggested that certain observed characteristics of ELF waveforms may be attributed to the inclination of the current channel in the lightning discharge.

1. Introduction

The propagation of electromagnetic waves at extremely low frequencies (ELF) has received considerable attention recently [1 to 5].[2] Unfortunately, the experimental data in this frequency range (1.0 cps to 3,000 cps) is still rather limited [8 to 15]. There is little doubt, however, that the bulk of the energy is transferred via a waveguide mode. The bounding surfaces of this waveguide are the ground and the lower edge of the ionospheric E region. In theoretical approaches to this problem, it is usually assumed that the ionosphere is homogeneous in its electrical properties. On comparing theoretical attenuation rates based on this assumption, it is apparent that the model must be refined to take account of the variation of the electron density with height. Such an extension to the mode theory has been made recently [16]. The influence of the earth's magnetic field is also expected to be important at ELF since the waves are reflected at a level where the collision frequency is of the same order as the gyrofrequency for electrons. The ionosphere is thus expected to be anisotropic. Unfortunately, the experimental data is not sufficient to confirm whether the anisotropy is significant.

Another feature of ELF propagation is that the distance between source and observer may be comparable to the wavelength. For example, at 300 cps the wavelength is 1,000 km. Many of the experiments are carried out at distances of this order and the computed attenuation rates must be considered with caution.

It is the purpose of this paper to survey the mode theory as it applies to ELF propagation and to develop several extensions. Many of the results are presented in graphical form as an aid to application.

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[2] Figures in brackets indicate the literature references at the end of this paper.

2. Basic Theoretical Model

The assumed theoretical model is taken to be a homogeneous conducting spherical earth of radius a surrounded by a concentric conducting ionospheric shell of inner radius $a+h$. It is convenient to introduce the usual spherical coordinate system (r, θ, ϕ). Using this model and assuming a vertical electric dipole source located at $\theta=0$, $r=a$, the expression for the vertical electric field, E_r, is given by an expression of the following form [4, 16]

$$E_r=\frac{Ids\eta}{2kha^2}\sum_{n=0}^{\infty}\delta_n\frac{\nu(\nu+1)}{\sin\nu\pi}P_\nu(-\cos\theta) \tag{1}$$

where

I—average current in the source dipole,
ds—length of source dipole,
η—intrinsic impedance of air space $\simeq 120\pi$ ohms,
$P_\nu(-\cos\theta)$—Legendre function of argument $-\cos\theta$ and (complex) order ν,
$\nu+\frac{1}{2}=ka\,S_n$ where S_n, for $n=0, 1, 2, \ldots$, is determined from the boundary condition and is described below, and

$$\delta_n\simeq\frac{1}{1+\frac{\sin 2kh\,C_n}{2\,kh\,C_n}}\simeq 1/2 \text{ for } n=0,\ \simeq 1 \text{ for } n=1, 2, 3, \ldots .$$

The individual terms in the above series correspond to the waveguide modes. In the general case, the factor S_n is obtained from the solution of a complicated transcendental equation which involves spherical Bessel functions of large argument and complex order. Certain aspects of this problem have been discussed by Schumann [1] and also the author [4]. For a homogeneous earth and a homogeneous ionosphere, both assumed isotropic, it has been shown that S_n may be approximated by [17]

$$S_n=\left\{1-\left(\frac{\pi n}{kh}\right)^2\frac{1}{4}\left[1+\sqrt{1+4i\frac{\Delta kh}{(\pi n)^2}}\right]^2\right\}^{\frac{1}{2}} \tag{2}$$

where

$$\Delta=\frac{1}{N_i}+\frac{1}{N_g}$$

in terms of the refractive indices, N_g and N_i, of the earth and the ionosphere respectively. This equation is valid subject to the condition $|\Delta|kh<<1$. The sign of the radical is chosen in the above equation which makes the real part positive. The values of S_n then are located in the fourth quadrant of the complex plane. When $n=0$, the above simplifies to

$$S_0=\left[1-i\frac{\Delta}{kh}\right]^{\frac{1}{2}}. \tag{3}$$

Now, since $|\Delta|kh<<1$, the radical in eq (1) can be expanded for $n>0$ to yield

$$S_n=\left[1-\left(\frac{\pi n}{kh}\right)^2-i\frac{2\Delta}{kh}\right]^{\frac{1}{2}} \text{ for } n=1, 2, 3, \ldots . \tag{4}$$

Furthermore, if in addition,

$$\frac{|\Delta|}{kh}<<1-\left(\frac{\pi n}{kh}\right)^2$$

it is permissible to write, for all n,

$$S_n \simeq \left[1-\left(\frac{\pi n}{kh}\right)^2\right]^{\frac{1}{2}} - i\,\frac{\epsilon_n}{2kh}\,\Delta\left[1-\left(\frac{\pi n}{kh}\right)^2\right]^{-\frac{1}{2}} \tag{5}$$

where $\epsilon_0=1$, $\epsilon_n=2$ (for $n=1, 2, 3, \ldots$). This latter equation is valid when the "waveguide modes" are not near cutoff and is in the form given by Schumann [2]. It is quite analogous to the standard results for the propagation constant in a rectangular waveguide with finitely conducting walls [18, 19].

3. Behavior of the Modes for the Spherical Earth

For purposes of computation of the mode series, several simplifications can be made. The asymptotic expansion for the Legendre function, given by

$$P_\nu(-\cos\theta) \simeq \left(\frac{2}{\pi\nu \sin\theta}\right)^{\frac{1}{2}} \cos\left[\left(\nu+\frac{1}{2}\right)(\pi-\theta)-\frac{\pi}{4}\right] \tag{6}$$

is valid if $|\nu| >> 1$ and θ not near 0 or π [20]. Thus, the modes are simply proportional to

$$\frac{1}{(\sin\theta)^{\frac{1}{2}}}\cos\left[kaS_n(\pi-\theta)-\frac{\pi}{4}\right]$$

which apart from a constant factor can be identified as the linear combination of two peripheral waves. These have the form

$$\frac{1}{(\sin\theta)^{\frac{1}{2}}}\,e^{-ikaS_n\theta}$$

and

$$\frac{1}{(\sin\theta)^{\frac{1}{2}}}\,e^{-ikaS_n(2\pi-\theta)}e^{i\pi/2}$$

where $\theta < \pi$.

These waves are traveling in opposing directions along the two respective great circle paths $a\theta$ and $a(2\pi-\theta)$, from the source to the observer. It is noticed that there is a $\pi/2$ phase advance which the wave traveling on the long great circle path picks up as it goes through the pole $\theta=\pi$. The linear combination of these two traveling waves is to form a standing wave pattern whose distance δ_m between minimums is approximately given by

$$k\delta_m \operatorname{Re} S_n = \pi \text{ or } \delta_m = \lambda/(2 \operatorname{Re} S_n)$$

subject to

$$\operatorname{Im} S_n \ll \operatorname{Re} S_n.$$

The preceding asymptotic expansion for $P_\nu(-\cos\theta)$ is not usable at and in the vicinity of the pole $\theta=\pi$. In this region a suitable representation is given by [20]

$$P_\nu(-\cos\theta) = J_o(\eta) + \sin^2\left(\frac{\pi-\theta}{2}\right)\left[\frac{J_1(\eta)}{2\eta} - J_2(\eta) + \frac{\eta}{6}J_3(\eta)\right] + 0\left(\sin^4\left(\frac{\pi-\theta}{2}\right)\right) \tag{7}$$

where $\eta=(2\nu+1)\sin((\pi-\theta)/2)$. $J_m(\eta)$, for $m=0, 1, 2,$ and 3, is the Bessel function of first type of argument η and order m. When $\pi-\theta$ is small, the first term is usually sufficient and furthermore

$$\eta \simeq \left(\nu+\frac{1}{2}\right)(\pi-\theta) \simeq kaS_n(\pi-\theta).$$

Thus for mode n, the field in the neighborhood of the pole is proportional to the Bessel function

$$J_0[kaS_n(\pi-\theta)].$$

It is then not surprising to see that the first term of the asymptotic expansion of J_0 is the same as that of P_ν $(-\cos\theta)$.

