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Terrestrial Propagation of Very-Low-Frequency Radio Waves

A Theoretical Investigation

James R. Wait*

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A self-contained treatment of the waveguide mode theory of the propagation of very-low-frequency radio waves is presented. The model of a flat earth with a sharply bounded and homogeneous ionosphere is treated for both vertical and horizontal dipole excitation. The properties of the modes are discussed in considerable detail.

The influence of earth curvature is also considered by reformulating the problem using spherical wave functions of complex order. The modes in such a curved guide are investigated and despite the initial complexity of the general solution, many interesting and limiting cases may be treated in simple fashion to yield useful and convenient formulas for calculation.

Other factors considered are the influence of the earth's magnetic field, antipodal effects, resonator type oscillations, and the influence of stratification at the lower edge of the ionosphere.

1. Introduction

The concept that radio waves are channeled between the earth and the ionosphere as in a waveguide has proved to be very useful at very low frequencies ($<$30 kc). In 1919, G. N. Watson [1] [1] employed this approach when he considered, at least in a formal way, the propagation of electromagnetic waves between an idealized homogeneous spherical earth and a concentric reflecting layer. Because of the extremely poor convergence of the exact series solutions, Watson devised a technique to convert this to a more rapidly converging series using function-theoretic means. The new representation corresponds to the sum of residues at poles in the complex plane and hence the name "residue series." The waves associated with these poles are the waveguide modes. Watson studied the numerical properties of these modes for the case of long waves or low frequencies and on assumption of a very highly conducting shell. This particular aspect of his investigation was prompted by the recent discoveries of Marconi that radio waves decay much more slowly with distance than predicted on the basis of classical diffraction theory in the absence of a reflecting shell. Watson found that the modes of low attenuation behaved like

$$\frac{1}{(\sin d/a)^{\frac{1}{2}}}\, e^{-\alpha\sqrt{f/\sigma}\, d}$$

where d is the great circle distance, f is the frequency, σ is the conductivity of the reflecting ionosphere, a is the radius of the earth, and α is a constant. For frequencies in the range from about 20 to 40 kc, observed field strengths behaved more or less in this fashion if the effective ionospheric conductivity was taken to be about 10^{-4} mhos/m or a conductivity of the same order as "tap water". Actually for frequencies in this range some 10 to 30 modes would be excited and if the complete mode sum were considered, the calculated field strength versus distance curve using such a model would show many rapid and violent undulations. Such a behavior is not observed under normal conditions and this fact alone is sufficient cause to reject this model even from a phenomenological viewpoint. The same model with certain refinements has been discussed more recently by Rydbeck [2] in a monograph, Bremmer [3] in his book, Schumann [4 to 6] in a series of papers, and most recently by Kaden [7]. From the frequency

* Central Radio Propagation Laboratory, National Bureau of Standards, Boulder, Colo.

[1] Figures in brackets indicate the literature references at the end of this paper.

analysis of atmospheric wave forms [8, 9], it is known that the attenuation rate does not vary like $\sqrt{f}$ except possibly at frequencies near 1 kc. Actually, the attenuation rate decreases with increasing frequency in the range from about 2 to 18 kc and thereafter increases. A behavior of this kind is highly suggestive of a Brewster angle effect. Such a proposal was first made by Namba [10] as far as this writer can ascertain. It thus appears than the ionosphere at vlf is behaving more like a magnetic wall (tangential H near zero) rather than an electric wall (tangential E near zero) as postulated by Watson, Bremmer, and Schumann.

Contributions to the waveguide mode concept have also been made by Budden [11 to 13 incl.] who unlike the early workers did not assume a highly conducting reflective layer at the outset. His model was a vertical electric dipole source in the space between the surface of a flat perfectly conducting ground and a sharply bounded homogeneous ionosphere. Various extensions and generalizations have been made by Al'pert [14, 15], Lieberman [16], Wait [17 to 22 incl.], Howe [23], and Friedman [24]. Their work is referred to from time to time in the text. It is the purpose of the present paper to present a unified treatment of the mode theory of vlf propagation. The results include much of the above work as special cases. In fact, in some instances, the analysis follows the work of Budden, Bremmer, and Rydbeck rather closely, although many new results are derived. While no attempt is made to present numerical results, limiting cases and simplified forms of the general solutions are discussed in some detail. An extensive bibliography is included in this paper for the convenience of those who are more interested in related numerical results and experimental data. It is intended that this paper will serve as a theoretical basis for subsequent papers by A. G. Jean, W. L. Taylor, A. D. Watt, and the author.

2. Basic Concepts

To introduce the subject a very simple model is chosen. The earth and the ionosphere are represented by perfectly conducting planes. In terms of a cylindrical coordinate system (ρ, ϕ, z) the ground surface is the plane $z=0$ and the lower boundary of the ionosphere is the plane $z=h$. The source is now considered to be a vertical electric dipole located on the ground. The electric field observed at some other point on the ground plane has only a vertical component and can be deduced by considering the images of the source dipole. These images are located at $z=\pm 2h$, $\pm 4h$, $\pm 6h$, etc., and all have equal sign and magnitude, because of the assumed perfect conductivity of the bounding walls. These images will always direct a wave broadside since the radiation from each image is in phase. At a distance which is large compared to h, this field can be calculated by replacing the line of dipole images by a continuous line source carrying an equivalent uniform current I_a. This current is the average current along the z axis and is given by

$$I_a = \frac{ds}{h} I$$

in terms of the height ds of the dipole and its current I. Now the field of a line source of current is well known and thus

$$E_z = \frac{I_a \mu\omega}{4} H_0^{(2)}(k\rho) = \frac{\mu\omega I ds}{4h} H_0^{(2)}(k\rho) \tag{2.1}$$

where $H_0^{(2)}(k\rho)$ is the Hankel function of the second kind of argument $k\rho$, μ is the permeability of the space, ω is the angular frequency, and $k=2\pi/\lambda$. When $\rho \gg \lambda$, the Hankel function can be replaced by the first term of its asymptotic expansion and this leads readily to

$$E_z \simeq \frac{\eta}{2} \frac{Ids}{h(\lambda\rho)^{\frac{1}{2}}} e^{i\pi/4} e^{-ik\rho} \tag{2.2}$$

where $\eta = (\mu/\epsilon)^{\frac{1}{2}} \simeq 120\pi$. As mentioned above, this field corresponds to the radiation directed broadside so the rays are parallel to the bounding walls. However, there will be other angles where the rays emanating from each of the dipoles in the line of images are also in phase.

Such a resonance condition exists when

$$2hC = n\lambda \tag{2.3}$$

where C is the cosine of the angle subtended by the rays and the z axis and n is an integer (see fig. 1). It is seen that for each value of n there are two families of rays which have the same radial phase velocity (i.e., $= c/S$) but with opposing vertical phase velocities (i.e., $= \pm c/C$). Again the radiation of these sets of waves (i.e., modes) can be imagined to originate from an equivalent line source. The strength of this line source is IS where S is the sine of the angle subtended by the rays and the vertical direction. To obtain the resultant vertical field, this must be again multiplied by S. Consequently the resultant field of all the families of rays or modes is obtained by summing over integral values of n from 0 to ∞ to give

$$E_z = \frac{\mu\omega I ds}{4h} \sum_{n=0}^{\infty} \epsilon_n S_n^2 H_0^{(2)}(kS_n\rho) \tag{2.4}$$

where $\epsilon_0 = 1$, $\epsilon_n = 2 (n = 1, 2, 3 \ldots)$ and $S_n = (1 - C_n^2)^{\frac{1}{2}}$ and $C_n = n\lambda/2h$. The term $n = 0$, corresponding to mode zero discussed above, is only included once in the summation, whereas the higher modes are included twice. In the far field, this expression for the field reads

$$E_z \simeq \frac{\eta I ds}{2h(\lambda\rho)^{\frac{1}{2}}} e^{i\pi/4} \sum_{n=0,1,2,\ldots}^{\infty} \epsilon_n S_n^{3/2} e^{-ikS_n\rho}. \tag{2.5}$$

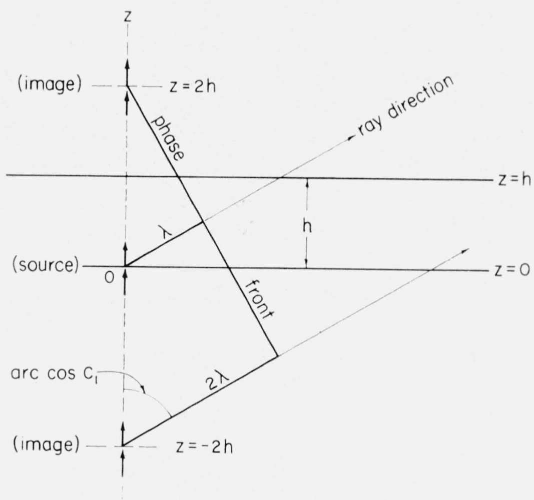

FIGURE 1. *Depicting ray-geometry corresponding to the first mode between parallel plates; for resonance, $\lambda = 2hC_1$.*

Up to this point the bounding walls have been assumed to be of perfect conductivity: The reflection coefficients for the rays are always $+1$. Another simple case is when the upper boundary has a reflection coefficient of -1 corresponding to a perfect magnetic conductor, and the lower boundary still has a reflection coefficient of $+1$ corresponding to the more common perfect electrical conductor. For this situation the images are now located at $z = \pm h, \pm 2h, \ldots$ but now they alternate in sign. It may be observed that there is no coherent family of rays directed broadside. This would have been the zero-order mode. The resonance condition for the modes is now

$$2hC = (n - \tfrac{1}{2})\lambda \tag{2.6}$$

where $n = 1, 2, 3, \ldots$. The corresponding expression for the vertical electric field is thus given by

$$E_z = \frac{\mu\omega I ds}{2h} \sum_{n=1,2,3,\ldots}^{\infty} S_n^2 H_0^{(2)}(kS_n\rho) \tag{2.7}$$

where the summation starts at $n=1$ and includes all positive integers. In the far field

$$E_z \simeq \frac{\eta I ds}{h(\lambda\rho)^{1/2}} e^{i\pi/4} \sum_{n=1,2,3,\ldots}^{\infty} S_n^{3/2} e^{-ikS_n\rho}. \tag{2.8}$$

In the foregoing discussion the observer and the source, which is a vertical electric dipole, are located on the ground plane. The above results are easily generalized to a finite source height, z_0, and a finite observer height, z, by inserting the factor cos (kz_0C_n) cos (kzC_n) inside the summations of eqs (4), (5), (7), and (8). This can be verified by returning to the image picture and noting now that they are located at $z=-h$, $\pm(2h+z_0)$, $\pm(2h-z_0)$, $\pm(4h+z_0)$, $\pm(4h-z_0)$, ... It may also be observed that the cos (kzC_n) when replaced by $[\exp(ikzC_n)+\exp(-ikzC_n)]/2$ can be identified as a family of upgoing and downcoming rays within the guide.

The important modifications of the preceding formulas as a result of imperfect reflection can be obtained by a rather simple physical argument. The complete treatment requires a more mathematical approach which is to be described later on.

The reflection coefficient for a ray incident on the ground plane at an angle (whose cosine is C) is denoted $R_g(C)$. The corresponding reflection coefficient for the upper boundary which is the lower edge of ionosphere is denoted $R_i(C)$. The resonance condition now has the form

$$R_g(C)R_i(C)e^{-i2khC}=e^{-i2\pi n} \tag{2.9}$$

which reduces to eq (3) if the reflection coefficients are both $+1$ and reduces to eq (6) if one reflection coefficient is $+1$ and the other -1. Physically, the above more general form can be the condition for a ray to traverse the guide twice, be reflected at each boundary, and yet still suffer a net phase shift of $2\pi n$ radians where n is an integer. Since $R_g(C)$ and $R_i(C)$ may be complex and less than unity the value of C (i.e., C_n) which satisfies the resonance equation may also be complex. The angle of incidence of these rays in the guide are thus also complex. The corresponding value of S_n is also complex and this results in attenuation of the wave in the radial direction. In fact, the attenuation constant is minus the imaginary part of kS_n in nepers per unit distance.

When the angle or its cosine C must be complex in order to satisfy a resonance equation, the resulting waves are damped. The numerical solution of such a complex resonance equation is quite difficult, in general, since it is not usually possible to obtain an explicit expression for C in terms of known parameters. This aspect of the problem is discussed in a later section.

3. Formulation for Flat Earth Case

3.1. Vertical Dipole Excitation

It is now desirable to formulate the problem in a more definite fashion. A vertical electric dipole of moment $I\,ds$ is placed in a homogeneous plane layer bounded by two plane interfaces (see fig. 2). The lower interface is at $z=0$ corresponding to the surface of a homogeneous ground of conductivity σ_g and dielectric constant ϵ_g. The upper interface at $z=h$ is the lower edge of a homogeneous ionosphere which for the moment is assumed to be isotropic and has effective electrical constants σ_i and ϵ_i. The fields in these regions can be derived from a Hertz vector which has only a z component, Π_z. Thus, for $h\geqq z\geqq 0$

$$\begin{aligned}
E_\rho &= \frac{\partial^2}{\partial\rho\partial z}\Pi_z & H_\rho &= 0 \\
E_\phi &= 0 & H_\phi &= -i\epsilon\omega\frac{\partial\Pi_z}{\partial\rho} \\
E_z &= \left(k^2+\frac{\partial^2}{\partial z^2}\right)\Pi_z & H_z &= 0
\end{aligned} \tag{3.1}$$

where $k=(\epsilon\mu)^{\frac{1}{2}}\omega=2\pi/\lambda$.

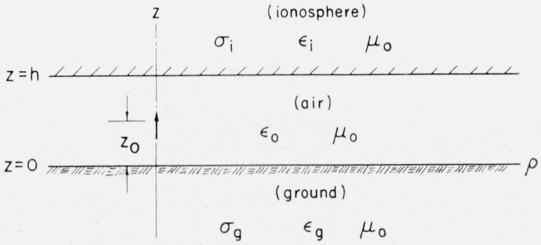

FIGURE 2. *Cylindrical coordinate system for the vertical dipole between the two plane interfaces.*

Similar expressions are applicable for the regions $z>h$ and $z<0$ where the z directed Hertz vectors are denoted $\Pi_z^{(i)}$ and $\Pi_z^{(g)}$, respectively, and the corresponding wave numbers are k_i and k_g, respectively. The formal solution of this problem is obtained in a straightforward fashion by requiring that the tangential field components E_ρ and H_ϕ are continuous across the two plane interfaces. An equivalent statement of these matching conditions is as follows:

$$\left.\begin{array}{l} k^2\Pi_z=k_g^2\Pi_z{}^{(g)} \\ \dfrac{\partial\Pi_z}{\partial z}=\dfrac{\partial\Pi_z{}^{(g)}}{\partial z} \end{array}\right] \quad \text{for } z=0, \tag{3.2}$$

$$\left.\begin{array}{l} k^2\Pi_z=k_i^2\Pi_z^{(i)} \\ \dfrac{\partial\Pi_z}{\partial z}=\dfrac{\partial\Pi_z^{(i)}}{\partial z} \end{array}\right] \quad \text{for } z=h. \tag{3.3}$$

To facilitate the solution, the primary excitation resulting from the source dipole is represented as a spectrum of plane waves. This well-known representation, for the primary Hertz function, is given as follows: [25]

$$\Pi_z^{(p)}=\frac{M\exp\{-ik[\rho^2+(z-z_0)^2]^{\frac{1}{2}}\}}{[\rho^2+(z-z_0)^2]^{\frac{1}{2}}}=\frac{Mik}{2}\int_\Gamma H_0^{(2)}(kS\rho)\exp[-ikC|z-z_0|]\,dC \tag{3.4}$$

where $M=I\,ds/(4\pi i\omega\epsilon)$ and $S=(1-C^2)^{\frac{1}{2}}$. The integration variable C can be regarded as the cosine of the angle of incidence of the plane waves in the spectrum. Γ is the contour of integration and it extends from $-\infty$ along the negative real axis to the origin, then out along the real axis to $+\infty$. It should be noted since C can be greater than unity complex angles in the spectrum occur. The above form for the primary excitation then suggests that the resultant Hertz function for the three regions can be written in the respective forms

(I)
$$\Pi_z=\Pi_z^{(p)}+\int_\Gamma[A(C)e^{-ikCz}+B(C)\,e^{+ikCz}]\,H_0^{(2)}\,kS\rho)\,dC \tag{3.5}$$

for $0\leqq z\leqq h$;

(II)
$$\Pi_z^{(g)}=\int_\Gamma G(C)e^{+ik_gC_gz}H_0^{(2)}(kS\rho)dC \tag{3.6}$$

for $z\leqq 0$; and

(III)
$$\Pi_z^{(i)}=\int_\Gamma I(C)e^{-ik_iC_iz}H_0^{(2)}(kS\rho)dC \tag{3.7}$$

for $z>h$. In the above, it can be verified that these Hertz functions satisfy the appropriate wave equation subject to the conditions that

$$N_g(1-C_g^2)^{\frac{1}{2}}=(1-C^2)^{\frac{1}{2}}=N_i(1-C_i^2)^{\frac{1}{2}} \tag{3.8}$$

where

$$N_g=\left(\frac{\sigma_g+i\epsilon_g\omega}{i\epsilon\omega}\right)^{\frac{1}{2}} \text{ and } N_i=\left(\frac{\sigma_i+i\epsilon_i\omega}{i\epsilon\omega}\right)^{\frac{1}{2}}.$$

Terms containing exp $(-ik_gC_gz)$ and exp (ik_iC_iz) are not permitted since they would violate the Sommerfeld radiation condition at $|z|\rightarrow\infty$.