At the lower end of the ELF band, where the wavelength is large compared to the height of the ionospheric reflecting layer, the electric field is essentially radial and only one waveguide-type mode is significant. The field is thus expressed by the first term of the mode series which reads

$$E_r = \frac{I\,ds\,\eta}{4kha^2}\frac{\nu(\nu+1)}{\sin\nu\pi}P_\nu(-\cos\theta) \tag{8}$$

where $\nu+\frac{1}{2}\simeq kaS_0$ and

$$S_0\simeq 1-\frac{i}{2kh}\left(\frac{1}{N_i}+\frac{1}{N_g}\right)$$

in terms of the relative refractive indices N_i and N_g of the homogeneous ionosphere and the homogeneous ground, respectively. Now as mentioned above, the factor $P_\nu(-\cos\theta)$ may be replaced by an asymptotic expansion if $ka\theta$ or $ka(\pi-\theta)$ is somewhat greater than unity. The field in this case may be regarded as two azimuthal-type traveling waves. Furthermore, at the pole (θ near π) where the second of these restrictions is violated, it is possible to use an equivalent representation which correctly accounts for the axial focusing. An alternate viewpoint which is suitable at ELF is to consider the field as a superposition of cavity-resonator type modes. It is expected that such a representation would be very good when the circumference of the earth is becoming comparable to the wavelength. A suggestion of this kind was apparently first put forth by Schumann.

The starting point is the expansion formula

$$\frac{P_\nu(-x)}{\sin\nu\pi}=-\frac{1}{\pi}\sum_{n=0}^{\infty}P_n(x)\,\frac{2n+1}{n(n+1)-\nu(\nu+1)} \tag{9}$$

where the summation is over integral values of n. This result follows directly from a formula given by Magnus and Oberhettinger [20] (p. 57) which is valid for $\nu\neq 0, \pm 1, \pm 2, \ldots$, and $0\leqq\theta<\pi$. The electric field, for $h/a<<1$, is thus written

$$E_r=\frac{I\,ds\,\nu(\nu+1)}{4\pi a^2\epsilon_0\omega h}\sum_{n=0}^{\infty}P_n(x)\,\frac{2n+1}{n(n+1)-\nu(\nu+1)} \tag{10}$$

where $x=\cos\theta$. The early terms of the series are then proportional to

$$P_0(x)=1,$$
$$P_1(x)=\cos\theta,$$
$$P_2(x)=\tfrac{1}{2}(3\cos^2\theta-1),$$

and so on.

Retaining just the first term it is seen that

$$E_r^\circ=E_r]_{n=0}=\frac{I\,ds}{4\pi a^2\epsilon_0 h}\frac{1}{i\omega} \tag{11}$$

which is independent of θ. Clearly this corresponds to a concentric spherical capacitor energized by a current $I ds/h$ resulting in a constant voltage hE_r° between the plates. On rewriting eq (11) in the form

$$hE^\circ=\frac{(I\,ds/h)}{i\omega C_e}$$

it is seen that

$$C_e = \frac{4\pi a^2 \epsilon_0}{h}$$

which can be identified as the capacity between the spherical surfaces whose areas are both $4\pi a^2$ within the approximation $h/a << 1$.

4. Earth-Flattening Approximation

The possibility that the curvature of the earth may be neglected is now investigated. To simplify field calculations at short distances, it would seem desirable to transform the mode series to a form where the first term corresponds to the model of a flat earth and the succeeding terms are corrections for curvature. This approach has been used by Pekeris [21] and more recently by Koo and Katzin [22] in their investigations of microwave duct propagation. From their work, it may be shown that

$$\frac{P_\nu(-\cos\theta)}{\sin\nu\pi} \simeq H_0^{(2)}(kS_n\rho) - \frac{1}{12}\left(\frac{\rho}{a}\right)^2 H_2^{(2)}(kS_n\rho) + \text{terms in } (\rho/a)^4, (\rho/a)^6, \text{etc.} \tag{12}$$

where $\nu + 1/2 = kS_n\rho$ and $\rho = a\theta$. $H_0^{(2)}$ and $H_2^{(2)}$ are Hankel functions of the second kind of order zero and two, respectively. When the great circle distance ρ is reasonably small compared to the earth's radius a, only the first term in the expansion need be retained.

For convenience in what follows, it is convenient to express the field components as a ratio to the quantity

$$E_0 = i(\eta/\lambda) Ids(e^{-ik\rho})/\rho \qquad (\eta \simeq 120\pi);$$

E_0 is the radiation field of the source at a distance ρ on a perfectly conducting ground. Thus, for both the source and the observer near the ground, it is not difficult to show that

$$E_z = WE_0$$

where

$$W \cong -i\pi \frac{\rho}{h} e^{ik\rho} \sum_{n=0}^{\infty} \delta_n S_n^2 H_0^{(2)}(kS_n\rho), \tag{13}$$

$$E_\rho = -SE_0$$

where

$$S \cong \frac{\pi}{N_g} \frac{\rho}{h} e^{ik\rho} \sum_{n=0}^{\infty} \delta_n S_n H_1^{(2)}(kS_n\rho) \tag{14}$$

and

$$H_\phi = -TE_0/\eta$$

where

$$T \cong -\pi \frac{\rho}{h} e^{ik\rho} \sum_{n=0}^{\infty} \delta_n S_n H_1^{(2)}(kS_n\rho) = -N_g S \tag{15}$$

where terms containing $(\rho/a)^2$, $(\rho/a)^4$, etc., have been neglected.

When $k\rho >> 1$, corresponding to the "far-zone," the above expressions may be simplified since the Hankel functions may be replaced by the first term of their asymptotic expansion. This leads to the compact result

$$\begin{bmatrix} W \\ S \\ T \end{bmatrix} \cong \frac{(\rho/\lambda)^{\frac{1}{2}}}{(h/\lambda)} e^{i\left[\frac{2\pi\rho}{\lambda} - \frac{\pi}{4}\right]} \sum_{n=0}^{\infty} \delta_n \begin{bmatrix} S_n^{\frac{3}{2}} \\ -S_n^{\frac{1}{2}}/N_g \\ S_n^{\frac{1}{2}} \end{bmatrix} e^{-i2\pi S_n\rho/\lambda} \tag{16}$$

which is valid for $\rho >> \lambda$. As expected, the ratio of W to T for a given mode is S_n which for low order or grazing modes is of the order of unity. The ratio of S to T, quite generally, is $-1/N_g$ which is very small compared to unity; in fact, it vanishes for a perfectly conducting ground as it must.

The excitation by a horizontal electric dipole may be treated in a similar manner. A complete discussion of the theory is given elsewhere [17]. The expression for the vertical field E_z of the horizontal dipole Ids which is located at the origin and oriented in direction $\phi=0$ may be written in the form

$$E_z=SE_0 \cos \phi \tag{17}$$

where S and E_0 are defined above. By using the reciprocity theorem, this result can also be deduced directly from eq (15). Comparing the above formula for E_z (for a horizontal dipole) with corresponding formula for E_z, for a vertical dipole, it may be easily shown that

$$\frac{E_z \text{ (for horizontal dipole)}}{E_z \text{ (for verticle dipole)}} \simeq \frac{\cos \phi}{N_g}\left(1-\frac{1}{N_g^2}\right)^{\frac{1}{2}} \simeq \frac{\cos \phi}{N_g} \tag{18}$$

being valid in the asymptotic or far zone. In terms of the ground conductivity σ_g and dielectric constant ϵ_g,

$$\frac{1}{N_g}=\left(\frac{i\epsilon_0\omega}{\sigma_g+i\epsilon_g\omega}\right)^{\frac{1}{2}} \cong \left(\frac{i\epsilon_1\omega}{\sigma_g}\right)^{\frac{1}{2}}.$$

This is a small quantity.

The above formulas for W, S, and T are valid only if $\rho<<a$. If the first curvature correction term is included, it is a simple matter to show, in the "far zone", that this amounts to multiplying the right-hand side by the factor

$$1+\frac{1}{12}\left(\frac{\rho}{a}\right)^2 \quad \text{or} \quad 1+\frac{\theta^2}{12}.$$

Now, returning to the mode series for a spherical earth and using the far field approximation for the Legendre function, it turns out that W, S, T have precisely the same form as in eq (16) except the factor $(\theta/\sin \theta)^{\frac{1}{2}}$ occurs in the right-hand side. Noting that

$$\left(\frac{\theta}{\sin \theta}\right)^{\frac{1}{2}} \cong 1+\frac{\theta^2}{12}+\text{terms in } \theta^4,\ \theta^6, \text{ etc.}$$

it is apparent to a second order that the derived curvature correction is the same as from the Hankel function series.

5. Distance and Frequency Dependence

Certain features of the mode series are best illustrated by a graphical plot of field strength versus distance under various values of the parameters. Since $N_g>>N_i$ it may be safely assumed for ELF that $\Delta=1/N_i$. Then on the assumption that the ionosphere is behaving like a conductor, it is possible to write

$$N_i=\left(1-i\frac{\omega_r}{\omega}\right)^{\frac{1}{2}} \tag{19}$$

where

$$\omega_r=\frac{\omega_0^2}{\nu}=\frac{(\text{plasma frequency})^2}{\text{collision frequency}}.$$

The above relation for N_i is an approximation which is valid when the collision frequency ν between electrons and neutral ions is much less than the angular frequency ω. Since ν is of the order of 10^7 in the lower E region, this is certainly valid at ELF. The (angular) plasma frequency ω_0 is given by

$$\omega_0^2=\frac{e^2N}{\epsilon_0 m}$$

where N is the electron density, e and m are charge and mass of the electron, and ϵ_0 is the dielectric constant of free space. ω_r, which is proportional to the ratio N/ν, is believed to be of the order of 10^5 in the lower region of the E layer. At least this appears to be the effective value deduced from vlf observations for highly oblique incidence [23]. At ELF it appears that the waves are reflected at higher levels in the ionosphere and the effective value of ω_r is somewhat larger, being possibly of the order of 10^6.

It should be noted in passing that ω_r may be replaced by σ_i/ϵ_0 where σ_i is the effective conductivity of the ionosphere and ϵ_0 is the dielectric constant of free space.

To indicate the variation of the field as a function of distance, the magnitude and phase of W and T are shown in figures 1 to 4 for the typical values: $\omega_r=5\times 10^5$ and $h=90$ km. Actually $|W|$ and $|T|$, both divided by the square root of the distance ρ (in kilometers), are plotted since the curves become linear at larger distances. The slope of these linear parts of the curves are proportional to the attenuation rate (in decibels per 1,000 km) as usually defined. It is seen immediately that at the shorter distances, the curves are no longer straight. This immediately indicates that considerable caution should be exercised in computing attenuation rates from spectral analyses of "sferics." For example, it is only when the distance ρ exceeds about one-sixth of a wavelength, is it permissible to assume that log $(|W|/\sqrt{\rho})$ or log $(|E_z|\times\sqrt{\rho})$ vary in a linear manner with distance ρ. Similarly, the phase of W (or E_z) varies in a linear manner only when the distance exceeds about one-half wavelength. Similar remarks apply to the magnetic field variations with distance.

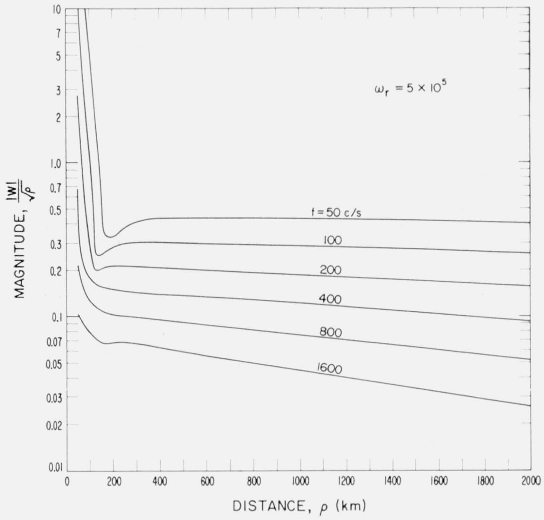

FIGURE 1. *The normalized magnitude of the electric field as a function of distance from the source for h=90 km.*

(ρ is expressed in kilometers in both ordinate and abscissa).

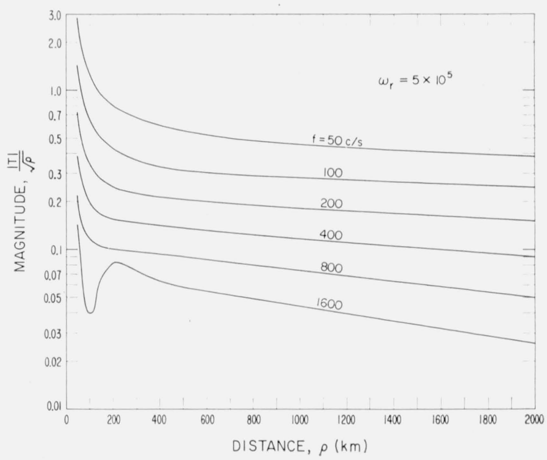

FIGURE 3. *The normalized magnitude of the magnetic field as a function of distance from the source for* h=*90 km.*