The form of the unknown function $A(C)$, $B(C)$, $G(C)$, and $I(C)$ can be obtained explicitly by using the four equations of continuity. This purely algebraic process is easily carried out and further details are omitted. The resultant Hertz function for the air region is explicitly given by

$$\Pi_z=\frac{ikM}{2}\int_\Gamma F(C)H_0^{(2)}(kS\rho)dC \tag{3.9}$$

where

$$F(C)=\frac{(e^{ikCz}+R_ge^{-ikCz})(e^{ikC(h-z_0)}+R_ie^{-ikC(h-z_0)})}{e^{ikCh}(1-R_gR_ie^{-2ikhC})} \tag{3.10}$$

and

$$R_g=R_g(C)=\frac{N_gC-C_g}{NC_g+C_g} \tag{3.11}$$

$$R_i=R_i(C)=\frac{N_iC-C_i}{N_iC+C_i}. \tag{3.12}$$

It can be immediately noted that the integrand has poles where

$$1-R_g(C)R_i(C)e^{2ikhC}=0.$$

This is the (complex) resonance equation obtained in the previous section by intuitive reasoning.

The integral may be evaluated by using function-theoretic means. The contour is transformed to the S plane. Thus eq (9) becomes

$$\Pi_z=\frac{ikM}{2}\int_\Gamma F(C)H_0^{(2)}(kS\rho)\frac{SdS}{C} \tag{3.13}$$

where the contour Γ may now be taken as the real axis from $-\infty$ to $+\infty$ in the S plane.

The contour is now closed by semicircles in the lower half-plane as indicated in figure 3. Because of the branch point at $S=+1$ and its associated branch line drawn vertically downward, the closing contour runs from one Riemann sheet to the other in the manner indicated. After making two circuits the contour closes on itself. The contours are indented at other branch points in the manner shown for B on the figure. These branch points are located well below the real axis (i.e., imaginary part of $S>>1$) and the corresponding branch cut integrations lead to waves which are heavily damped provided

$$\frac{h}{\rho}<<\left|\frac{1-N_i^2}{N_i}\right| \text{ and } k\rho|N_i^2-1|>>1. \tag{3.14}$$

Now the line integral around the complete circuit in the two sheeted Riemann surface is equal to $-4\pi i$ times the sum of the residues of the poles of the integrand. The poles which occur in pairs are located on both Riemann sheets. For highly conducting walls, a number (at least one) is located just below the real axis between the origin and the branch point at $+1$. The remainder are located along or near the negative imaginary axis. The contribution from these latter poles is very small and they correspond to the waveguide modes beyond "cut-off."

The contribution along the semicircles is seen to vanish if the radius R approaches infinity. This is assured by the presence of the Hankel function $H_0^{(2)}(kS\rho)$ which is exponentially decreasing in the lower half-plane of S. Consequently, each of the two integrations along the

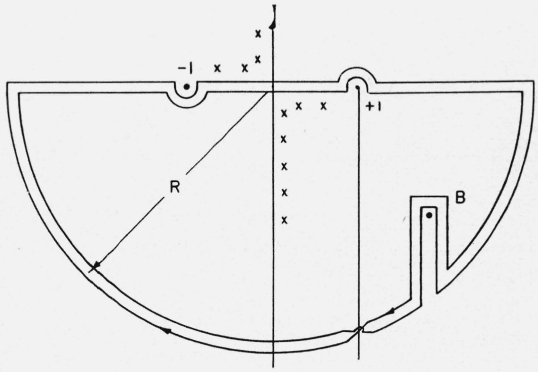

FIGURE 3. *S plane.*
Branch point; x, poles.

real axis are approximately equal to $-2\pi i$ times the sum of the residues at $S=S_n$. When the integral is expressed in the original C plane the residue series may be regarded as the contributions from the poles at $C=C_n$ which for $n=0, 1, 2, \ldots$ are in the first quadrant and for $n=-1, -2, -3, \ldots$ are in the third quadrant. This leads to

$$\Pi_z=\pi kM\sum_{n=-\infty}^{\infty}\left[\frac{1}{\frac{\partial}{\partial C}\frac{1}{F(C)}}\right]_{C=C_n}H_0^{(2)}(kS_n\rho) \tag{3.15}$$

where the square bracket term is the residue of the function $F(C)$ at the pole $C=C_n$. Carrying out the differentiation and making use of the resonance condition

$$R_g(C_n)R_i(C_n)e^{-2ikhC_n}=1 \tag{3.16}$$

leads readily to

$$\Pi_z\simeq\frac{i\pi M}{h}\sum_{n=-\infty}^{+\infty}H_0^{(2)}(kS_n\rho)\,f_n(z_0)\,f_n(z)\,\delta_n(C) \tag{3.17}$$

where

$$\delta_n(C)=\left[1+i\frac{\partial[R_i(C)R_g(C)]/\partial C}{2khR_i(C)R_g(C)}\right]^{-1}_{C=C_n} \tag{3.18}$$

where

$$f_n(z)=\frac{e^{ikC_nz}+R_g(C_n)e^{-ikC_nz}}{2[R_g(C_n)]^{\frac{1}{2}}} \tag{3.19}$$

and $f_n(z_0)$ has exactly the same form.

When the walls are perfectly conducting $R_i(C)=R_g(C)=1$, the factor $\delta_n(C)$ becomes unity if $n=1, 2, 3, \ldots$ and becomes ½ if $n=0$, and $f_n(z)=\cos kC_nz$. The above expression can then be written

$$\Pi_z=\frac{i\pi M}{h}\sum_{n=0,1,2,\ldots}^{\infty}\epsilon_nH_0^{(2)}(kS_n\rho)\cos(kC_nz_0)\cos(kC_nz) \tag{3.20}$$

where $\epsilon_0=1$, $\epsilon_n=2(n\neq 0)$. The corresponding value of the electric field component E_z can be expressed

$$E_z=\left(k^2+\frac{\partial^2}{\partial z^2}\right)\Pi_z=\frac{\mu\omega Ids}{4h}\sum_{n=0,1,2,\ldots}^{\infty}\epsilon_nS_n^2H_0^{(2)}(kS_n\rho)\cos(kC_nz_0)\cos(kC_nz) \tag{3.21}$$

which is in agreement with eq (2.4) obtained from physical intuitive reasoning.

The extension to the case when the source is a vertical magnetic dipole is simple. Formally the above results are still valid if I is replaced by the magnetic current K. E_z then becomes H_z and the field is essentially horizontally polarized. The reflection coefficients $R_g(C)$ and $R_i(C)$ are to be replaced by their counterparts for horizontal polarization.

Explicitly these are given by

$$R_g^h(C)=\frac{C-N_gC_g}{C+N_gC_g} \tag{3.22}$$

and

$$R_i^h(C)=\frac{C-N_iC_i}{C+N_iC_i}. \tag{3.23}$$

It may be noted, since $R(-C)\ R(C)=1$ where R is any of the reflection coefficients R_i, R_g, R_i^h, or R_g^h, that for negative values of n

$$C_{-1}=-C_0$$

$$C_{-2}=-C_1$$

$$C_{-(n+1)}=-C_n.$$

It then follows that the summations $\sum_{-\infty}^{+\infty}\ldots$ may be replaced by $2\sum_{0}^{\infty}\ldots$ everywhere since $\delta_n(C)=\delta_n(-C)$.

For convenience in numerical computation, it is convenient to express the field components as a ratio to the quantity

$$E_0=i(\eta/\lambda)Ids(e^{-ik\rho})/\rho \qquad (\eta\cong 120\pi). \tag{3.24}$$

E_0 is the field of the source at a distance ρ on a perfectly conducting ground. Thus for both the source and the observer near the ground it is not difficult to show by means of eqs (1), (17), and (19) that

$$E_z=WE_0$$

where

$$W\cong -i\pi\frac{\rho}{h}e^{ik\rho}\sum_{n=0}^{\infty}\delta_nS_n^2H_0^{(2)}(kS_n\rho) \tag{3.25}$$

$$E_\rho=SE_0$$

where

$$S\cong\frac{\pi}{N_g}\frac{\rho}{h}e^{ik\rho}\sum_{n=0}^{\infty}\delta_nS_nH_1^{(2)}(kS_n\rho) \tag{3.26}$$

and

$$H_\phi=TE_0/\eta$$

where

$$T\cong-\pi\frac{\rho}{h}e^{ik\rho}\sum_{n=0}^{\infty}\delta_nS_nH_1^{(2)}(kS_n\rho). \tag{3.27}$$

In the above it has been assumed that $|N_g|^2>>1$.

When $k\rho>>1$, corresponding to the "far-zone," the above expressions may be simplified since the Hankel functions may be replaced by the first term of their asymptotic expansion.[2]

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[2] The relevant expansions are

$$H_0^{(2)}(x)\simeq\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}e^{-ix}e^{i\frac{\pi}{4}}\left[1-\frac{1}{8ix}+\frac{9}{2(8ix)^2}\cdots\right]$$

$$H_1^{(2)}(x)\simeq\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}e^{-ix}e^{i\frac{3\pi}{4}}\left[1+\frac{3}{8ix}-\frac{15}{2(8ix)^2}\cdots\right].$$

This leads to the compact result

$$\begin{bmatrix} W \\ S \\ T \end{bmatrix} \simeq \frac{(\rho/\lambda)^{\frac{1}{2}}}{(h/\lambda)} e^{i\left[\frac{2\pi\rho}{\lambda}-\frac{\pi}{4}\right]} \sum_{n=0}^{\infty} \delta_n \begin{bmatrix} S_n^{3/2} \\ -S_n^{\frac{1}{2}}/N_g \\ S_n^{\frac{1}{2}} \end{bmatrix} e^{-i2\pi S_n\rho/\lambda} \tag{3.28}$$

which is valid for $\rho >>> \lambda$. As expected, the ratio of W to T for a given mode is S_n which for low order or grazing modes is of the order of unity. The ratio of S to T for a given mode is $-1/N_g$ which is very small compared to unity; in fact, it vanishes for a perfectly conducting ground as it must.

3.2. Horizontal Dipole Excitation

The previous section contains the formulation for a vertical dipole source. The corresponding treatment for a horizontal dipole is also quite straightforward although the lack of symmetry increases the complexity. Often in the radio propagation literature the statement is made that the fields of a *horizontal electric* dipole are the same as, or proportional to, the fields of a *vertical magnetic* dipole at the same location. This is only true broadside to the horizontal dipole where the field is purely TE (transverse electric) or horizontally polarized. For other directions, the field has a TM (transverse magnetic) component corresponding to vertical polarization. The modes corresponding to the TM waves may have much smaller attenuation than the modes of the TE type and thus it is desirable to formulate the problem directly with a horizontal dipole source.

As in the previous section the earth and the ionosphere are assumed to be bounded by parallel planes separated by a distance h. Choosing a rectangular coordinate system (x, y, z), the dipole is located at $z=z_0$ and is parallel to the x axis (see fig. 4).

The solution for a horizontal dipole over a homogeneous flat earth with no ionosphere (i.e., $h\to\infty$) was obtained by Sommerfeld many years ago. The generalization for the two interfaces is quite straightforward. A Hertz vector is introduced which has both an x component Π_x and a z component Π_z. The fields in terms of these are

$$\begin{aligned}
E_x &= k^2\Pi_x + \frac{\partial}{\partial x}\left[\frac{\partial \Pi_x}{\partial x} + \frac{\partial \Pi_z}{\partial z}\right], \\
E_y &= \frac{\partial}{\partial y}\left[\frac{\partial \Pi_x}{\partial x} + \frac{\partial \Pi_z}{\partial z}\right], \\
E_z &= k^2\Pi_z + \frac{\partial}{\partial z}\left[\frac{\partial \Pi_x}{\partial x} + \frac{\partial \Pi_z}{\partial z}\right], \\
H_x &= i\epsilon\omega \frac{\partial \Pi_z}{\partial y}, \\
H_y &= -i\epsilon\omega\left[\frac{\partial \Pi_z}{\partial x} - \frac{\partial \Pi_x}{\partial z}\right], \\
H_z &= -i\epsilon\omega \frac{\partial \Pi_x}{\partial y}.
\end{aligned} \tag{3.29}$$

As before a subscript g or i is added to these quantities when specific reference is made to the ground or the ionosphere, respectively.

The boundary conditions at the interfaces $z=0$ and $z=h$ are that tangential components of the fields are continuous. This, in turn, requires that $k^2\Pi_x$, $\partial\Pi_x/\partial x + \partial\Pi_z/\partial z$, $i\epsilon\omega\Pi_z$ and $i\epsilon\omega\partial\Pi_x/\partial z$ are each continuous at these interfaces. Integral representations of Π_x and Π_z which

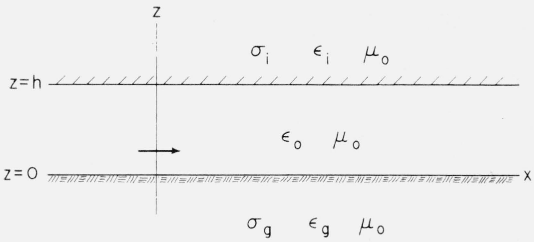

FIGURE 4. *Rectangular coordinate system for the horizontal dipole between two plane interfaces.*

are suitable for matching are

$$\Pi_x = M\frac{e^{-ikR}}{R} + \int_\Gamma [U(C)e^{-ikCz} + V(C)e^{+ikCz}] H_0^{(2)}(kS\rho)dC \tag{3.30}$$

$$\Pi_z = \frac{\partial}{\partial x}\int_\Gamma [X(C)e^{-ikCz} + Y(C)e^{+ikCz}] H_0^{(2)}(kS\rho)dC \tag{3.31}$$

for $0 < z < h$. Similar expressions are used for the x and z components of the Hertz vector in the spaces $z < 0$ and $z > h$. Applying the boundary conditions involving Π_x only, leads directly to the following solutions for the unknown coefficients in Π_x:

$$U(C) = \left[\frac{R_g^h + R_g^h R_i^h \exp[-2ikC(h-z_0)]}{1 - R_g^h R_i^h \exp(-2ikCh)}\right] \exp(-ikCz_0) \tag{3.32}$$

$$V(C) = \left[\frac{R_i^h + R_g^h R_i^h \exp[-2ikCz_0]}{1 - R_g^h R_i^h \exp(-2ikCh)}\right] \exp[-ikC(2h-z_0)] \tag{3.33}$$

where R_g^h and R_i^h are the complex reflection coefficients given by eqs (22) and (23) and they are also functions of C.