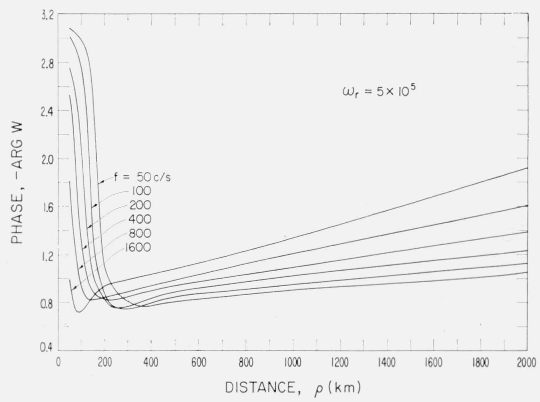

FIGURE 2. *The normalized phase (lag), in radians, of the electric field as a function of distance for* h=*90 km.*

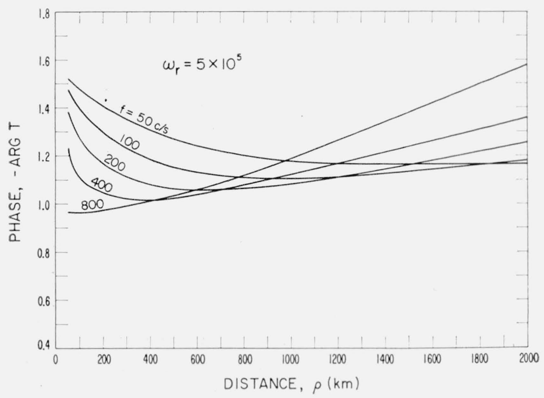

FIGURE 4. *The normalized phase (lag), in radians, of the magnetic field as a function of distance for* h=*90 km.*

At the extremely short distances where ρ is somewhat less than the ionospheric reflecting height (i.e., 90 km), the magnitude of W varies essentially as the inverse square of the distance and the phase approaches 180°. In this same region the quantity T varies as the inverse of the distance and the phase approaches 90°. The general characteristics of the field at these extremely short distances are in accord with the near field behavior of a dipole on a flat ground plane when the presence of the ionosphere is neglected. In this rather trivial case

$$W=\left(1-\frac{i}{k\rho}-\frac{1}{k^2\rho^2}\right) \tag{20}$$

and

$$T=\left(1-\frac{i}{k\rho}\right). \tag{21}$$

To shed further light on the nature of the ELF fields in the waveguide, the radial impedance of the wave is now considered. Noting that

$$-\left.\frac{E_z}{\eta H_\phi}\right]_{z=0}=\frac{W}{T} \tag{22}$$

which is by definition the (normalized) impedance of the wave looking in the radial or ρ direction. First it should be noted that for $k\rho>>1$,

$$\frac{W}{T}\simeq 1$$

and for $k\rho << 1$ and $\rho << h$,

$$\frac{W}{T}\simeq\frac{1}{ik\rho}.$$

For the intermediate and interesting range, $k\rho$ is comparable to unity and ρ is somewhat greater than h. Using the numerical values of W and T mentioned above, the ratio $|W/T|$ and the phase (lag) defined by arg T- arg W are plotted in figures 5 and 6 and as a function of distance for various frequencies. As before, $\omega_r=5\times 10^5$ and $h=90$ km.

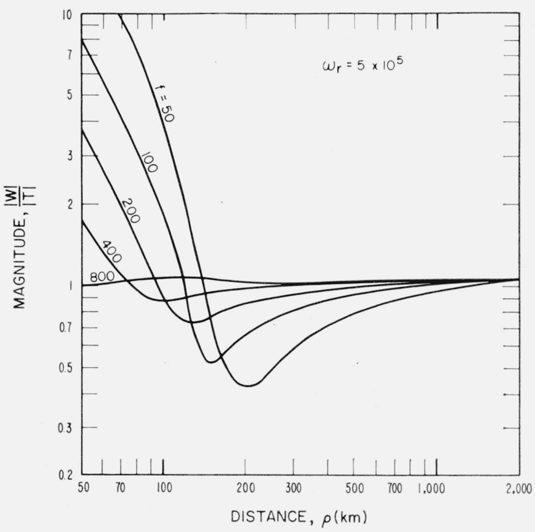

FIGURE 5. *The magnitude of the radial wave impedance.*

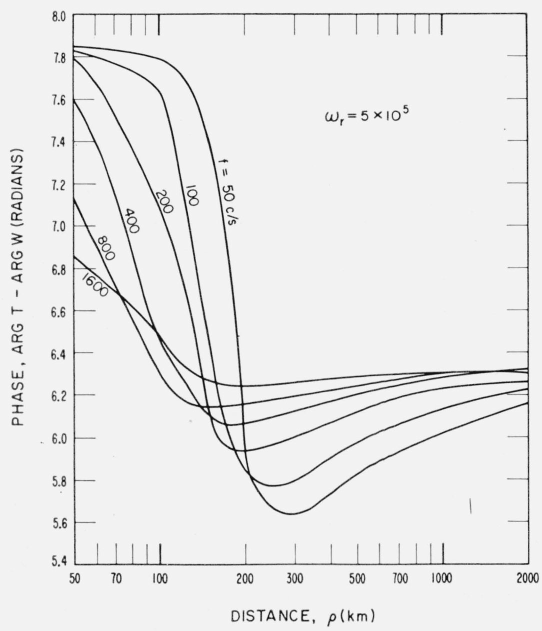

FIGURE 6. *The phase (lag) of the radial wave impedance.*

The impedance ratio W/T is independent of the frequency spectrum of the source provided, of course, it may be represented by an equivalent vertical electric dipole. In fact, this complex ratio could be easily calculated from the frequency spectra of the waveforms of the vertical electric field and the horizontal magnetic field of an atmospheric. The observed variation of magnitude or phase of W/T as a function of frequency should then provide a basis for distance measuring. Such a scheme, while admittedly crude, requires only one receiving station equipped with a vertical whip and a loop antenna.

6. Effect of the Earth's Magnetic Field

In the preceding it has been tacitly assumed that the ionosphere is behaving as an isotropic homogeneous conductor. In this case, the refractive index N_i of the ionosphere may be written [24]

$$N_i \simeq \left[1 - i\frac{\omega_r}{\omega}\right]^{\frac{1}{2}} \simeq \left(\frac{\omega_r}{i\omega}\right)^{\frac{1}{2}} \tag{23}$$

where

$$\omega_r = \frac{\omega_0^2}{\nu} = \frac{(\text{plasma frequency})^2}{(\text{collision frequency})}.$$

When the earth's magnetic field is steeply dipping, it is appropriate to invoke the quasi-longitudinal approximation of Booker [24]. In this case the refractive index is double valued, one corresponds to the ordinary and the other the extraordinary, thus

$$N_i \simeq \left[1 - i\frac{\Omega_r}{\omega}\exp(\pm i\tau)\right]^{\frac{1}{2}} \tag{24}$$

where

$$\tan\tau = \frac{\omega_L}{\nu} = \frac{\text{longitudinal component of gyrofrequency}}{\text{collision frequency}}$$

and

$$\Omega_r = \frac{\omega_0^2}{(\nu^2 + \omega_L^2)^{\frac{1}{2}}}.$$

Using this model for the ionosphere, the reflection coefficient for a sharp boundary has been derived by Budden [25]. Adapting this result to ELF it has been shown [17] that the formula for S_n has the same form as the isotropic medium if Δ is defined by

$$\Delta = \frac{1}{N_g} + \left(\frac{i\omega}{\Omega_r}\right)^{\frac{1}{2}} \cos\tau/2. \tag{25}$$

Since $\Omega_r = \omega_r \cos\tau$ it is possible to write the results in the same form as the isotropic ionosphere if we set

$$\Delta \simeq \frac{1}{N_g} + \frac{1}{(N_i)_{\text{eff}}}$$

where

$$(N_i)_{\text{eff}} = \left[\frac{(\omega_r)_{\text{eff}}}{i\omega}\right]^{\frac{1}{2}}.$$

$(N_i)_{\text{eff}}$ and $(\omega_r)_{\text{eff}}$ are the effective values of N_i and ω_r, respectively.