The remaining two boundary conditions, namely, the continuity of $i\epsilon\omega\Pi_z$ and $\partial\Pi_z/\partial z + \partial\Pi_x/\partial x$ enable the coefficients $X(C)$ and $Y(C)$ to be found in terms of $U(C)$ and $V(C)$. The connecting relations are

$$X(C) = \frac{P - QR_g \exp(-ikCh)}{1 - R_i R_g \exp(-2ikCh)} \tag{3.34}$$

and

$$Y(C) = \frac{PR_i \exp(-2ikCh) - Q\exp(-ikCh)}{1 - R_i R_g \exp(-i2kCh)} \tag{3.35}$$

where

$$P = [e^{-ikCz_0} + U + V]\left(\frac{1+R_g}{2ikC}\right) \tag{3.36}$$

and

$$Q = [e^{-ikC(h-z_0)} + Ue^{-ikCh} + Ve^{ikCh}]\left(\frac{1+R_i}{2ikC}\right). \tag{3.37}$$

It is understood that R_i, R_g are functions of C and are defined by eqs (11) and (12).

The integral for Π_x can be observed to have precisely the same form as the z component of a (magnetic) Hertz vector for a vertical magnetic dipole. This in turn has the same general form as the z component of the (electric) Hertz vector for a vertical electric dipole. The residue series representation for Π_x is given by

$$\Pi_x \simeq \frac{\pi i M}{h}\sum_m H_0^{(2)}(kS_m\rho) f_m^h(z_0) f_m^h(z) \delta_m^h(C) \tag{3.38}$$

where

$$\delta_m^h(C) = \left[1 + i\frac{dR_i^h(C)R_g^h(C)/dC}{2khC}\right]_{C=C_m}$$

where the summation is over the poles of the integrand at $C=C_m$ of the integral in eq (30). These are solutions of

$$R_i^h(C)R_g^h(C)e^{-2ikhC}=e^{-i2\pi m} \tag{3.39}$$

for integral values of m.

The height functions have the form

$$2f_m^h(z)=[R_g^h(C)]^{-\frac{1}{2}}e^{+ikCz}+[R_g^h(C)]^{\frac{1}{2}}e^{-ikCz}. \tag{3.40}$$

Of particular interest is the vertical magnetic field component. It is given by

$$\begin{aligned} H_z&=-i\epsilon\omega\frac{\partial\Pi_x}{\partial y}=-i\epsilon\omega\frac{\partial\Pi_x}{\partial\rho}\sin\phi \\ &=\sin\phi\,\frac{Idsk}{4h}\sum_m S_m H_1^{(2)}(kS_m\rho)f_m^h(z_0)f_m^h(z)\delta_m^h(C). \end{aligned} \tag{3.41}$$

When $|kS_m\rho|>>1$ or when $\rho>>\lambda$, the first term of the asymptotic expansion of the Hankel function $H_1^{(2)}(kS_m\rho)$ may be employed. This leads to

$$H_z\cong\frac{\sin\phi}{\eta}E_0\frac{(\rho/\lambda)^{\frac{1}{2}}}{(h/\lambda)}\frac{e^{i\pi/4}}{2}\sum_m S_m^{\frac{1}{2}}f_m(z_0)f_m(z)e^{ik(1-S_m)\rho}\,\delta_m^h(C). \tag{3.42}$$

The other field components involve integrals which may be treated in the same way. Also of great interest is the vertical electric field. It is not difficult to show that

$$E_z=-\frac{\partial}{\partial x}\left[\frac{kM\pi}{h}\sum_n H_0^{(2)}(kS_n\rho)\,g_n(z_0)f_n(z)\,\delta_n(C)\right] \tag{3.43}$$

where

$$2g_n(z_0)=C_n[R_g(C_n)]^{-\frac{1}{2}}\exp(ikC_nz)-C_n[R_g(C_n)]^{\frac{1}{2}}\exp(-ikC_nz). \tag{3.44}$$

It may be noted that

$$kg_n(z_0)=-i\frac{\partial}{\partial z_0}f_n(z_0)\cdot \tag{3.45}$$

The summation is now over the roots $C=C_n$ of the equation

$$R_i(C)R_g(C)e^{-i2\pi kCh}=e^{-i2\pi n}. \tag{3.46}$$

When $\rho>>\lambda$ the above expression simplifies to

$$E_z\cong\cos\phi\,E_0\frac{(\rho/\lambda)^{\frac{1}{2}}}{(h/\lambda)}\frac{e^{-i\pi/4}}{2}\sum_n S_n^{\frac{1}{2}}f_n(z)g_n(z_0)e^{ik(1-S_n)\rho}\delta_n(C). \tag{3.47}$$

When $|kC_nz_0|\ll1$ and $|kC_nz|\ll1$ the preceding simplifies even further to

$$E_z\cong\cos\phi\,E_0\frac{1}{N_g}\frac{(\rho/\lambda)^{\frac{1}{2}}}{(h/\lambda)}\frac{e^{-i\pi/4}}{2}\sum_n S_n^{\frac{1}{2}}e^{ik(1-S_n)\rho}\delta_n(C). \tag{3.48}$$

4. Properties of the Modes for Flat Earth Case

4.1. Vertical Polarization

Much has been written in the literature on the numerical characteristics of the modes. Controversy concerning the method of numbering the modes has also arisen. It is the opinion that much of this discussion has been unnecessarily involved. The important thing is to sum over all modes which are excited by the dipole. Consequently only these modes need be considered. Because of the form of the integration contour Γ the relevant solutions must satisfy

$$R_g(C)R_i(C)e^{-2ikhC}=e^{-i2\pi n} \tag{4.1}$$

and have their real and imaginary parts positive. That is, C_n is located in the first quadrant. The numbering is then assigned in such a way that there is continuity in the limiting case of perfect conductivity (i.e., $R_g=R_i=1$).

$$C_n=\frac{\pi n}{kh} \text{ with } n=0,1,2,3, \ldots .$$

As Dr. H. H. Howe points out, this is not quite unambiguous when both the ground and the ionosphere are both imperfectly conducting. The more general statement of the rule is [23]:

For a fixed value of kh, determine n on the assumption of perfectly conducting walls, then σ_g and σ_i in turn are to decrease *continuously* to their prescribed values while C varies continuously. For walls of high but finite conductivity this means that mode 0 has a minimum attenuation and the other modes have successively higher attenuation as n increases. For poorly conducting walls, this is not necessarily so, and in fact, in cases of most practical interest for the vlf band the mode of lowest attenuation is of order one.[3]

Numerical values for C_n are available and will not be quoted here. Some properties of the modes, however, may be simply obtained without resorting to a full numerical solution. For example, if the walls are highly conducting the reflection coefficients may be approximated as follows:

$$R_g(C)=\frac{N_gC-C_g}{N_gC+C_g}\simeq 1-\frac{2C_g}{N_gC}\simeq \exp\left(-\frac{2}{N_gC}\right) \tag{4.2}$$

$$R_i(C)=\frac{N_iC-C_i}{N_iC+C_i}\simeq 1-\frac{2C_i}{N_iC}\simeq \exp\left(-\frac{2}{N_iC}\right) \tag{4.3}$$

subject to $|C|^2>>|N_g|^{-2}$ and $|N_i|^{-2}$. Then the resonance equation is simplified to

$$khC=\pi n+i\frac{\Delta}{C} \tag{4.4}$$

where $\Delta=(1/N_i+1/N_g)$.

Regarding Δ/C as a small quantity, this can be solved to give

$$S_n\cong\left[1-\left(\frac{\pi n}{kh}\right)^2\right]^{\frac{1}{2}}-i\,\frac{\epsilon_n}{2kh}\,\Delta\left[1-\left(\frac{\pi n}{kh}\right)^2\right]^{-\frac{1}{2}} \tag{4.5}$$

where $\epsilon_0=1$, $\epsilon_n=2$, $(n\neq 0)$.

The magnitude of the second term must be small compared to the first term for the above perturbation method to be valid. This restriction and the previous one are both met if simultaneously

$$kh|\Delta|<<1 \text{ and } \frac{|\Delta|}{kh}<<1-\left(\frac{\pi n}{kh}\right)^2.$$

Now for highly conducting walls $\sigma_g>>\epsilon_g\omega$ and $\sigma>>\epsilon_i\omega$ and thus

$$\Delta\cong\sqrt{i}|\Delta|\cong\sqrt{i}\left[\left(\frac{\epsilon\omega}{\sigma_g}\right)^{\frac{1}{2}}+\left(\frac{\epsilon\omega}{\sigma_i}\right)^{\frac{1}{2}}\right]. \tag{4.6}$$

[3] The mode numbering system described above is somewhat different from Budden. For a fixed value of σ_i he starts with a very small value of kh, increases it continuously and requires that C varies continuously for the same n value. [13]

Consequently

$$\operatorname{Re} S_n \cong \left[1-\left(\frac{\pi n}{kh}\right)^2\right]^{\frac{1}{2}} + \frac{\epsilon_n |\Delta|}{2\sqrt{2}kh}\left[1-\left(\frac{\pi n}{kh}\right)^2\right]^{-\frac{1}{2}} \tag{4.7}$$

and

$$\operatorname{Im} S_n \cong -\frac{\epsilon_n |\Delta|}{2\sqrt{2}kh}\left[1-\left(\frac{\pi n}{kh}\right)^2\right]^{-\frac{1}{2}}. \tag{4.8}$$

The influence of finite conductivity is thus to increase the real part of S_n and consequently the phase velocity $c/\operatorname{Re} S_n$ is decreased relative to the free space value c. As expected the finite conductivity produces damping and the resulting attenuation factor is $-k \operatorname{Im} S_n$ in nepers per unit distance, to this approximation.

The above approximate formulas for the real and imaginary parts of S_n are the ones usually encountered. They have been quoted by Schumann [5] for example. It is not also appreciated that they are not applicable for a mode which is near cutoff. This should be evident, however, from the second inequality given above. To relax this restriction, the resonance equation

$$khC = \pi n + i\frac{\Delta}{C} \text{ (valid for } |\Delta| kh << 1) \tag{4.9}$$

is solved as a quadratic in C to yield

$$2C_n = \left(\frac{\pi n}{kh}\right) \pm \left[\left(\frac{\pi n}{kh}\right)^2 + 4i\frac{\Delta}{kh}\right]^{\frac{1}{2}}. \tag{4.10}$$

The positive sign before the radical is chosen since it reduces to $C=(\pi n/kh)$ when Δ approaches zero as it must. The corresponding form for S_n is then given by

$$S_n = \left\{1-\left(\frac{\pi n}{kh}\right)^2 \frac{1}{4}\left[1+\sqrt{1+4i\frac{\Delta kh}{(\pi n)^2}}\right]^2\right\}^{\frac{1}{2}}. \tag{4.11}$$

When $n=0$, this simplifies to

$$S_0 = \left[1 - i\frac{\Delta}{kh}\right]^{\frac{1}{2}}$$

which reduces to eq (5) when $n=0$ and $|\Delta| << kh$. Now since $|\Delta| kh << 1$ the radical can be expanded for $n>0$ to yield

$$S_n \cong \left[1-\left(\frac{\pi n}{kh}\right)^2 - i\frac{2\Delta}{kh}\right]^{\frac{1}{2}} \qquad \text{for } n=1,2,3,\ \ldots \tag{4.12}$$

The preceding discussion concerns walls which are highly conducting. The approximate solution obtained would indicate that the attenuation increases indefinitely as the conductivity of the walls decreases. Such is true as long as $|\Delta| << 1$. For very poor conductivities this condition becomes violated. When $|\Delta| kh$ is of the order of unity, it is apparently necessary to solve the resonance equation by numerical or graphical means. This approach is described briefly in a later section. As it turns out, for a given value of n, the attenuation reaches a maximum value as $|\Delta|$ is continuously increased and thereafter diminishes and approaches a broad minimum. To illustrate this interesting phenomenon the resonance equation

$$R_g(C) R_i(C) e^{-2ikhC} = e^{-2\pi i n} \tag{4.13}$$

is solved approximately under the condition that

$$|N_i C| << 1 \qquad \text{and } |N_g C| >> 1.$$

Thus

$$R_i(C) \cong -e^{-2N_iC/C_i} \text{ and } R_g(C) \cong e^{-2C_g/N_gC} \tag{4.14}$$

and therefore

$$\frac{C_g}{N_gC}+\frac{N_iC}{C_i}+ikhC=\pi i(n-\tfrac{1}{2}). \tag{4.15}$$

The zero-order approximation is obtained by replacing $R_i(C)$ by -1 and $R_g(C)$ by $+1$. This would yield

$$C=\overline{C}_n=(n-\tfrac{1}{2})\pi/(kh),\ n=1, 2, 3, \ldots \tag{4.16}$$

as mentioned in section 2. For the first-order perturbation

$$C_i=\left(1-\frac{S^2}{N_i^2}\right)^{\frac{1}{2}} \cong \frac{1}{N_i}\,[N_i^2-1+(\overline{C}_n)^2]^{\frac{1}{2}} \tag{4.17}$$

and

$$C_g=\left(1-\frac{S^2}{N_g^{\ 2}}\right)^{\frac{1}{2}} \cong 1$$

since $|N_gC|>>1$. With these simplifications it readily follows that

$$C_n \cong \frac{\pi(n-\frac{1}{2})+\dfrac{i}{N_g\overline{C}_n}}{kh-iN_i^2[N_i^2-1+(\overline{C}_n)^2]^{-\frac{1}{2}}} \tag{4.18}$$

and

$$S_n=(1-C_n^2)^{\frac{1}{2}}.$$

When the upper medium is an ionized region, it is convenient to write

$$N_i^2=1-\frac{i}{L}.$$

It may be shown that for vlf, L is approximately real and has a magnitude of the order of unity. Furthermore, for a highly conducting ground

$$1/N_g \cong G^{\frac{1}{2}}e^{i\pi/4}$$

where

$$G=\frac{\epsilon\omega}{\sigma_g}.$$

Then

$$C_n \cong \frac{\pi(n-\frac{1}{2})+e^{+i3\pi/4}G^{\frac{1}{2}}(\bar{C}_n)^{-1}}{kh-i\left(1-\dfrac{i}{L}\right)\left[(\bar{C}_n)^2-\dfrac{i}{L}\right]^{-\frac{1}{2}}}. \tag{4.19}$$

Assuming $(\overline{C}_n)^2<<L$ (which is true for low order modes), and that L is real, the real and imaginary parts of S_n can be written

$$\text{Re } S_n=\overline{S}_n+\frac{1}{2\sqrt{2}\pi(h/\lambda)\overline{S}_n}\left[(\overline{C}_n)^2\left(\sqrt{L}-\frac{1}{\sqrt{L}}\right)+\sqrt{G}\right] \tag{4.20}$$

and

$$\text{Im } S_n=\frac{-1}{2\sqrt{2}\pi(h/\lambda)\overline{S}_n}\left[(\overline{C}_n)^2\left(\sqrt{L}+\frac{1}{\sqrt{L}}\right)+\sqrt{G}\right] \tag{4.21}$$

where

$$\bar{C}_n = \frac{\pi(n-\frac{1}{2})}{kh} = \frac{(n-\frac{1}{2})}{(2h/\lambda)}$$

and

$$\bar{S}_n = [1-(\bar{C}_n)^2]^{\frac{1}{2}}.$$

It may be observed that for a fixed value of h/λ and $\sqrt{G}$ the attenuation factor, $-k \text{ Im } S_n$, has a broad minimum when $L=1$. For L somewhat less than unity, the attenuation factor varies as $L^{-\frac{1}{2}}$ or directly as the square root of the effective conductivity of the ionosphere. On the other hand, for L somewhat greater than unity the attenuation factor varies as $L^{\frac{1}{2}}$ or inversely as the square root of the effective conductivity.