Specifically,

$$(\omega_r)_{\text{eff}} \simeq \omega_r \frac{\cos\tau}{(\cos\tau/2)^2}. \tag{26}$$

In other words, the effective conductivity of the ionosphere is modified by the factor $\cos \tau/(\cos \tau/2)^2$ which varies from unity to zero as τ varies from 0 to $\pi/2$. Thus, the attenuation is increased as a result of a steeply dipping (or vertical) magnetic field. In fact

$$\frac{\text{attenuation with magnetic field}}{\text{attenuation without magnetic field}} \simeq \frac{\cos (\tau/2)}{(\cos \tau)^{\frac{1}{2}}}. \tag{27}$$

If $\tau \ll 1$ (i.e., $\omega_L \ll \nu$), this ratio becomes unity and the influence of the earth's magnetic field vanishes. At the level in the ionosphere where ELF waves are reflected, it is expected that ω and ν are comparable, both being of the order of 10^6. Assuming that they were actually equal, τ becomes 45° and the ratio of the attenuation rates is 1.05. Thus, it is only when ω_L is greater than ν does the earth's magnetic field appreciably influence the attenuation. The ratio is plotted in figure 7 as a function of $\tan \tau$ or ω_L/ν. This illustrates the situation clearly.

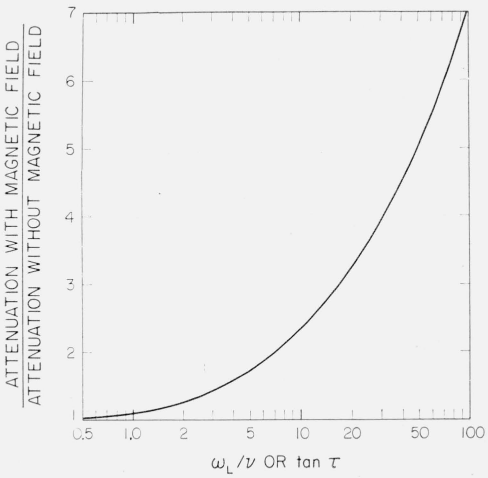

Figure 7. *The influence of earth magnetic field on attenuation.*

The abscissa is the ratio of the vertical or longitudinal component of the gyrofrequency, ω_L, and the collision frequency ν.

The preceding results are subject to the validity of Booker's quasi-longitudinal approximation of the Appleton-Hartree equation [24]. The validity of this approximation requires that

$$\frac{\omega_T^4}{4\omega^2\omega_L^2} << \left| \left(1 - \frac{\omega_0^2}{\omega^2} - i\frac{\nu}{\omega}\right)^2 \right| \tag{28}$$

where ω_L and ω_T are the longitudinal and transverse components of the (angular) gyrofrequency. Clearly this condition is violated when the transverse component of the earth's magnetic field is large such as for propagation along the magnetic equator. This case, however, has been considered by Barber and Crombie [26] who derived explicit results for the reflection coefficient at a sharply bounded ionosphere with a purely transverse magnetic field. Adapting their results to ELF, it is not difficult to show that Δ should now be replaced by [17]

$$\Delta = \frac{1}{N_g} + \left(\frac{i\omega}{\omega_r}\right)^{\frac{1}{2}} X \tag{29}$$

where

$$X = \frac{\left(1 + \frac{i\omega}{\omega_r}\right)^{\frac{1}{2}} \left(1 + i\frac{\omega}{\omega_r} + i\frac{\omega_T^2 \omega}{\nu^2 \omega_r}\right)^{\frac{1}{2}} - \frac{\omega_T}{\nu}\left(i\frac{\omega}{\omega_r}\right)^{\frac{1}{2}}}{\left(1 + i\frac{\omega}{\omega_r}\right)^2 - \frac{\omega_T^2}{\nu^2}\frac{\omega^2}{\omega_r^2}}.$$

Now at ELF, $\omega \ll \omega_r$ and thus X is very close to unity since ω_T and ν are of the same order of magnitude.[3] Consequently, a transverse magnetic field appears to have a negligible effect on the attenuation and the phase for ELF.[4]

A reasonable conclusion from the above is that only the vertical component of the earth's magnetic field is effective in ELF propagation. Furthermore, the ionosphere is *effectively* an isotropic conductor even in the presence of the earth's magnetic field.

7. Effect of an Inhomogeneous Ionosphere

The electron density in the actual ionosphere usually increases with height in the E region. Furthermore, the collision frequency decreases with height. Thus the effective value of the refractive index cannot be assumed constant. An approach is to let the refractive index increase or decrease from some initial value $\overline{N}$ at height h in a monotonic fashion.[5] Choosing an exponential variation the refractive index as a function of height is explicitly

$$
\begin{aligned}
N(z) &= 1.0 \quad \text{for } 0<z<h \\
&= \overline{N} \exp\,[(z-h)/l] \quad \text{for } z>h
\end{aligned}
\tag{30}
$$

where l is a scale factor. If $l>0$, the refractive index is increasing with height and, if $l<0$, the refractive index is decreasing with height. It has been shown elsewhere [16] that the factor Q has the forms

$$
Q = \frac{K_0(i\overline{N}kl)}{K_1(i\overline{N}kl)} \quad \text{if } l>0 \tag{31}
$$

and

$$
Q = \frac{I_0(-i\overline{N}kl)}{I_1(-i\overline{N}kl)} \quad \text{if } l<0. \tag{32}
$$

where I_0, I_1, K_0, and K_1 are modified Bessel functions with their conventional meaning.

The preceding results are valid for $|\overline{N}|>>1$ which is well satisfied at ELF. To the same approximation

$$
\overline{N} \cong \left[1 - i\,\frac{\overline{\omega}_r}{\omega}\right]^{\frac{1}{2}} \cong \left(\frac{\overline{\omega}_r}{\omega}\right)^{\frac{1}{2}} e^{-i\pi/4} \tag{33}
$$

where $\overline{\omega}_r = \overline{\omega}_0^2/\overline{\nu}$ in terms of the plasma frequency $\overline{\omega}_0$ and collision frequency $\overline{\nu}$. It then follows that the imaginary and real parts of the propagation constant kS_0 are given by

$$
\operatorname{Im} kS_0 = -\frac{1}{2h}\left(\frac{\omega}{2\omega_r}\right)^{\frac{1}{2}} P(x) \tag{34}
$$

and

$$
\operatorname{Re} kS_0 = k + \frac{1}{2h}\left(\frac{\omega}{2\omega_r}\right)^{\frac{1}{2}} P'(x) \tag{35}
$$

where

$$
P(x) = |Q| \sin\left(\frac{\pi}{4} - \arg Q\right)\sqrt{2}
$$

$$
P'(x) = |Q| \cos\left(\frac{\pi}{4} - \arg Q\right)\sqrt{2}
$$

---

[3] Note the positive values of ω_T correspond to propagation from east-to-west while negative values correspond to propagation from west-to-east.

[4] If ω_T is somewhat greater than ν, it is seen that to a first order

$$X \cong 1 - (\omega_T/\nu)\,(i\omega/\omega_r)^{1/2}$$

which will modify the Re S_0 to a slight extent but not Im S_0.

[5] In this section the influence of the earth's magnetic field is neglected.

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and where

$$x=|kl\overline{N}|=|kl|(\overline{\omega}_r/\omega)^{\frac{1}{2}}.$$

The functions $P(x)$ and $P'(x)$ approach unity if x is sufficiently large. This limiting case corresponds to the homogeneous ionosphere. Thus $P(x)$ is the modification of the attenuation and $P'(x)$ is the modification of the phase resulting from a nonhomogeneous ionosphere. It may be observed that the attenuation is generally lower if the refractive index is an increasing function with height. On the other hand, the attenuation is increased where the refractive index is decreasing with height.

To illustrate the behavior of the attenuation rates as a function of frequency, they are plotted in figures 8a and 8b in terms of decibels per 1,000 km of path length for a frequency scale from 50 cps to 1.5 kc. Values of the scale distance l are indicated in kilometers. The height h of the lower edge of the ionosphere is taken to be 90 km. The values chosen for $\overline{\omega}_r$ are 10^5 and 10^6 which are not inconsistent with previous work.

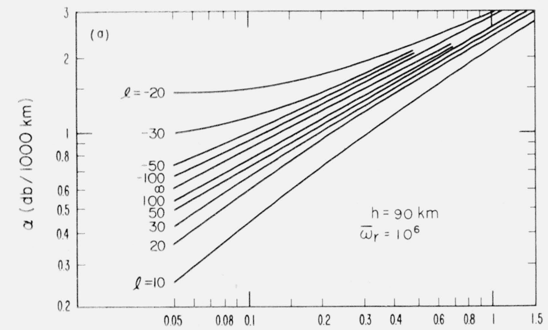

FIGURE 8a. *Attenuation at ELF for an exponential profile,* h=*90 km and* $\overline{\omega}_r=10^6$.

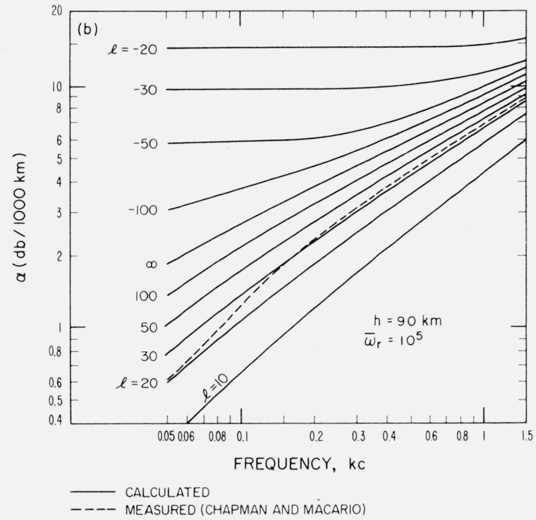

FIGURE 8b. *Attenuation at ELF for an exponential profile,* h=*90 km,* $\overline{\omega}_r=10^5$.

The dashed curve corresponds to nighttime experimental data of Chapman and Macario.

It is seen that the curve for $l=\infty$ corresponding to a homogeneous ionosphere has a slope of 1/2 as it should. Depending on the sign of l and its magnitude, the slope may be modified considerably for the inhomogeneous ionosphere. The dotted curve in figure 8b corresponds to nighttime experimental data from Chapman and Macario [27] who obtained it from the spectral analyses of large numbers of atmospherics recorded in London. Clearly this experimental curve appears to fit fairly well on the $l=30$-km curve, at least in the range 100 cps to 1.5 kc. There is no reason to expect any better fit than this because of the idealized profile assumed. Furthermore, the atmospherics analyzed by Chapman and Macario were often quite near the receiving location in terms of a wavelength.

8. Propagation of ELF Pulses

In the foregoing discussion, it has been assumed that the source is time harmonic. In most cases of practical interest, the current in the source dipole is of a transient nature such as a surge. The radiated fields are also transient in nature. While the resulting waveforms may be transformed to the frequency plane via spectral analyses, it is often more convenient to study the waveforms themselves [28 to 31].

The source is assumed again to be equivalent to a vertical electric dipole on the earth's surface, but now its current-moment is a function of time and is denoted $p(t)$ which is zero for $t<0$. At a distance ρ along the earth's surface, the resulting vertical electric field is $e_z(t)$ and the horizontal magnetic field is $h_\phi(t)$. The Laplace Transform of the current moment is denoted $P_0(s)$ or more explicitly

$$P_0(s)=Lp(t)=\int_0^\infty e^{-st}p(t)dt \tag{36}$$

where s is the transform variable and may be formally identified with $i\omega$. Also

$$E_z(s)=Le_z(t)$$

and

$$H_\phi(s)=Lh_\phi(t)$$

where L is the Laplace-Transform operator defined above. Now it is assumed that the important frequencies in the spectra are sufficiently low that only the zero-order mode need be retained. Thus

$$\begin{matrix} E_z(s) \\ -\eta H_\phi(s) \end{matrix} \simeq \frac{\mu s}{2\pi h} P_0(s) \begin{matrix} K_0 \\ K_1 \end{matrix} (sS_0\rho/c) \tag{37}$$

where

$$S_0=\left[1+\frac{c}{h}\left(\frac{\epsilon_0}{s\sigma_i}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

and where K_0 and K_1 are modified Bessel functions of argument $sS_0\rho/c$. These may be asymptotically represented by

$$K_0(x)\simeq\left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{-x}\left[1+\frac{A_0}{x}+\frac{B_0}{x^2}+\ldots\right] \tag{38}$$

and

$$K_1(x)\simeq\left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{-x}\left[1+\frac{A_1}{x}+\frac{B_1}{x^2}+\ldots\right] \tag{39}$$

for large x, where

$$A_0=-1/8,\ B_0=9/128,\ A_1=3/8,\ B_1=-15/128.$$

Thus

$$\begin{matrix} E_z(s) \\ -\eta H_\phi(s) \end{matrix} \simeq \frac{\mu}{2\pi h}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{c}{\rho}\right)^{\frac{1}{2}} s^{\frac{1}{2}} P_0(s) e^{-sS_0\rho/c} \begin{bmatrix} 1+\frac{A_0}{s}\frac{c}{\rho}+\frac{B_0}{s^2}\left(\frac{c}{\rho}\right)^2+\ldots \\ 1+\frac{A_1}{s}\frac{c}{\rho}+\frac{B_1}{s^2}\left(\frac{c}{\rho}\right)^2+\ldots \end{bmatrix} \tag{40}$$

where x has been replaced by $sS_0\rho/c$ in the exponent but is replaced by $s\rho/c$ elsewhere. This is justified since S_0 is near unity. In fact, the binomial expansion

$$S_0=1+\frac{c}{2h}\left(\frac{\epsilon_0}{s\sigma_i}\right)^{\frac{1}{2}}-\frac{1}{8}\left(\frac{c}{h}\right)^2\frac{\epsilon_0}{s\sigma_i}+\frac{1}{16}\left(\frac{c}{h}\right)^3\left(\frac{\epsilon_0}{s\sigma_i}\right)^{3/2}+\ldots \tag{41}$$

may be truncated beyond the second term in most cases of interest, although the third term is retained since it doesn't complicate matters.