The excitation of the modes is proportional to the quantity

$$\delta_n(C_n) = \left[1 + i\,\frac{\partial[R_g(C)R_i(C)]/\partial C}{2khR_g(C)R_i(C)}\right]^{-1}_{C=C_n}. \tag{4.22}$$

When $kh|\Delta| << 1$, where $\Delta = 1/N_g + 1/N_i$, it follows from eqs (2) and (3) that

$$R_g(C_n)R_i(C_n) \simeq \exp\left[-2\Delta/C_n\right],$$

and

$$[\partial[R_g(C)R_i(C)]/\partial C]_{C=Cn} = \frac{2\Delta}{C_n^2} \exp\left[-2\Delta/C_n\right]$$

and thus

$$\delta_n(C_n) = \left[1 + i\,\frac{\Delta}{khC_n^2}\right]^{-1}. \tag{4.23}$$

Now the resonance condition states that

$$ikhC_n + \Delta/C_n = i2\pi n \tag{4.24}$$

and for $n=0$ this leads immediately to

$$\delta_0(C_0) = 1/2$$

while for $n = 1,2,3, \ldots$

$$\delta_n(C_n) = \left(2 - \frac{\pi n}{khC_n}\right)^{-1} \simeq 1.$$

On the other hand, if the upper medium is very poorly conducting such that

$$|N_iC_n| \ll 1$$

and the lower medium is highly conducting

$$|N_gC_n| \gg 1$$

it follows that

$$R_g(C_n)R_i(C_n) \cong -\exp\left[-\frac{2N_iC_n}{C_i} - \frac{2}{N_gC_n}\right] \tag{4.25}$$

and

$$\left[\frac{\partial}{\partial C} R_g(C)R_i(C)\right]_{C=C_n} \cong -\left[\frac{2N_i}{C_i} - \frac{2}{N_gC_n^2}\right] R_g(C_n)R_i(C_n). \tag{4.26}$$

Thus

$$\delta_n(C_n) \cong \left[1 - \frac{i}{kh}\left(\frac{N_i}{C_i} - \frac{1}{N_gC_n^2}\right)\right]^{-1} \cong 1 \tag{4.27}$$

for $n=1,2,3, \ldots$ since the term in parentheses is always small compared to unity.

4.2. Horizontal Polarization

In the case when the excitation is by a vertical magnetic dipole or horizontal electric dipole the modes excited may be of a transverse electric (TE) type. The appropriate modal equation is

$$R_g^h(C)\,R_i^h(C)\,e^{-i2khC}=e^{-i2\pi(m-1)} \tag{4.28}$$

where $m=1,2,3,\ \ldots$.

Now if $|C/N_g|$ and $|C/N_i|\ll 1$

$$R_g^h(C)\cong -\exp\left(-2C/N_gC_g\right) \tag{4.29}$$

and

$$R_i^h(C)\cong -\exp\left(-2C/N_iC_i\right)\cdot \tag{4.30}$$

The modal equation is thus simplified to

$$C\left[\frac{1}{N_iC_i}+\frac{1}{N_gC_g}\right]+ihkC=i\pi m \tag{4.31}$$

remembering that

$$C_i=\left(1-\frac{S^2}{N_i^2}\right)^{\frac{1}{2}},\qquad C_g=\left(1-\frac{S^2}{N_g^2}\right)^{\frac{1}{2}},$$

and $S^2=1-C^2$.

A first order solution is obtained by replacing S^2 in the expressions for C_i and C_g by the zero order value, e.g.,

$$S^2\cong 1-\left(\frac{\pi m}{kh}\right)^2\cdot \tag{4.32}$$

The approximate solution of the mode equation is then given by

$$C_m\cong\frac{i\pi m}{\Delta_h^m+ikh} \tag{4.33}$$

where

$$\Delta_h^m\cong\left[N_g^2-1+\left(\frac{\pi m}{kh}\right)^2\right]^{-\frac{1}{2}}+\left[N_i^2-1+\left(\frac{\pi m}{kh}\right)^2\right]^{-\frac{1}{2}} \tag{4.34}$$

and $S_m=(1-C_m^2)^{\frac{1}{2}}$. When $|N_i^2|$ and $|N_g^2|\gg 1-\left(\frac{\pi m}{kh}\right)^2$

it is seen that

$$\Delta_h^m\cong\frac{1}{N_g}+\frac{1}{N_i}=\Delta\cdot$$

For $|\Delta_h^m|\ll kh$,

$$C_m\cong\frac{\pi m}{kh}\left(1+i\frac{\Delta}{kh}\right) \tag{4.35}$$

and

$$S_m\cong\left[1-\left(\frac{\pi m}{kh}\right)^2\right]^{\frac{1}{2}}-i\frac{\Delta}{kh}\left(\frac{\pi m}{kh}\right)^2\left[1-\left(\frac{\pi m}{kh}\right)^2\right]^{-\frac{1}{2}} \tag{4.36}$$

which is valid when the modulus of the second term is small compared to the first.

For highly conducting walls

$$\Delta\cong\sqrt{i}\,|\Delta|=\sqrt{i}\left[\left(\frac{\epsilon\omega}{\sigma_g}\right)^{\frac{1}{2}}+\left(\frac{\epsilon\omega}{\sigma_i}\right)^{\frac{1}{2}}\right] \tag{4.37}$$

and therefore

$$\mathrm{Re}\,S_m\cong\left[1-\left(\frac{\pi m}{kh}\right)^2\right]^{\frac{1}{2}}+\frac{|\Delta|}{\sqrt{2}\,kh}\left[1-\left(\frac{\pi m}{kh}\right)^2\right]^{-\frac{1}{2}}\left(\frac{\pi m}{kh}\right)^2 \tag{4.38}$$

and

$$\operatorname{Im} S_m \cong -\frac{|\Delta|}{\sqrt{2}\,kh}\left[1-\left(\frac{\pi m}{kh}\right)^2\right]^{-\frac{1}{2}}\left(\frac{\pi m}{kh}\right)^2. \tag{4.39}$$

In summary, these are valid when

$$|\Delta| \ll kh\left[1-\left(\frac{\pi m}{kh}\right)^2\right]$$

and $\pi m/kh<1$.

It is rather interesting to note that the above expressions for Re S_m and Im S_m are very similar to the corresponding expressions derived for Re S_n and Im S_n in the case of vertical polarization. [For example compare with eqs (7) and (8).] In the present case, of course, there is no zero order mode but apart from this, the perturbation term involving $|\Delta|$ now has an additional factor $[(\pi m/kh)]^2$ which is less than unity if the mode is above "cutoff." Thus, everything else considered equal, the attenuation factor of the TE mode is decreased relative to the TM mode in the earth-ionosphere waveguide with walls assumed to be of high conductivity.

In the earlier notation it was convenient to represent the refractive index in the form

$$N_i^2 = 1-\frac{i}{L}$$

where L is a real number which may be comparable to or much less than unity. Thus

$$\Delta_h^m \cong \left[\left(\frac{\pi m}{kh}\right)^2-\frac{i}{L}\right]^{-\frac{1}{2}}+\frac{1}{N_g}. \tag{4.40}$$

The corresponding solution for the modal equation is obtained from

$$S_m = \left[1-\left(\frac{\pi m}{kh-i\Delta_h^m}\right)^2\right]^{\frac{1}{2}}. \tag{4.41}$$

When $L<<1$ this reduces to eq (36).

5. Influence of Earth Curvature

The curvature of the earth has been neglected up to this point. The problem is now formulated in terms of spherical coordinates (r, θ, ϕ), with the earth idealized as a homogeneous sphere of radius a, of conductivity σ_g, and dielectric constant ϵ_g. The lower edge of the assumed homogeneous ionosphere is located at $r=a+h$. The source vertical electric dipole is then located at $r=a+z_0$ and the observer is at $r=a+z$ (see fig. 5). In view of the intrinsic spherical symmetry the fields can be represented in terms of a single scalar function, ψ, as follows [26]

$$\begin{aligned}
E_r &= \frac{i}{r}\,\eta\,\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\psi}{\partial\theta}\right)\\
E_\theta &= -\frac{i}{r}\,\eta\,\frac{\partial^2}{\partial\theta\partial r}\,(r\psi)\\
H_\phi &= -k\,\frac{\partial\psi}{\partial\theta}\\
E_\phi &= H_r = H_\theta = 0
\end{aligned} \tag{5.1}$$

where $\eta=(\mu/\epsilon)^{\frac{1}{2}}$ and $k^2=\omega^2(\epsilon-i\sigma/\omega)\mu$.

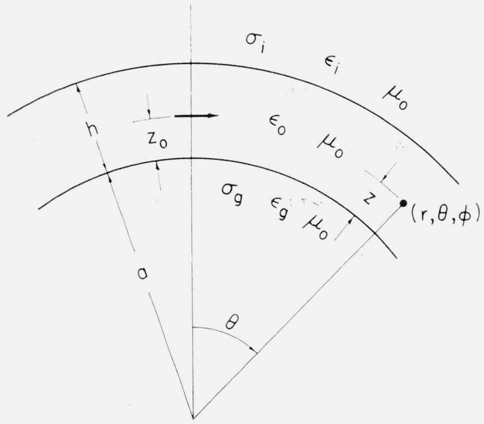

FIGURE 5. *Spherical coordinate system for vertical electric dipole between concentric spherical interfaces.*

As usual, the permeability is taken to be the same as free space for all the regions ($\mu = 4\pi \times 10^{-7}$). A subscript g is affixed to σ, ϵ, etc., when reference is made to the ground and a subscript i for the ionosphere. Since ψ is a solution the wave equation appropriate for the regions, the solution may be represented in terms of spherical wave functions,

$$a_\nu^{(1)} h_\nu^{(1)}(k_g r) P_\nu(-\cos\theta) \qquad \text{for } r<a$$

$$[b_\nu^{(1)} h_\nu^{(1)}(kr) + b_\nu^{(2)} h_\nu^{(2)}(kr)] P_\nu(-\cos\theta) \qquad \text{for } (a+h)>r>a$$

$$c_\nu^{(2)} h_\nu^{(2)}(k_i r) P_\nu(-\cos\theta) \qquad \text{for } r>(a+h).$$

In the above

$$h_\nu^{(1,2)}(kr) = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} H_{\nu+\frac{1}{2}}^{(1,2)}(kr) \tag{5.2}$$

where $H_{\nu+\frac{1}{2}}^{(1,2)}(kr)$ is the Hankel function of the first or second kind of order $\nu+\frac{1}{2}$ with argument kr. $P_\nu(-\cos\theta)$ is the hypergeometric series which is a special case of the hypergeometric function $F(\alpha, \beta, \gamma, z)$ namely

$$P_\nu(-\cos\theta) = F\left(-\nu, \nu+1, 1, \frac{1+\cos\theta}{2}\right). \tag{5.3}$$

The reason $P_\nu(-\cos\theta)$ is employed rather than $P_\nu(+\cos\theta)$ is due to the fact that ψ must be regular on a ray $\theta=\pi$, whereas $\theta=0$ is to contain the singularity which is the source of the field. Sommerfeld [26] has pointed out that

$$\lim_{\theta\to 0} \mathrm{F}\left(-\nu, \nu+1, 1, \frac{1+\cos\theta}{2}\right) \to \frac{\sin \nu\pi}{\pi} \log \theta^2 \tag{5.4}$$

which illustrates the singular nature of ψ along the polar axis.

The quantity ν is to be found from the boundary conditions that the fields E_θ and H_ϕ are continuous at $r=a$ and $a+h$. This, in turn, requires that $(\eta/r)\,\partial(r\psi)/\partial r$ and $k\psi$ are continuous. Thus, four linear equations in the coefficients $a_\nu^{(1)}$, $b_\nu^{(1)}$, $b_\nu^{(2)}$, and $c_\nu^{(2)}$ are obtained. In order that these yield a nontrivial solution, the four by four determinant of the coefficients should vanish. This requirement is explicitly given by eq (5) as follows:

$$\begin{vmatrix} \left(\frac{\eta_g}{a}\right)\left[\frac{\partial r h_\nu^{(1)}(k_g r)}{\partial r}\right] & -\left(\frac{\eta}{a}\right)\left[\frac{\partial r h_\nu^{(1)}(kr)}{\partial r}\right]_a & -\left(\frac{\eta}{a}\right)\left[\frac{\partial r h_\nu^{(2)}(kr)}{\partial r}\right]_a & 0 \\ k_g h_\nu^{(1)}(k_g a) & -k h_\nu^{(1)}(ka) & -k h_\nu^{(2)}(ka) & 0 \\ 0 & \left(\frac{\eta}{a+h}\right)\left[\frac{\partial r h_\nu^{(1)}(kr)}{\partial r}\right]_{a+h} & \left(\frac{\eta}{a+h}\right)\left[\frac{\partial r h_\nu^{(2)}(kr)}{\partial r}\right]_{a+h} & \left(\frac{\eta_i}{a+h}\right)\left[\frac{\partial r h_\nu^{(2)}(k_i r)}{\partial r}\right]_{a+h} \\ 0 & k h_\nu^{(1)}[k(a+h)] & k h_\nu^{(2)}[k(a+h)] & k_i h_\nu^{(2)}[k_i(a+h)] \end{vmatrix} = 0 \tag{5.5}$$

Such an equation as this was obtained by G. N. Watson in 1919. To solve it for ν without approximation does not seem to be possible, although if the general spherical Hankel functions of complex order and argument could be programed for a computer, an exact numerical solution might be obtained. In view of the idealizations of the model and the uncertainty of the effective electrical constants of the lower ionosphere, however, it does not seem warranted to expend too much effort in this direction. As is so often desirable in physical problems, asymptotic approximations to the rigorous wave functions are introduced which greatly simplify the problem but at the same time lose some generality.

The Debye-Watson representation of the Hankel functions are [26]

$$kr\, h_{\nu a}^{\binom{(1)}{(2)}}(kr) = \left(\frac{\pi kr}{2}\right)^{\frac{1}{2}} H_{\nu+\frac{1}{2}}^{\binom{(1)}{(2)}}(kr) \cong \left[1-\frac{(\nu+\frac{1}{2})^2}{(kr)^2}\right]^{-\frac{1}{4}} \exp\left(\mp i\,\frac{\pi}{4}\right) \exp\left[\pm i\int_{\nu+\frac{1}{2}}^{kr}\left[1-\frac{(\nu+\frac{1}{2})^2}{x^2}\right]^{\frac{1}{2}} dx\right] \tag{5.6}$$

when $|(\nu+\frac{1}{2})/kr|<1$ but not near 1. Also $|\nu+\frac{1}{2}|$ and kr must be large compared to unity. The upper (and lower) signs are to be considered together. This is really a W.K.B. (Wentzel, Kramers, and Brillouin) approximation to the radial part of the wave equation. It is not difficult to show that the resonance equation involving spherical Hankel functions can now be expressed in the equivalent form

$$R_g R_i \exp\left\{-i2\int_{ka}^{k(a+h)}\left[1-\frac{(\nu+\frac{1}{2})^2}{x^2}\right]dx\right\} = \exp(-i2\pi n) \tag{5.7}$$

where $n=0, 1, 2, \ldots$.

$$R_g = \frac{k_g\left[1-\frac{(\nu+\frac{1}{2})^2}{(ka)^2}\right]^{\frac{1}{2}} - k\left[1-\frac{(\nu+\frac{1}{2})^2}{(k_g a)^2}\right]^{\frac{1}{2}}}{k_g\left[1-\frac{(\nu+\frac{1}{2})^2}{(ka)^2}\right]^{\frac{1}{2}} + k\left[1-\frac{(\nu+\frac{1}{2})^2}{(k_g a)^2}\right]^{\frac{1}{2}}} \tag{5.8}$$

and

$$R_i = \frac{k_i\left[1-\frac{(\nu+\frac{1}{2})^2}{k^2(a+h)^2}\right]^{\frac{1}{2}} - k\left[1-\frac{(\nu+\frac{1}{2})^2}{k_i^2(a+h)^2}\right]^{\frac{1}{2}}}{k_i\left[1-\frac{(\nu+\frac{1}{2})^2}{k^2(a+h)^2}\right]^{\frac{1}{2}} + k\left[1-\frac{(\nu+\frac{1}{2})^2}{k_i^2(a+h)^2}\right]^{\frac{1}{2}}}. \tag{5.9}$$

The functions R_g and R_i quoted above can readily be identified as Fresnel reflection coefficients for complex angles of incidence $\cos^{-1}C$ and $\cos^{-1}C'$, respectively. Furthermore

$$C=[1-S^2]^{\frac{1}{2}} \qquad \text{where } S=\frac{\nu+\frac{1}{2}}{ka}$$

and

$$C'=[1-(S')^2]^{\frac{1}{2}} \qquad \text{where } S'=\frac{\nu+\frac{1}{2}}{k(a+h)}.$$

The resonance equation can thus be written

$$R_g(C)R_i(C')\ \exp\left[-i2k\int_0^h\left[C^2+\frac{2z}{a}\,S^2\right]^{\frac{1}{2}}dz\right]=e^{-2\pi i n} \tag{5.10}$$

where

$$R_g(C)=\frac{N_gC-C_g}{N_gC+C_g}\qquad \text{with } C_g=\left[1-\frac{S^2}{N_g^2}\right]^{\frac{1}{2}} \tag{5.11}$$

and

$$R_i(C')=\frac{N_iC'-C_i'}{N_iC'+C_i'}\qquad \text{with } C_i'=\left[1-\frac{(S')^2}{N_i^2}\right]^{\frac{1}{2}}. \tag{5.12}$$

It can be seen that, in view of the relation

$$(a+h)\ S'=aS,$$

the resonance equation reduces to its flat earth counterpart as h/a tends to zero. In fact, it appears that if $|C|>>(h/a)^{\frac{1}{2}}\simeq 1/10$, the effect of curvature can be disregarded. This condition is violated for most of the numerical results given by Al'pert in the region from 15 to 30 kc. He assumed that the modes could be calculated on the basis of a flat earth in all his work [14, 15].