The current moment is now taken to be an impulse, that is

$$p(t)=p_0\delta(t)$$

where $\delta(t)$ is the unit impulse function. Thus

$$P_0(s) = p_0.$$

The transforms may now be readily inverted

$$\begin{matrix} e_z^0(t) \\ -\eta h_\phi^0(t) \end{matrix} \simeq \frac{\mu}{2\pi^2 h}\left(\frac{c}{2\rho}\right)^{\frac{1}{2}} \frac{p_0}{\beta^{3/2}} \exp\left[\frac{\rho c}{8h^2}\left(\frac{\epsilon_0}{\sigma_i}\right)\right]\left[P\left(\frac{t'}{\beta}\right) + \frac{A_0}{A_1}\left(\frac{ct'}{\rho}\right)P_a\left(\frac{t'}{\beta}\right) + \frac{B_0}{B_1}\left(\frac{ct'}{\rho}\right)^2 P_b\left(\frac{t'}{\beta}\right) + \ldots\right] u(t-\rho/c) \tag{42}$$

where

$$P(T) = \left(\frac{1}{2T} - 1\right)\left(\frac{1}{T}\right)^{3/2} \exp\left(-\frac{1}{4T}\right) \tag{43}$$

$$P_a(T) = 2\left(\frac{1}{T}\right)^{3/2} \exp\left(-\frac{1}{4T}\right) \tag{44}$$

$$P_b(T) = 4\left(\frac{1}{T}\right)^{3/2} \exp\left(-\frac{1}{4T}\right) - 2\frac{1}{T^2}\pi^{\frac{1}{2}} \operatorname{erfc}\left(\frac{1}{2T^{\frac{1}{2}}}\right) \tag{45}$$

and

$$t' = t - \rho/c \quad \text{and} \quad \beta^{\frac{1}{2}} = \frac{\rho}{2h}\left(\frac{\epsilon_0}{\sigma_i}\right)^{\frac{1}{2}}.$$

It is immediately seen that if $(ct'/\rho) << 1$, the field responses are both proportional to $P(T)$. To illustrate the nature of this function, it is plotted in figure 9 as a function of T. To facilitate the application a multiple scale is included which relates actual time t' in microseconds with the parameter T for distances ρ of 1,000 to 4,000 km. The height of the ionospheric reflecting layer is taken as 70 and 90 km. The quantity B indicated on the curves is related to the ionospheric conductivity by

$$B = \frac{1}{60\sigma_i h} = \frac{2\pi c}{\omega_r h}.$$

The values of B shown, namely 0.05 and 0.1, are typical. For $h = 90$ km, these correspond to ω_r values of 4×10^5 and 2×10^5, respectively.

The responses $e_z(t)$ and $h_\phi(t)$ to a general source $p(t)$ can be expressed in terms of the impulse responses $e_z^0(t)$ and $h_\phi^0(t)$, respectively, by using the convolution theorem. Thus

$$e_z(t) = \frac{1}{p_0}\int_{\rho/c}^{t} p(t-\tau)\, e_z^0(\tau)\, d\tau \cdot u(t-\rho/c) \tag{46}$$

and similarly for $h_\phi(t)$. For example, if a special analytical form for $p(t)$ is chosen such as

$$p(t) = \frac{p_0}{2}\left(\frac{6t_m}{\pi}\right)^{\frac{1}{2}} \frac{1}{t^{3/2}} \exp\left(-\frac{3t_m}{2t}\right), \tag{47}$$

the convolution integral may be evaluated to give

$$e_z(t) \simeq -\eta h_\phi(t) \simeq \frac{\mu}{2\pi^2 h}\left(\frac{c}{2_\rho}\right)^{\frac{1}{2}} \frac{p_0}{\gamma^{3/2}} \exp\left[\frac{\rho c}{8h^2}\left(\frac{\epsilon_0}{\sigma}\right)\right] P(t'/\gamma) \tag{48}$$

where

$$\gamma^{\frac{1}{2}} = \beta^{\frac{1}{2}} + (6t_m)^{\frac{1}{2}} = \frac{\rho}{2h}\left(\frac{\epsilon_0}{\sigma_i}\right)^{\frac{1}{2}} + (6t_m)^{\frac{1}{2}}.$$

The function $P(T)$ was defined above and is the same as the one plotted in figure 9. Now, however, T is to be identified with t'/γ rather than t'/β. The special form for $p(t)$ given above is a unidirectional pulse which reaches its maximum value at $t = t_m$ and decays to 7 percent of

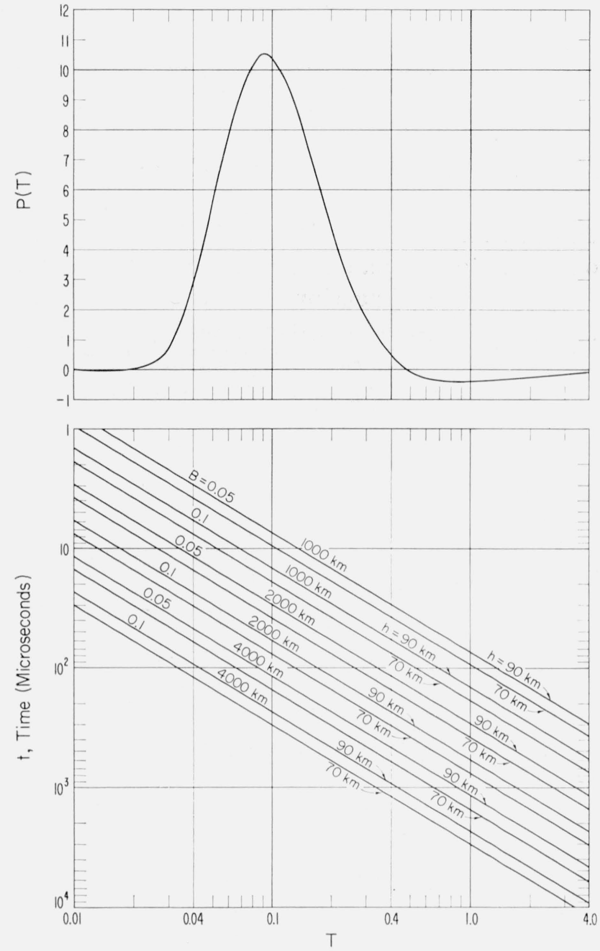

FIGURE 9. *Transient response of the zero-order mode for an impulsive vertical dipole source.*

The lower chart facilitates conversion from the parameter T to actual time t'.

its maximum value at $t=3t_m$. If the duration, $3t_m$, of the source pulse is much less than β the response $e_z(t)$ approaches $e^0(t)$. This is not surprising since $p(t)$ approaches $p_0\delta(t)$ if t_m approaches zero.

The general effect of the finite duration of the source pulse is seen to "stretch" the waveform of the electric and magnetic field. This effect has been discussed at some length in a recent paper by the author entitled "On the theory of the slow-tail portion of atmospheric waveforms." This is to appear in the July issue of the Journal of Geophysical Research.

While most cloud-to-ground lightning strokes may be represented by a vertical electric dipole, it is believed that cloud-to-cloud discharges are better represented by a horizontal electric dipole [32]. In the ELF portion of the spectrum the height of the discharge is very small compared with the wavelength, thus eq (17) is valid. If the source moment is again $p(t)$ then the transform for the vertical electric field, in the far zone, is given by

$$E_z(s) \simeq \frac{\mu}{2\pi h}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{c}{\rho}\right)^{\frac{1}{2}} sP_0(s)e^{-sS_0\rho/c}\left(\frac{\epsilon_0}{\sigma_i}\right)^{\frac{1}{2}}\cos\phi$$

where ϕ is the angle subtended by the horizontal dipole and the direction to the observer. Letting the source be an impulse [i.e., $p(t)=p_0\delta(t)$], the transform may be readily inverted to give

$$e_z(t) \simeq \frac{\mu}{2\pi^2\rho}\left(\frac{c}{2\rho}\right)^{\frac{1}{2}}\frac{p_0}{\beta^{3/2}}\left(\frac{\sigma_i}{\sigma_g}\right)^{\frac{1}{2}} Q(T) \tag{49}$$

where

$$Q(T)=\frac{1}{T^{5/2}}\left(\frac{1}{2T}-3\right)\exp\left(-\frac{1}{4T}\right), \tag{50}$$

and where, as before,

$$T=t'/\beta \quad\text{and}\quad \beta^{\frac{1}{2}}=\frac{\rho}{2h}\left(\frac{\epsilon_0}{\sigma_i}\right)^{\frac{1}{2}}.$$

An immediate generalization is a source which may be imagined as an inclined electric dipole. Lightning discharges from cloud-to-ground and cloud-to-cloud would be seldom purely vertical or horizontal. In view of the small dimensions of the discharge paths in terms of a wavelength, it is permissible to replace the inclined channel by superimposed vertical and horizontal electric dipoles. The resulting responses of the radiated field are thus obtained by superposition. For example, if the current moment of the vertical electric dipole is $p_0\delta(t)$ then the current moment of the horizontal electric dipole is $gp_0\delta(t)$ where g is a positive or negative dimensionless number. For lightning strokes with long horizontal sections, g could be large compared to unity [32]. As before, the direction $\phi=0$ corresponds to the direction of the horizontal dipole component.

Invoking the above mentioned principle of superposition, it readily follows that the far zone expression for the inclined dipole is given by

$$e_z(t)\simeq\frac{\mu p_0}{2\pi^2 h}\left(\frac{c}{2\rho}\right)^{\frac{1}{2}}[P(T)+GQ(T)]\beta^{-3/2}$$

where

$$G=g\left(\frac{\sigma_i}{\sigma_g}\right)^{\frac{1}{2}}\frac{h}{\rho}\cos\phi.$$

To indicate the bahavior of the waveform, the quantity

$$S(T)=\frac{P(T)+GQ(T)}{1+|G|}$$

is plotted in figure 10a and 10b for both positive and negative values of G. The quantity $S(T)$ characterizes the waveform of the inclined dipole energized by an impulsive current. The special case $G=0$, such that $S(T)=P(T)$, corresponds to a vertical dipole source, whereas the special case $G=\infty$ corresponds to a purely horizontal dipole source. Such a range of G values can be expected since cloud-to-ground and intra-cloud discharges are primarily vertical, whereas the cloud-to-cloud discharges and air flashes are mainly horizontal and may have either polarity.

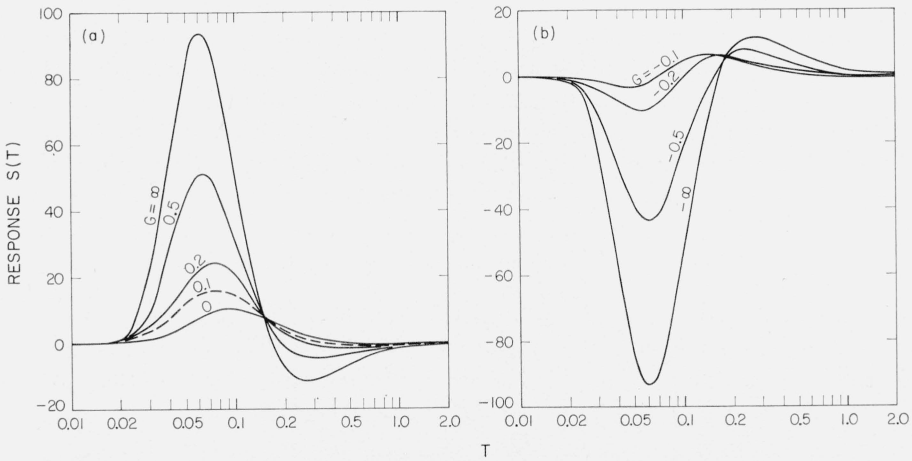

FIGURE 10. *Transient response of the zero-order mode for an inclined dipole source.*
When $G=0$ the dipole is vertical and when $G=\pm\infty$ the dipole is horizontal.

The modification of the pulse shape of the ELF waveforms, as a result of the inclination of the current channel, would appear to be an important factor in interpreting observed data. For example, Pierce [9], Lieberman [29], Tepley [31], and Jean [33] have all observed ELF waveforms which are strikingly similar to those shown in figures 10a and 10b. For example, the observed reception of a pulse which has a second half-cycle of the same magnitude as the first half-cycle can only be reconciled with an inclined source with G equal to about ± 0.2. For certain values of G (e.g., small negative values), a third half-cycle of relatively small amplitude is also produced.

9. Concluding Remarks

In the present theoretical study it has been assumed that the ELF signal propagates entirely in the space between the earth and a concentric ionosphere. It is quite possible that whistler type propagation may also be important. Furthermore, sources of ELF energy may be present in the exosphere due to the possible existence of ionized hydrogen whose gyrofrequency is of the order of 900 cps [34]. Other types of ions such as sodium may also be significant as pointed out by Aarons [14]. Despite these extraneous factors, the evidence that the "slow tails" are propagated in the earth-ionosphere waveguide is overwhelming.

It is the opinion of the author that the only really questionable assumption is the neglect of heavy ions on the constitutive properties of the lower edge of the E region or the top of the D region. Normally, it would be expected that the constitutive properties of the ionosphere would only be influenced by the electrons because of the extremely large ratio between the masses (i.e., of the order of 10^7). However, if the ratio of the number of heavy ions to the number of electrons is of the order of 10^3 or 10^4, as has been suggested by Scott [35] for the Arctic E layer, the influence of heavy ions may be significant at frequencies less than 100 cps. In particular, it should be noted that the gyrofrequency of the sodium ion is approximately 30 cps. However, if the operating frequency is somewhat greater than the gyrofrequency of heavy ions, their influence is expected to be minor.

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Additional numerical data are available in, Field strength calculations for ELF radio waves, by J. R. Wait and N. F. Carter, NBS Tech. Note 52 (PB161553) 1960; available from the Government Printing Office, Washington 25, D.C., for 50 cents.

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