The resonance equation quoted above for a curved earth is only valid if the W.K.B. or second-order approximations to the spherical wave functions are valid. In a later section the corresponding form of the mode equation based on the Airy or third-order approximation is developed following the work of Rydbeck. It is indicated from this more involved analysis that the second-order approximation is valid if

$$\left(\frac{ka}{2}\right)^{\frac{1}{3}}C<<1.$$

As will be seen, this is met for most cases of practical interest if the frequency is less than about 15 kc.

6. Mode Series for Curved Earth

Following the suggestion of Sommerfeld, the field is written as a sum of modes. Thus

$$\psi(r,\theta)=\sum_n D_\nu z_\nu(kr)P_\nu(-\cos\theta) \tag{6.1}$$

for $a<r<a+h$, where

$$z_\nu(kr)=b_\nu^{(1)}h_\nu^{(1)}(kr)+b_\nu^{(2)}h_\nu^{(2)}(kr). \tag{6.2}$$

The factor D_ν is to be determined by insisting that the function $\psi(r,\theta)$ has the proper behavior at the source. The summation is over all integral values of n and the corresponding (complex) values of ν are obtained from the resonance eq (5.5) as described in the previous section.

Invoking the W.K.B. or second order approximation for the spherical wave functions, it follows that

$$\begin{aligned} z_\nu(kr)\simeq \text{const}\Bigg\{ & R_g^{-\frac{1}{2}}(C_n)\ \exp\left[+ik\int_0^z(C_n^2+\frac{2z}{a}\,S_n^2)^{\frac{1}{2}}\,dz\right] \\ & +R_g^{\frac{1}{2}}(C_n)\ \exp\left[-ik\int_0^z(C_n^2+\frac{2z}{a}\,S_n^2)^{\frac{1}{2}}\,dz\right]\Bigg\} \end{aligned} \tag{6.3}$$

This can be identified immediately as a combination of a downcoming and an upgoing wave.

The ratio of these two at the earth's surface ($z=0$) is $R_g(C_n)$. An alternate representation is

$$z_\nu(kr) \cong \text{const}\left\{ R_i^{-\frac{1}{2}}\,(C'_n) \quad \mathrm{p}\left[+ik\int_z^h \left(C_n^2+\frac{2z}{a}\,S_n^2\right)^{\frac{1}{2}} dz\right]\right.$$

$$\left. +R_i^{\frac{1}{2}}\,(C'_n)\,\exp\left[-ik\int_z^h \left(C_n^2+\frac{2z}{a}\,S_n^2\right)^{\frac{1}{2}} dz\right]\right\} \tag{6.4}$$

which is a combination of an upgoing wave and a downcoming wave. The ratio of these two at the lower edge of the ionosphere ($z=h$) is $R_i(C'_n)$. It should be noted that

$$C'_n \cong \left(C_n^2+\frac{2h}{a}\,S_n^2\right)^{\frac{1}{2}}$$

since $(a+h)\,S'_n=aS_n$ and $h/a << 1$. The internal consistency of these two representations at $z=0$ and h for $z_\nu(kr)$ can be readily demonstrated from the relation

$$1=R_g^{\frac{1}{2}}(C_n)\;R_i^{\frac{1}{2}}(C'_n)\;\exp\left[-ik\int_0^h \left(C_n^2+\frac{2z}{a}\,S_n^2\right)^{\frac{1}{2}} dz\right] \tag{6.5}$$

which also indicates that the multiplicative constant is the same for the two representations. In what follows the constant can be absorbed into the factor D_ν.

To study the orthogonality properties of the modes, the following integral is considered

$$I=\int_{ka}^{k(a+h)} z_\nu(\rho)\,z_\mu(\rho)\,d\rho \tag{6.6}$$

where ν and μ are two sets of modes. Now quite generally the function $z_\nu(\rho)$ satisfies

$$\rho\,\frac{d^2}{d\rho^2}\,(\rho z_\nu)+[\rho^2-\nu(\nu+1)]z_\nu=0 \tag{6.7}$$

and there is a similar relation for z_μ. These two equations are now multiplied by z_μ and z_ν, respectively and integrated over the domain ka to kb, to obtain

$$I=\frac{\rho\left(z_\mu\,\dfrac{d}{d\rho}\,(\rho z_\nu)-z_\nu\,\dfrac{d}{d\rho}\,(\rho z_\mu)\right)\Bigg|_{ka}^{kb}}{\nu(\nu+1)-\mu(\mu+1)}. \tag{6.8}$$

For the important modes, the right hand side of eq (8) is negligibly small if $\mu\neq\nu$ since the numerator vanishes at the limits ka and kb when R_g and R_i approach ± 1. For the important case $\mu=\nu$, a normalization factor is defined by

$$N_\nu=\lim_{\mu\to\nu}\int_{ka}^{kb} z_\mu(\rho)\,z_\nu(\rho)\,d\rho$$

$$=\int_{ka}^{kb}[z_\nu(\rho)]^2 d\rho \cong \frac{2kh}{\delta_n},\ \text{where } \delta_n=\frac{1}{1+\dfrac{\sin 2kh\,C_n}{2kh\,C_n}}. \tag{6.9}$$

It should be remarked at this point that the modes are not strictly orthogonal since the right-hand side of eq (8) does not vanish identically although it is small compared to N_ν. As the conductivity of the bounding walls approaches infinity the modes would be completely orthogonal.

Multiplying both sides of eq (1) by $z_\nu(\rho)$ and then integrating with respect to ρ from ka to kb, leads to the following formula for D_ν:

$$D_\nu=\frac{1}{N_\nu P_\nu(-\cos\theta)}\int_{ka}^{kb} z_\nu(\rho)\,\psi(r,\theta)\,d\rho. \tag{6.10}$$

To actually evaluate D_ν, it is desirable to let $r \to r_0$ and $\theta \to 0$, in which case $\psi(r,\theta) \to \psi_0(r,\theta)$ where ψ_0 is the primary influence which is singular at $(r_0,0)$. For a vertical electric dipole consisting of an infinitesimal element of length ds and carrying a current, I, it is well known that

$$\psi_0(r,\theta) = \frac{Ids}{4\pi r_0 k} \frac{e^{-ikR}}{R} \tag{6.11}$$

where $R = (r_0^2 + r^2 - 2rr_0 \cos \theta)^{\frac{1}{2}}$.

Following the process suggested by Sommerfeld for the determination of the Green's function for the perfectly conducting sphere, the integration in eq (10) is carried out in the immediate neighborhood of the source. For example, $r = r_0(1+\eta)$ where $-\epsilon < \eta < +\epsilon$, $\epsilon << 1$ and $dr = r_0 d\eta$, $z_\nu(kr) \simeq z_\nu(kr_0)$, $e^{-ikR} \cong 1$, while

$$\frac{1}{R} \simeq \frac{1}{r_0}\left[(1+\eta)^2 + 1 - 2(1+\eta)\left(1 - \frac{\theta^2}{2}\right)\right]^{-\frac{1}{2}} \simeq \frac{1}{r_0(\eta^2+\theta^2)^{\frac{1}{2}}}. \tag{6.12}$$

Therefore

$$N_\nu D_\nu \cong \frac{Ids}{4\pi r_0} z_\nu(kr_0) \lim_{\theta \to 0} \left[\frac{1}{P_\nu(-\cos\theta)} \int_{-\epsilon}^{+\epsilon} \frac{d\eta}{(\eta^2+\theta^2)^{\frac{1}{2}}}\right]. \tag{6.13}$$

Now

$$\lim_{\theta \to 0} [P_\nu(-\cos \theta)] \to \frac{\sin \nu\pi}{\pi} \log \theta^2$$

and

$$\lim_{\theta \to 0} \int_{-\epsilon}^{+\epsilon} \frac{d\eta}{(\eta^2+\theta^2)^{\frac{1}{2}}} = \lim_{\theta \to 0} \left\{\log\left[\epsilon + \sqrt{\epsilon^2+\theta^2}\right] - \log\left[-\epsilon + \sqrt{\epsilon^2+\theta^2}\right]\right\} \to \log \theta^2. \tag{6.14}$$

It then follows that

$$D_\nu = \frac{i}{2kh} \frac{z_\nu(kr_0)}{\sin \nu\pi} \frac{Ids}{4r_0} \delta_n. \tag{6.15}$$

The final form of the function ψ is thus given by

$$\psi(r,\theta) \cong \frac{Idsi}{2khr_0} \sum_{n=0,1,2,\ldots}^{\infty} \frac{z_\nu(kr_0)}{z_\nu(ka)} \frac{z_\nu(kr)}{z_\nu(ka)} \frac{P_\nu(-\cos\theta)}{\sin \nu\pi} \delta_n \tag{6.16}$$

where the second-order or W.K.B. representations may be used for radial functions $z_\nu(kr)$, etc. As can be seen from eq (4) these can be greatly simplified if $z/a << |C_n^2|$, for then

$$\frac{z_\nu(kr)}{z_\nu(ka)} \simeq \frac{e^{ikC_n z} + R_g(C_n)e^{-ikC_n z}}{2[R_g(C_n)]^{\frac{1}{2}}} = f_n(z) \tag{6.17}$$

which is the same height-gain function obtained for the flat earth case. For heights even as great as 10 km and frequencies less than 20 kc, this is an excellent approximation. Similarly,

$$\frac{z_\nu(kr_0)}{z_\nu(ka)} \cong f_n(z_0) \tag{6.18}$$

where $z_0 = r_0 - a$ is the height of the source dipole.

The radial field component is of most practical interest and, for the moment, attention will be confined to it. Since

$$E_r = \frac{i}{r}\eta \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin \frac{\partial\psi}{\partial\theta}\right) \tag{6.19}$$

and

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial P_\nu(-\cos\theta)}{\partial\theta}\right] + \nu(\nu+1)P_\nu(-\cos\theta) = 0 \tag{6.20}$$

it follows that

$$E_r \cong \frac{Ids\eta}{2khr_0 r} \sum_{n=0}^{\infty} \delta_n f_n(z_0) f_n(z) \frac{\nu(\nu+1)}{\sin \nu\pi} P_\nu(-\cos\theta) \tag{6.21}$$

with $\nu+\frac{1}{2}\cong kaS_n$. This is the final solution of the problem being valid for the air space between the earth and the ionosphere.[4]

For purposes of computation several simplifications can be made. The asymptotic expansion for the Legendre function, given by

$$P_\nu(-\cos\theta) \cong \left(\frac{2}{\pi\nu \sin\theta}\right)^{\frac{1}{2}} \cos\left[\left(\nu+\frac{1}{2}\right)(\pi-\theta)-\frac{\pi}{4}\right] \tag{6.22}$$

is valid if $|\nu| >> 1$ and θ not near 0 or π. Since the imaginary part of $\nu(\pi-\theta)$ is also large for $\pi-\theta$ greater than about 10° or 20°, it follows that

$$P_\nu(-\cos\theta) \cong \left(\frac{1}{2\pi\nu \sin\theta}\right)^{\frac{1}{2}} \exp\left[i\left(\nu+\frac{1}{2}\right)(\pi-\theta)-i\pi/4\right]. \tag{6.23}$$

Furthermore, the source and observer heights are usually sufficiently low that kh_0C_n and $kh_1C_n \ll 1$ and $r_0 \cong r \cong a$.

The simplified form of the field can now be written

$$E_r = E_0 \left[\frac{d/a}{\sin d/a}\right]^{\frac{1}{2}} \frac{(d/\lambda)^{\frac{1}{2}}}{(h/\lambda)} e^{i[2\pi(d/\lambda)-(\pi/4)]} \sum_{n=0}^{\infty} \delta_n S_n^{\frac{3}{2}} e^{-i2\pi S_n(d/\lambda)} \tag{6.24}$$

where $d=a\theta$, the arc length between the source and the observer, λ is the free-space wavelength, and $S_n=(1-C_n^2)^{\frac{1}{2}}$. E_0 is the field of the source at a distance d on a flat perfectly conducting earth. For $d/\lambda \gg 1$,

$$E_0 = i(\eta/\lambda)\, Ids\, (e^{-i2\pi d/\lambda})/d. \tag{6.25}$$

As the radius a of the earth tends to infinity it is immediately evident that the flat earth formula given by eq (2.5) is recovered.

7. Antipodal Effects

The general form for the field in the space $a<r<a+h$ for a vertical dipole source has the form

$$E_r = \frac{Ids\eta}{2kha^2} \sum_{n=0}^{\infty} \delta_n \frac{\nu(\nu+1)}{\sin\nu\pi} P_\nu(-\cos\theta) \tag{7.1}$$

where $\delta_0=\frac{1}{2}$, $\delta_n=1\,(n\neq 0)$, and

$$\nu(\nu+1) \cong (\nu+\tfrac{1}{2})^2 \cong kaS_n^2. \tag{7.2}$$

Now as mentioned above when θ is not near 0 or π, the Legendre function may be replaced by the first term of its asymptotic expansion. This result quoted above is valid if

$$\frac{1}{|\nu|} \gg (\pi-\theta) \qquad \text{and } \frac{1}{|\nu|} \gg \theta. \tag{7.3}$$

In this region, the modes are simply proportional to

$$\frac{1}{(\sin\theta)^{\frac{1}{2}}} \cos\left[kaS_n(\pi-\theta)-\frac{\pi}{4}\right] \tag{7.4}$$

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[4] From an analysis by Pekeris; Phys. Rev. **70**, 518 (1946) it may be shown that

$$\frac{P_\nu(-\cos\theta)}{\sin\nu\pi} \simeq H_0^{(2)}(kS_n\rho) - \frac{1}{12}\left(\frac{\rho}{a}\right)^2 H_2^{(2)}(kS_n\rho)$$

plus terms in $(\rho/a)^4$, $(\rho/a)^6$, etc., where $\nu+\frac{1}{2}=kS_n\rho$. Inserting this in eq (21) leads directly back to eq (3.21) when the curvature correction terms are neglected.

which apart from a constant factor can be identified as the linear combination of two peripheral waves of the form

$$\frac{1}{(\sin\theta)^{\frac{1}{2}}}\, e^{-ikaS_n\theta}$$

and

$$\frac{1}{(\sin\theta)^{\frac{1}{2}}}\, e^{-ikaS_n(2\pi-\theta)}e^{i\pi/2}$$

where $\theta<\pi$.

These waves are traveling in opposing directions along the two respective great circle paths $a\theta$ and $a(2\pi-\theta)$ from the source to the observer. It is noticed that there is a $\pi/2$ phase advance which the wave traveling on the long great circle path picks up as it goes through the pole $\theta=\pi$. The linear combination of these two traveling waves is to form a standing wave pattern whose distance Δ_m between minimums is approximately given by

$$k\Delta_m \mathrm{Re} S_n=\pi \qquad \text{or } \Delta_m=\lambda/(2 \text{ Re } S_n)$$

subject to

$$-\text{Im } S_n \ll \text{Re } S_n.$$

As one approaches the pole $\theta=\pi$, the first term in the asymptotic expansion for the Legendre function is inadequate. A more general form is the asymptotic series [27]

$$P_\nu(-\cos\theta)\simeq\frac{2}{\pi^{1/2}}\frac{\Gamma(\nu+1)}{\Gamma(\nu+3/2)}\sum_l \frac{[(1/2)l]^2}{(\nu+3/2)_l l!}\frac{\cos\left[\left(\nu+\frac{2l+1}{2}\right)(\pi-\theta)-\left(\frac{2l+1}{4}\right)\pi\right]}{(2\sin\theta)^{l+1/2}} \tag{7.5}$$

where

$$(\alpha)_l=\alpha(\alpha+1)\,(\alpha+2)\ .\ .\ .\ (\alpha+l-1),$$

for example, $\alpha_0=1$, $\alpha_1=\alpha$, $\alpha_2=\alpha(\alpha+1)$, $\alpha_3=\alpha(\alpha+1)\,(\alpha+2)$, etc.

Since $|\nu|\gg 1$ the factorial functions may be replaced by the first two terms of their asymptotic expansions; this leads to

$$\frac{\Gamma(\nu+1)}{\Gamma(\nu+3/2)}\sim\frac{1}{\nu^{\frac{1}{2}}}\left(1-\frac{3}{8\nu}+0\left(\frac{1}{\nu^2}\right)\right). \tag{7.6}$$

The preceding asymptotic expansion for $P_\nu(-\cos\theta)$ is not usable at and in the vicinity of the pole $\theta=\pi$. In this region a suitable representation is given by [27]

$$P_\nu(-\cos\theta)=J_0(\eta)+\sin^2\left(\frac{\pi-\theta}{2}\right)\left[\frac{J_1(\eta)}{2\eta}-J_2(\eta)+\frac{\eta}{6}J_3(\eta)\right]+0\left(\sin^4\left(\frac{\pi-\theta}{2}\right)\right) \tag{7.7}$$

where $\eta=(2\nu+1)\sin[(\pi-\theta)/2]$. $J_m(\eta)$, for $m=0, 1, 2,$ and 3, is the Bessel function of first type of argument η and order m. When $\pi-\theta$ is small the first term is usually sufficient and furthermore

$$\eta\simeq(\nu+\tfrac{1}{2})\,(\pi-\theta)\simeq kaS_n(\pi-\theta).$$

Thus for mode n, the field in the neighborhood of the pole is proportional to the Bessel function

$$J_0[kaS_n(\pi-\theta)].$$

It is then not surprising to see that the first term of the asymptotic expansion of J_0 is the same as that of $P_\nu(-\cos\theta)$.

8. Resonator Type Oscillations Between Earth and the Ionosphere

At extremely low frequencies (elf), where the wavelength is large compared to the height of the ionospheric reflecting layer, the electric field is essentially radial and only one waveguide type mode is significant. The field is thus expressed by the first term of the mode series which reads

$$E_r = \frac{Ids\eta}{4kha^2} \frac{\nu(\nu+1)}{\sin \nu\pi} P_\nu(-\cos\theta) \tag{8.1}$$

where $\nu+\frac{1}{2} \simeq kaS_0$ and

$$S_0 \cong 1 - \frac{i}{2kh}\left(\frac{1}{N_i}+\frac{1}{N_g}\right) \tag{8.2}$$

in terms of the relative refractive indices N_i and N_g of the homogeneous ionosphere and the homogeneous ground, respectively. Now at elf $|N_g| >> |N_i|$ and furthermore,

$$N_i \cong \left(\frac{\sigma_i + i\omega\epsilon_i}{i\omega\epsilon}\right)^{\frac{1}{2}} \cong \left(\frac{\sigma_i}{i\omega\epsilon}\right)^{\frac{1}{2}}.$$

Thus

$$S_0 \cong 1 - \frac{i^{3/2}}{2(\sigma_i\mu\omega)^{\frac{1}{2}}h}.$$

Now as mentioned in section 7, the factor $P_\nu(-\cos\theta)$ may be replaced by an asymptotic expansion if $ka\theta$ or $ka(\pi-\theta)$ is somewhat greater than unity. The field in this case may be regarded as two azimuthal-type traveling waves. Furthermore at the pole (θ near π) where the second of these restrictions is violated, it is possible to use an equivalent representation which correctly accounts for the axial focusing. An alternate viewpoint which is suitable at elf is to consider the field as a superposition of cavity-resonator type modes. It is expected that such a representation would be very good when the circumference of the earth is becoming comparable to the wavelength. A suggestion of this kind was apparently first put forth by Schumann [28].

The starting point is the expansion formula

$$\frac{P_\nu(-x)}{\sin \nu\pi} = -\frac{1}{\pi}\sum_{n=0}^{\infty} P_n(x) \frac{2n+1}{n(n+1)-\nu(\nu+1)} \tag{8.3}$$

where the summation is over integral values of n. This result follows directly from a formula given by Magnus and Oberhettinger [27] (p. 57) which is valid for $\nu \neq 0, \pm 1, \pm 2, \ldots$, and $0 \leqq \theta < \pi$. The electric field, for $h/a << 1$, is thus written

$$E_r = \frac{I\,ds\,\nu(\nu+1)}{4\pi a^2 \epsilon\omega h} \sum_{n=0}^{\infty} P_n(x) \frac{2n+1}{n(n+1)-\nu(\nu+1)} \tag{8.4}$$

where $x=\cos\theta$. The early terms of the series are then proportional to

$$P_0(x)=1,$$

$$P_1(x)=\cos\theta,$$

$$P_2(x)=\tfrac{1}{2}(3\cos^2\theta-1), \tag{8.5}$$

and so on. The configuration of the electric field in the first three cavity modes is depicted in figure 6.

Retaining just the first term it is seen that

$$E_r^\circ = E_r\Big]_{n=0} = \frac{Ids}{4\pi a^2\epsilon h}\frac{1}{i\omega} \tag{8.6}$$

which is independent of θ. Clearly this corresponds to a concentric spherical capacitor energized by a current Ids/h resulting in a constant voltage hE_r° between the plates. On rewriting eq (6) in the form

$$hE_r^\circ = \frac{(Ids/h)}{i\omega C_e} \tag{8.7}$$

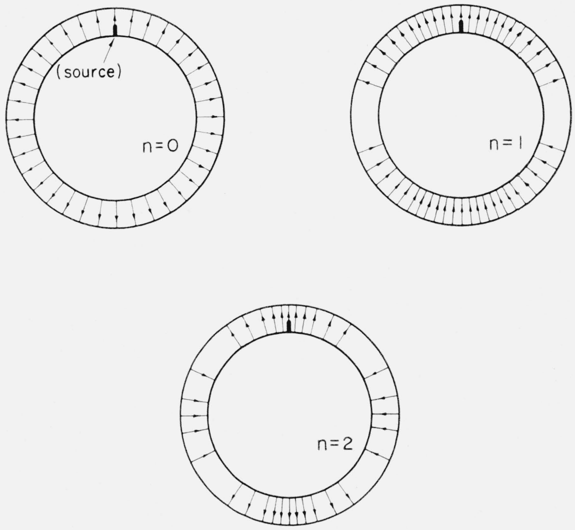

FIGURE 6. *Depicting electric field lines in the first three cavity resonator modes.*

it is seen that

$$C_e = \frac{4\pi a^2 \epsilon}{h}$$

which can be identified as the capacity between the spherical surfaces whose areas are both $4\pi a^2$ within the approximation $h/a << 1$.

It has often been suggested that the omnipresent constant voltage gradient in the atmosphere results from the accumulated action of lightning strokes which impart a charge to this earth-ionosphere condenser. For example, when a current surge flows say for 10^{-3} sec with an average of 10^3 amp with an average column height of 3 km (i.e., $ds \simeq 2\times 10^3$, then for $h \simeq 70$ km it readily follows that

$$hE_r^\circ \cong 1.3 \text{ v} \qquad \text{or } E_r^\circ \cong 2\times 10^{-5} \text{ v/m}.$$

Presumably, many such charges are required to build the field up to its observed value.

Of somewhat more interest are the cavity-resonator oscillations which may be excited. Using the notation of the operational calculus $i\omega$ is formally replaced by p then

$$\nu(\nu+1) \simeq -p^2 - p^{\frac{3}{2}}\alpha \tag{8.8}$$

where $\alpha = 1/[h(\sigma_i\mu)^{1/2}]$. The source dipole moment Ids is in general a function of time. For purposes of illustration, consider

$$Ids = (Ids)_0\, u(t) \tag{8.9}$$

where $u(t)$ is the unit step function at $t=0$. The Laplace transform of the source moment is thus given by

$$\int_0^\infty Ids e^{-pt} dt = \frac{(Ids)_0}{p}. \tag{8.10}$$

The Laplace transform of the field is given by

$$E_r(p) = \frac{(Ids)_0}{h^2 C_e}\, (p + \alpha p^{\frac{1}{2}}) \sum_{n=0}^{\infty} P_n(x)\, \frac{2n+1}{\omega_n^2 + p^2 + \alpha p^{\frac{3}{2}}} \tag{8.11}$$

where $\omega_n^2=(a/c)^2\, n(n+1)$. The actual time response of the electric field is denoted $e_r(t)$ and is zero for $t<0$. It is related to $E_r(p)$ by

$$E_r(p)=\int_0^\infty e_r(t)e^{-pt}dt. \tag{8.12}$$

The inversion of this integral equation is a standard problem in operational calculus and has been carried out explicitly by Schumann [27] for a transform which has the form of eq (11). In the present discussion a much simpler approach is used which is justified when the damping is small. It should be noted that $\alpha p^{\frac{1}{2}}$ has already been assumed small compared to p, thus a perturbation method is in order.

When $\alpha=0$ corresponding to no dissipation (i.e., perfectly conducting boundaries)

$$E_r^{(n)}(p)=\frac{(I\,ds)_0}{h^2C_e}P_n(x)\frac{(2n+1)p}{\omega_n^2+p^2}. \tag{8.13}$$

The poles in the p plane are thus at $p=\pm i\omega_n$. The inversion to the time domain gives

$$e_r^{(n)}(t)=\frac{(I\,ds)_0}{h^2C_e}P_n(x)(2n+1)\cos\omega_n t \tag{8.14}$$

which may be verified by noting that the above expressions for $E_r^{(n)}(p)$ and $e_r^{(n)}(t)$ satisfy eq (12).

A step-function dipole source thus excites the static field (i.e., $\omega_0=0$) and the cavity-resonator modes ($n=1, 2, 3, \ldots$). For $a=6{,}400$ km

$$\omega_1/2\pi=10.6 \text{ cps}$$

$$\omega_2/2\pi=18.3 \text{ cps}$$

$$\omega_3/2\pi=25.9 \text{ cps}.$$

To account for finite conductivity it is necessary to solve the equation

$$p^2+p^{3/2}\alpha+\omega_n^2=0 \tag{8.15}$$

which gives the poles for the function $E_r(p)$ in the case when $\alpha\neq0$. Remembering that $\alpha p^{\frac{1}{2}}<<p$, it readily follows that

$$p\cong i\omega_n\left[1+\frac{\alpha}{2\omega_n^{1/2}}e^{i3\pi/4}\right]\cong i\omega_n'-\Omega_n \tag{8.16}$$

where $\omega_n'=\omega_n\left(1-\frac{\alpha}{2^{3/2}\omega_n^{1/2}}\right)$ is the resonant frequency and

$$\Omega_n=\frac{\alpha\omega_n^{1/2}}{2^{3/2}}$$

is the damping coefficient. It then easily follows that $\cos\omega_n t$ is to be replaced by

$$e^{-\Omega_n t}\cos\omega_n' t.$$

To this approximation, the effect of finite conductivity is to exponentially damp the oscillations with time. For $a=6{,}400$ km, $h\sim100$ km, $\sigma_i\sim10^{-4}$ mhos/m, the time constant is given by

$$\frac{1}{\Omega_n}\cong\frac{4}{\sqrt{n(n+1)}}\text{ sec } (n=1,2,3,\ldots) \tag{8.17}$$

which is rather interesting.

The total field is thus given by

$$e_r(t) \cong \frac{(I\,ds)_0}{h^2 C_e} \sum_{n=0}^{\infty} P_n(x)(2n+1)e^{-\Omega_n t} \cos \omega_n' t \tag{8.18}$$

which is valid for $\alpha^2 t >> 1$ or $t >> 1/(h^2 \sigma_i \mu)$.

9. Excitation by Horizontal Dipoles for the Curved Earth

The formulation of the theory for a horizontal dipole is similar to that for a vertical dipole. The complexity of the equations is greater, however, because of the nonsymmetry of the problem. Schumann [6] uses this approach in his analysis but his results are not complete as discussed below. The deficiency arises when the eigenfunction series is matched to the source singularity. In the case of the vertical dipole as outlined in the previous sections, this process is relatively straightforward but in the case of the horizontal dipole there is coupling between TE and TM modes which apparently is not accounted for using this technique. An alternative is to set up the problem in terms of a harmonic series representation wherein the summation is over integral values of n, the index of the spherical wave functions. This series is poorly convergent, however, and the Watson technique must be used to transform it to a series of residues at the complex poles ν. Such a procedure was used by Wait [29] for a horizontal dipole over an earth with a homogeneous atmosphere. It would not be difficult to generalize these results to include the influence of the ionospheric reflecting layer. In the present work, however, it seems more instructive to use a different method which makes use of the reciprocity theorem and the results for vertical electric and vertical magnetic dipoles.

For the first part of the problem a vertical magnetic dipole of moment $K\,ds$ is considered. It is located at $r=r_0$ on the polar axis. Due again to the intrinsic symmetry of the problem the fields can be obtained from a single scalar function ψ^h as follows:

$$\begin{aligned} H_r &= \frac{i}{r\eta} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\psi^h}{\partial\theta}\right) \\ H_\theta &= -\frac{i}{r\eta} \frac{\partial^2}{\partial\theta\partial r}(r\psi^h) \\ E_\phi &= k \frac{\partial\psi^h}{\partial\theta} \end{aligned} \tag{9.1}$$

and $H_\phi = E_r = H_\theta = 0$. Such fields are purely of the TE type whereas they were of the TM type for a vertical electric dipole source.

The solution precedes in the same manner as for the vertical electric dipole. Now, however, the boundary conditions are that $(1/\eta r)\partial(r\psi^h)/\partial r$ and $k\psi^h$ are continuous at the concentric spherical interfaces. In a form suitable for application to vlf propagation, the final result is given by [5]

$$\psi^h = \frac{K\,ds}{2khr_0} \sum_{m=1,2,3,\ldots}^{\infty} \frac{\delta_m^h f_m^h(z_0) f_m^h(z)}{\sin \bar{\nu}\pi} P_{\bar{\nu}}(-\cos\theta) \tag{9.2}$$

where

$$\delta_m^h \cong \frac{1}{1 - \dfrac{\sin 2khC_m}{2khC_m}}, \tag{9.3}$$

$$2f_m^h(z) = [R_g^h(C_m)]^{-\frac{1}{2}} \exp[ikC_m z] + [R_g^h(C_m)]^{\frac{1}{2}} \exp[-ikC_m z] \tag{9.4}$$

[5] To conform with standard waveguide practice, the TE mode of lowest attenuation is denoted $m=1$.

and similarly for $f_m^h(z_0)$. The modal equation has the form

$$R_g^h(C)R_i^h(C') \exp\left\{-i2k\int_0^h\left[C^2+\frac{2z}{a}S^2\right]^{\frac{1}{2}}dz\right\}=e^{-2\pi im} \tag{9.5}$$

where

$$R_g^h(C)=\frac{C-N_gC_g}{C+N_gC_g} \qquad \text{with } C_g=\left[1-\frac{S^2}{N_g^2}\right]^{\frac{1}{2}}$$

and

$$R_i^h(C')=\frac{C'-N_iC_i'}{C'+N_iC_i'} \text{ with } C_i'=\left[1-\frac{(S')^2}{N_i^2}\right]^{\frac{1}{2}}.$$

The electric field component E_ϕ is thus given by

$$E_\phi=\frac{Kds}{2hr_0}\sum_m \delta_m^h \frac{f_m^h(z_0)f_m^h(z)}{\sin \bar{\nu}\pi}\frac{\partial P_{\bar{\nu}}(-\cos\theta)}{\partial\theta}. \tag{9.6}$$

Now when the source is a small loop of area da carrying an average circulating current I it follows that

$$Kds=i\mu\omega Ida. \tag{9.7}$$

Furthermore, if the receiving antenna is a horizontal electric antenna of effective length dl, the voltage at the terminals is given by

$$v=E_\phi dl \sin\phi \tag{9.8}$$

where ϕ is the angle subtended by the receiving antenna and the arc joining the two antennas (see fig. 7). Thus the mutual impedance Z_m between the source loop of area da and

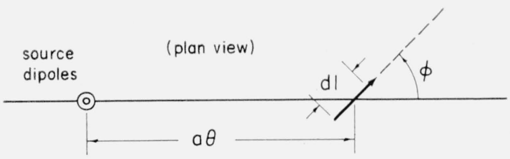

FIGURE 7. *Source vertical magnetic and electric dipoles and receiving horizontal electric dipole for mutual impedance calculation.*

the horizontal receiving antenna of length dl is

$$Z_m=\frac{v}{I}=\frac{i\mu\omega\, da\, dl\sin\phi}{2hr_0}\sum_m\cdots \tag{9.9}$$

where the summand is the same as in eq (6).

The mutual impedance Z_e between a vertical electric dipole source at $\theta=0$, $r=r_0$, and the horizontal receiving antenna is also required. This may be obtained from the scalar function ψ previously obtained. In particular

$$E_\theta=-\frac{i}{r}\eta\frac{\partial^2}{\partial\theta\partial r}(r\psi)\simeq -i\eta\frac{\partial^2}{\partial\theta\partial z}\psi \tag{9.10}$$

$$\simeq\eta\frac{Ids}{2hr_0}\sum_n\delta_n\frac{f_n(z_0)g_n(z)}{\sin\nu\pi}\frac{\partial}{\partial\theta}P_\nu(-\cos\theta) \tag{9.11}$$

where

$$2g_n(z)=\left[C_n^2+\frac{2z}{a}S_n^2\right]^{\frac{1}{2}}\left\{R_g^{-1/2}\exp\left[+ik\int_0^z\left(C_n^2+\frac{2z}{a}S_n^2\right)^{\frac{1}{2}}dz\right]\right.$$

$$\left.-R_g^{1/2}\exp\left[-ik\int_0^z\left(C_n^2+\frac{2z}{a}S_n^2\right)^{\frac{1}{2}}\right]dz\right\}. \quad (9.12)$$

When $z/a<<|C_n^2|$

$$2g_n(z)\simeq C_n[R_g^{-1/2}e^{ikC_nz}-R_g^{1/2}e^{-ikC_nz}]. \quad (9.13)$$

Furthermore if $|kC_nz|<<1$ which is the usual case

$$g_n(z)\simeq\Delta_nf_n(z) \quad (9.14)$$

where

$$\Delta_n=\frac{1}{N_g}\left(1-\frac{S_n^2}{N_g^2}\right)^{\frac{1}{2}}\simeq\frac{1}{N_g}.$$

The mutual impedance Z_e between the vertical electric dipole and the horizontal receiving antenna dl is thus given by

$$Z_e\simeq\frac{\eta\, ds\, dl\cos\phi}{2hr_0}\sum_n \cdots \quad (9.15)$$

where the summand is the same as in eq (11).

It is now a simple matter to write down the field expressions when the source is a horizontal electric antenna carrying a current I of length dl. The antenna or dipole now is considered to be located at $r=r_0$ and $\theta=0$ and oriented in the direction $\phi=0$. The vertical magnetic field at (r, θ, ϕ) is obtained from the relation

$$i\mu\omega H_rda=IZ_m \quad (9.16)$$

which relates the total magnetic flux in a small loop of area da at (r, θ, ϕ) and the vertical magnetic field at the same point. Using eq (9.) it is seen that

$$H_r=\frac{Idl}{2hr}\sum_m\delta_m^h\frac{f_m^h(z_0)f_m^h(z)}{\sin\bar{\nu}\pi}\frac{\partial P_{\bar{\nu}}(-\cos\theta)}{\partial\theta}\sin\phi. \quad (9.17)$$

In a similar fashion, the vertical electric field at (r, θ, ϕ) is obtained from the relation

$$E_rds=IZ_e \quad (9.18)$$

which relates the voltage in the small vertical antenna of length ds at (r, θ, ϕ) and the vertical electric field at the same point. Thus

$$E_r=\eta\frac{Idl}{2hr}\sum_n\frac{\delta_nf_n(z)g_n(z_0)}{\sin\nu\pi}\frac{\partial}{\partial\theta}P_\nu(-\cos\theta)\cos\phi. \quad (9.19)$$

The other field components can be found from the above expressions for E_r and H_r. Quite generally the field components in spherical coordinates (see fig. 8) can be written in terms of a set of purely TE and TM modes derivable from scalar functions U and V. Thus

$$E_r=\left[\frac{\partial^2(Ur)}{\partial r^2}+k^2rU\right] \quad (9.20)$$

$$E_\theta=\frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}(rU)-\frac{i\mu\omega}{r\sin\theta}\frac{\partial(rV)}{\partial\phi} \quad (9.21)$$

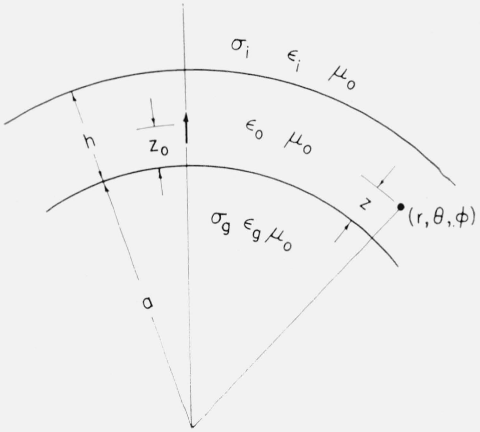

FIGURE 8. *Spherical coordinate system for horizontal electric dipole between concentric spherical interfaces.*

$$E_\phi = \frac{1}{r \sin\theta} \frac{\partial^2}{\partial\phi\partial r}(rU) + \frac{i\mu\omega}{r} \frac{\partial(rV)}{\partial\theta} \tag{9.22}$$

$$H_r = \frac{\partial^2(rV)}{\partial r^2} + k^2 rV \tag{9.23}$$

$$H_\theta = \frac{1}{r} \frac{\partial^2}{\partial r\partial\theta}(rV) + \frac{i\epsilon\omega}{r\sin\theta} \frac{\partial}{\partial\phi}(rU) \tag{9.24}$$

$$H_\phi = \frac{1}{r\sin\theta} \frac{\partial^2}{\partial r\partial\phi}(rU) - \frac{i\epsilon\omega}{r} \frac{\partial}{\partial\theta}(rU). \tag{9.25}$$

Since U and V satisfy the equations

$$(\nabla^2 + k^2)\begin{smallmatrix}U\\V\end{smallmatrix} = 0 \tag{9.26}$$

in the space $a<r<a+h$ they must be made up of solutions of the form

$$\begin{matrix} h_\nu^{(1)}(kr) & & \cos q\phi \\ & P_\nu^q(\cos\theta) & \\ h_\nu^{(2)}(kr) & & \sin q\phi \end{matrix} \tag{9.27}$$

where q is an integer. Since the field for E_r and H_r has already been prescribed, $q=1$. With some consideration it is seen that

$$U = \sum_n A_n f_n(z) \frac{\partial P_\nu(-\cos\theta)}{\partial\theta} \cos\phi \tag{9.28}$$

and

$$V = \sum_m B_m f_m^h(z) \frac{\partial P_\nu(-\cos\theta)}{\partial\theta} \sin\phi. \tag{9.29}$$

Further, on noting

$$\left[k^2 + \frac{\partial^2}{\partial r^2} - \frac{\nu(\nu+1)}{r^2}\right][rh_\nu^{(1,2)}(kr)] = 0 \tag{9.30}$$

it is seen that

$$E_r = \sum_n \frac{\nu(\nu+1)}{r} A_n f_n(z) \frac{\partial P_\nu(-\cos\theta)}{\partial\theta} \cos\phi \tag{9.31}$$

and

$$H_r = \sum_m \frac{\bar{\nu}(\bar{\nu}+1)}{r} B_m f_m^h(z) \frac{\partial P_{\bar{\nu}}(-\cos\theta)}{\partial\theta} \sin\phi. \tag{9.32}$$

On comparing eqs (30) and (31) with (17) and (19), it follows that

$$A_n = \frac{\eta I dl}{2h\nu(\nu+1)} \frac{\delta_n g_n(z_0)}{\sin \nu\pi} \tag{9.33}$$

and

$$B_m = \frac{I dl}{2h\bar{\nu}(\bar{\nu}+1)} \frac{\delta_m^h f_m^h(z_0)}{\sin \bar{\nu}\pi}. \tag{9.34}$$

When $|\nu| >> 1$ and θ is not near 0 or π the following asymptotic expansion is valid:

$$\frac{1}{\sin \nu\pi} \frac{\partial}{\partial\theta} P_\nu(-\cos\theta) \simeq \frac{2e^{-i\pi/4}}{(\sin\theta)^{\frac{1}{2}}} \left(\frac{a}{\lambda}\right)^{\frac{1}{2}} S_n^{1/2} e^{-ika\theta S_n} \tag{9.35}$$

where use has been made of the relation

$$\nu \simeq kaS_n.$$

The height function $g(z_0)$ occurring in the expression for E_r can be simplified at low heights. For example, if $|kC_n z_0| << 1$ which is the usual case

$$g_n(z_0) \simeq \frac{1}{iN_g} \left(1 - \frac{S_n^2}{N_g^2}\right)^{\frac{1}{2}}. \tag{9.36}$$

Thus the vertical electric field of a horizontal dipole is well approximated by

$$E_r \simeq E_0 \frac{\cos\phi}{N_g} \left[\frac{d/a}{\sin d/a}\right]^{\frac{1}{2}} \frac{(d/\lambda)^{\frac{1}{2}}}{(h/\lambda)} e^{i[(2\pi d/\lambda)-(\pi/4)]} \sum_{n=0}^{\infty} \delta_n S_n^{1/2} e^{-i2\pi S_n (d/\lambda)} \tag{9.37}$$

where

$$E_0 = i(\eta/\lambda) Ids(e^{-i2\pi d/\lambda})/d. \tag{9.38}$$

It is of interest to compare this with eq (6.24) for the vertical electric field of a vertical dipole with the same moment. It is seen for a given mode

$$\frac{E_r^{(n)}\text{ (for horizontal dipole)}}{E_r^{(n)}\text{ (for vertical dipole)}} \simeq \frac{\cos\phi}{S_n N_g} \left(1 - \frac{S_n^2}{N_g^2}\right)^{\frac{1}{2}}. \tag{9.39}$$

Since $S_n \simeq 1$, it is seen that the ratio does not depend critically on mode number n, thus

$$\frac{E_r\text{ (for horizontal dipole)}}{E_r\text{ (for vertical dipole)}} \simeq \frac{\cos\phi}{N_g} \left(1 - \frac{1}{N_g^2}\right)^{\frac{1}{2}}. \tag{9.40}$$

In most cases $|N_g| >> 1$ so the ratio is of the order of $1/N_g$ which is small. In particular, at very low frequencies

$$\frac{\cos\phi}{N_g} \simeq \left(\frac{\epsilon\omega}{\sigma_g}\right)^{\frac{1}{2}} e^{i\pi/4} \cos\phi \tag{9.41}$$

which indicates that the ratio varies as the square root of the frequency. This is in disagreement with the results of Schumann who finds that dependence is the inverse first power of frequency. In the direction $\phi = 0$, of course, the ratio derived here turns out to be nothing more than the "wave-tilt" for a vertically polarized plane wave at grazing incidence on a flat earth. Thus, Schumann's results [7] for the horizontal dipole would seem to be in error.

The horizontal dipole also, of course, radiates horizontal polarization. The simplified expression for H_r can be written by employing the single term asymptotic representation described above. Thus

$$H_r \simeq \frac{E_0 \sin\phi}{\eta} \frac{(d/\lambda)^{\frac{1}{2}}}{(h/\lambda)} \left[\frac{d/a}{\sin d/a}\right]^{\frac{1}{2}} e^{i[(2\pi d/\lambda)-(\pi/4)]} \sum_{m=1}^{\infty} \delta_m^h e^{-ika\theta S_m} S_m^{1/2} f_m^h(z_0) f_m^h(z). \tag{9.42}$$

10. Influence of the Earth's Magnetic Field

In the preceding analysis the earth's magnetic field has tacitly been neglected. To indicate its effect the reflection coefficient for sharply bounded ionosphere with the magnetic field included shall be discussed. The derivation of the general formulas is due to Budden [30] but for highly oblique incidence great simplifications to his results can be made.

The starting point is the magneto-ionic formula of Appleton and Hartree for the complex refractive index μ for a homogeneous ionized medium with superimposed magnetic field. In the region from 70 to 90 km in the ionosphere where very low frequencies are reflected, it is often permissible to employ the quasi-longitudinal approximation of Booker. It is now implied that the waves after they are transmitted into the ionosphere are steeply refracted toward the vertical. Essentially this means that the refractive index does not depend to any great extent on the direction of propagation for temperate and polar latitudes so that

$$\mu^2 \cong 1 - i(\omega_r/\omega) \exp(\pm i\tau) \tag{10.1}$$

where

$$\tan \tau = \omega_L/\nu$$

and

$$\omega_r = \omega_0^2(\nu^2 + \omega_L{}^2)^{-\frac{1}{2}}.$$

In the above

$\omega_0^2 = Ne^2/\epsilon m$,
N = number of electrons per meter^3,
e and m = charge and mass of electrons,
$\epsilon = 8.854 \times 10^{-12}$,
ν = collision frequency,
$\omega_L = (4\pi \times 10^{-7}) He/m$, and
H = effective strength of the earth's magnetic field, (i.e., the longitudinal component for propagation in the ionosphere).

It is now desirable to consider four reflection coefficients ${}_{||}R_{||}$, ${}_{||}R_{\perp}$, ${}_{\perp}R_{||}$, and ${}_{\perp}R_{\perp}$ to indicate the complex ratio of a specified electric field in the wave after reflection to a specified electric field in the wave before reflection. The first subscript denotes whether the electric field in the incident wave is parallel ($||$) or perpendicular ($\perp$) to the plane of incidence and the second subscript refers in the same way to the reflected wave. A cartesian coordinate system (x, y, z) is now taken with z measured vertically upwards. The incident wave has its normal in the xz plane inclined at an angle θ to the z axis. The components of the electric field are $E_{||}$ in the xz plane and $E_{\perp}$ perpendicular to this plane (i.e., in the direction of increasing y). When the + sign is taken in eq (1), the refractive index is denoted μ_0 corresponding to the ordinary wave, and when the − sign is taken the refractive index is denoted μ_e corresponding to the extraordinary wave. With this convention, it can be shown that

$$E_{\perp 0}/E_{||0} = -i \text{ and } E_{\perp e}/E_{||e} = i \tag{10.2}$$

in the northern hemisphere.

The incident wave is now characterized by a factor $\exp[-ik(x \sin\theta + z\cos\theta)x]$ and the reflected wave, therefore, contains a factor $\exp[-ik(x\sin\theta - z\cos\theta)x]$. Furthermore, the transmitted waves have factors $\exp[-ik(x\sin\theta_0 + z\cos\theta_0)\mu_0 x]$ and $\exp[-ik(x\sin\theta_e + z\cos\theta_e)\mu_e x]$. The reflection coefficients are now obtained by matching tangential field components at the air-ionosphere interface. The results, expressed in a form suitable for computation are listed below.

$$ {}_{||}R_{||} = [(\mu_0 + \mu_e)(C^2 - C_0C_e) + (\mu_0\mu_e - 1)(C_0 + C_e)C]/D \tag{10.3}$$

$$ {}_{||}R_{\perp} = 2iC(\mu_0 C_0 - \mu_e C_e)/D \tag{10.4}$$

$$ {}_{\perp}R_{||} = 2iC(\mu_0 C_e - \mu_e C_0)/D \tag{10.5}$$

$$ {}_{\perp}R_{\perp} = [(\mu_0 + \mu_e)(C^2 - C_0C_e) - (\mu_0\mu_e - 1)(C_0 + C_e)C]/D \tag{10.6}$$

where

$$D = (\mu_0 + \mu_e)(C^2 + C_0 C_e) + (\mu_0 \mu_e + 1)(C_0 + C_e) C \tag{10.7}$$

and

$$C = \cos\theta, C_0 = \cos\theta_0, C_e = \cos\theta_e.$$

Numerical values based on these formulas are available.

Now for highly oblique incidence the value of $|C|$ is small. Thus for $|C^2| \ll 1$

$$_{||}R_{||} \cong -\left[\frac{1 - \dfrac{(\mu_0\mu_e - 1)(C_0 + C_e)}{(\mu_0 + \mu_e) C_0 C_e} C}{1 + \dfrac{(\mu_0\mu_e + 1)(C_0 + C_e)}{(\mu_0 + \mu_e) C_0 C_e} C}\right]. \tag{10.8}$$

Furthermore if $|\mu_0\mu_e C^2| \ll 1$,

$$_{||}R_{||} \cong -\left[1 - \frac{2\mu_0\mu_e}{\mu_0 + \mu_e} \frac{C_0 + C_e}{C_0 C_e} C\right] \tag{10.9}$$

$$\cong -\exp(-2\beta_{||} C)$$

where

$$\beta_{||} = \frac{\mu_0\mu_e}{\mu_0 + \mu_e} \frac{C_0 + C_e}{C_0 C_e}. \tag{10.10}$$

To the same approximation

$$_{\perp}R_{\perp} \cong -\left[1 - \frac{2}{\mu_0 + \mu_e} \frac{C_0 + C_e}{C_0 C_e} C\right] \cong -\exp(-2\beta_{\perp} C) \tag{10.11}$$

where

$$\beta_{\perp} = \frac{1}{\mu_0 + \mu_e} \frac{C_0 + C_e}{C_0 C_e}. \tag{10.12}$$

Also

$$_{||}R_{\perp} \cong \frac{2iC}{C_0 C_e} \frac{\mu_0 C_0 - \mu_e C_e}{\mu_0 + \mu_e} \tag{10.13}$$

and

$$_{\perp}R_{||} \cong \frac{2iC}{C_0 C_e} \frac{\mu_0 C_e - \mu_e C_0}{\mu_0 + \mu_e}. \tag{10.14}$$

It is immediately evident from the above that as θ tends to $\pi/2$ (i.e., grazing incidence), the reflection coefficients $_{||}R_{||}$ and $_{\perp}R_{\perp}$ are both approaching -1 whereas the conversion coefficients $_{||}R_{\perp}$ and $_{\perp}R_{||}$ are both approaching zero. In this sense a sharply bounded ionosphere behaves as an equivalent isotropic medium for highly oblique incidence.

Some further simplifications are possible when the ionosphere is effectively a good conductor. For example, if

$$|\omega_r/\omega| \gg 1$$

$$\mu_0^2 = 1 - i(\omega_r/\omega) e^{i\tau} \cong -i(\omega_r/\omega) e^{i\tau}, \tag{10.15}$$

$$\mu_e^2 \cong 1 - i(\omega_r/\omega) e^{-i\tau} \cong -i(\omega_r/\omega) e^{-i\tau}. \tag{10.16}$$

Consequently,

$$\begin{matrix}_{||}\\ _{\perp}\end{matrix} R \begin{matrix}_{||}\\ _{\perp}\end{matrix} \cong \frac{(1+i)(\omega/\omega_r)^{\frac{1}{2}} \cos(\tau/2)(\mathrm{C}^2 - 1) \pm 2^{\frac{1}{2}}[1 - i\omega/\omega_r]\,\mathrm{C}}{(1+i)(\omega/\omega_r)^{\frac{1}{2}} \cos(\tau/2)(\mathrm{C}^2 + 1) + 2^{\frac{1}{2}}[1 + i\omega/\omega_r]\,\mathrm{C}} \tag{10.17}$$

and

$$\begin{matrix}_{||}\\ _{\perp}\end{matrix} R \begin{matrix}_{\perp}\\ _{||}\end{matrix} \cong \frac{-2(\omega/\omega_r)^{\frac{1}{2}}(1+i)\,\mathrm{C} \sin(\tau/2)}{(1+i)(\omega/\omega_r)^{\frac{1}{2}} \cos(\tau/2)(\mathrm{C}^2 + 1) + 2^{\frac{1}{2}}[1 + i\omega/\omega_r]\,\mathrm{C}}. \tag{10.18}$$

These may be further approximated, for $\cos\theta << 1$, by

$$1-{}_{\|}R_{\|} \simeq 2\left(\frac{i\omega}{\omega_r}\right)^{\frac{1}{2}} \frac{\cos(\tau/2)}{\cos\theta}, \quad {}_{\perp}R_{\perp}+1 \simeq 2\left(\frac{i\omega}{\omega_r}\right)^{\frac{1}{2}} \frac{\cos\theta}{\cos(\tau/2)} \tag{10.19}$$

provided the right-hand sides of these equations are small compared to one. To the same approximation

$$\left.\begin{matrix} {}_{\|}R_{\perp} \\ {}_{\perp}R_{\|} \end{matrix}\right. \simeq -2(i\omega/\omega_r)^{\frac{1}{2}} \sin(\tau/2). \tag{10.20}$$

The quasi-longitudinal approximation used above is only valid when

$$\frac{\omega_T^4}{4\omega^2\omega_L^2} << \left|\left(1-\frac{\omega_0^2}{\omega^2}-i\frac{\nu}{\omega}\right)^2\right|$$

where ω_L and ω_T are the longitudinal and transverse components of the (angular) gyro frequency. Clearly, this condition is violated when the transverse component of the earth's magnetic field is large such as for propagation around the magnetic equator.

The case of a purely horizontal and transverse field has been considered by N. F. Barber and D. D. Crombie (to be published in Journal of Atmospheric and Terrestrial Physics). Their results, applicable to a sharply bounded ionosphere, may be written in the following form

$${}_{\|}R_{\|} = \frac{C-\Delta}{C+\Delta}, \quad {}_{\|}R_{\perp}=0 \tag{10.21}$$

where

$$\Delta = \frac{\left[C^2+\dfrac{1+iL}{iL-L^2-\gamma^2}\right]^{\frac{1}{2}}(iL-L^2-\gamma^2)-i\gamma S}{(1+iL)^2-\gamma^2}$$

where $L=\omega/\omega_r$ and $\gamma=\omega_T\omega/\omega_0^2$. For east-to-west propagation (along the magnetic equator), γ is positive, while for west-to-east propagation, γ is negative. For highly oblique incidence, the above simplifies to

$$\frac{1}{\Delta} \simeq \beta_{\|} \tag{10.22}$$

where

$$\frac{1}{\beta_{\|}} = \frac{(1+iL)^{\frac{1}{2}}(iL-L^2-\gamma^2)^{\frac{1}{2}}-i\gamma}{(1+iL)^2-\gamma^2}.$$

Furthermore if, $|\beta_{\|}C| << 1$

$${}_{\|}R_{\|} \simeq -e^{-2\beta_{\|}C} \tag{10.23}$$

which has the same form as equation (11)

The exact determination of the reflection coefficients for any orientation of the earth's magnetic field may be carried out using a method outlined by Bremmer [3]. This has been done by Johler and Walters whose results are to be published in the following issue of this journal.

11. Mode Series for an Anisotropic Ionosphere

In this section the formalism for the mode theory is developed for a plane earth and an anisotropic sharply-bounded ionosphere. The geometry is the same as in section (3) where the ionosphere was assumed to be isotropic. In the present case the reflection coefficient $[R_i]''$ is regarded as a matrix and written in the form

$$[R_i]''=\begin{bmatrix} {}_{\|}R_{\|} & {}_{\perp}R_{\|} \\ {}_{\|}R_{\perp} & {}_{\perp}R_{\perp} \end{bmatrix}'' \tag{11.1}$$

where the two primes are to indicate that it is a two column matrix. The individual coefficients ${}_{\|}R_{\|}$, ${}_{\perp}R_{\|}$, ${}_{\|}R_{\perp}$, and ${}_{\perp}R_{\perp}$ discussed earlier, indicate the complex ratio of an electric field component in the wave after reflection to an electric field component in the wave before reflection. The first subscript denotes whether the electric field specified in the incident wave is parallel ($\|$) or perpendicular ($\perp$) to the plane of incidence. The second subscript refers in the same way to the electric field in the reflected wave.

When the ionosphere becomes isotropic corresponding to a zero magnetic field, the reflection coefficient in matrix notation becomes simply

$$[R_i]''=\begin{bmatrix} R_i & 0 \\ 0 & R_i^h \end{bmatrix}'' \tag{11.2}$$

where R_i and R_i^h are the complex scalar reflection coefficients for vertically-polarized and horizontally-polarized waves, respectively. The corresponding [matrix] reflection for the ground is

$$[R_g]''=\begin{bmatrix} R_g & 0 \\ 0 & R_g^h \end{bmatrix}''. \tag{11.3}$$

The case of two successive reflections, the first from the ground and the second from the anisotropic ionosphere, is represented by the matrix

$$[R_gR_i]''=[R_i]''\times[R_g]''=\begin{bmatrix} R_g\,{}_{\|}R_{\|} & R_g^h\,{}_{\perp}R_{\|} \\ R_g\,{}_{\|}R_{\perp} & R_g^h\,{}_{\perp}R_{\perp} \end{bmatrix}''. \tag{11.4}$$

In the case of the isotropic ionosphere this reduces to

$$[R_gR_i]''=\begin{bmatrix} R_gR_i & 0 \\ 0 & R_g^hR_i^h \end{bmatrix}''. \tag{11.5}$$

The arguments employed here are virtually identical to those of Budden who, however, assumes a perfectly conducting ground, such that

$$R_g=+1 \text{ and } R_g^h=-1.$$

In the previous formulation for a vertical electric dipole between the plane ground surface and a sharply bounded isotropic ionosphere, the fields could be completely derived from an electric Hertz vector which has only a z component. Of course, if the source was not symmetrical it was necessary to introduce an additional component of the electric Hertz vector. When the upper boundary is anisotropic, the single component Hertz vector is not adequate even for a vertical electric dipole source. This is not surprising since the TM modes are coupled to the TE modes by the anisotropic boundary conditions.

Any electromagnetic field, in such a parallel plate region, can be obtained from a superposition of TM and TE modes which are derived from electric and magnetic Hertz vectors.

These are denoted individually by $\vec{\Pi}_{\|}$ and $\vec{\Pi}_{\perp}$ or, collectively by the matrix

$$[\vec{\Pi}]''=\begin{bmatrix}\vec{\Pi}_{\|}\\ \vec{\Pi}_{\perp}\end{bmatrix}' \tag{11.6}$$

where the single prime is to indicate that it is a single column matrix. $\vec{\Pi}_{\perp}$ which is a magnetic Hertz vector is often referred to as a Fitzgerald vector. Furthermore, the electric and magnetic fields can also be written as single column matrices in the manner

$$[\vec{E}]'=\begin{bmatrix}\vec{E}_{\|}\\ \vec{E}_{\perp}\end{bmatrix}', [\vec{H}]'=\begin{bmatrix}\vec{H}_{\|}\\ \vec{H}_{\perp}\end{bmatrix}' \tag{11.7}$$

where

$$\vec{E}_{\|}=(k^2+\text{grad div})\vec{\Pi}_{\|} \tag{11.8}$$

$$\vec{H}_{\|}=i\epsilon\omega \text{ curl } \vec{\Pi}_{\|} \tag{11.9}$$

$$\eta\vec{H}_{\perp}=(k^2+\text{grad div})\vec{\Pi}_{\perp} \tag{11.10}$$

$$\eta\vec{E}_{\perp}=-i\mu\omega \text{ curl } \vec{\Pi}_{\perp}. \tag{11.11}$$

The intrinsic impedance η is introduced in the latter two equations to make $\vec{\Pi}_{\|}$ and $\vec{\Pi}_{\perp}$ of the same dimensions. The Hertz vector in matrix form corresponding to the primary excitation is then written

$$[\vec{\Pi}_p]'=\begin{bmatrix}\vec{\Pi}_p\\ 0\end{bmatrix}' \tag{11.12}$$

where $\vec{\Pi}_p$ has only a z component $\Pi_z^{(p)}$. To match boundary conditions in the case of azimuthal symmetry it is only necessary that the vectors $\vec{\Pi}_{\|}$ and $\vec{\Pi}_{\perp}$ have a z component. The condition of azimuthal symmetry is achieved if the reflection coefficients themselves are independent of the azimuthal coordinate ϕ.

Formally the solution has the same form as the isotropic case if the appropriate reflection coefficients are now regarded as matrices. For example for the space $o<z<h$ the (matrix) Hertz vector

$$[\Pi_z]'=\frac{ik[M]'}{2}\int_{\Gamma}[F(C)]''H_0^{(2)}(kS\rho)dC \tag{11.13}$$

where

$$[M]=\begin{bmatrix}M\\ 0\end{bmatrix}' \text{ with } M=\frac{I\,ds}{4\pi i\epsilon\omega}$$

and where

$$[F(C)]''=\frac{(e^{ikCz}+[R_g]''e^{-ikCz})(e^{ikC(h-z_0)}+[R_i]''e^{-ikC(h-z_0)})}{e^{ikCh}(1-[R_gR_i]''e^{-2ikhC})}. \tag{11.14}$$

It should be noted that the denominator in the above expression is also a two column matrix. Inverting this, following the usual rules for such operations, leads to

$$[F(C)]''=(e^{ikCz}+[R_g]''e^{-ikCz})(1/\Delta)[N]''\{e^{i2kCh}+[R_i]''\} \tag{11.15}$$

for $z_0=0$, where

$$\Delta=\begin{vmatrix} e^{2iCkh}-{}_{\|}R_{\|}\ R_g & {}_{\perp}R_{\|}\ \ R_g^h \\ {}_{\|}R_{\perp}\ R_g & e^{2iCkh}-{}_{\perp}R_{\perp}\ R_g^h \end{vmatrix} \tag{11.16}$$

and

$$[N]''=\begin{bmatrix} e^{2iCkh}-{}_{\perp}R_{\perp}\, R_g^h & {}_{\perp}R_{\parallel}\, R_g^h \\ {}_{\parallel}R_{\perp}\, R_g & e^{2iCkh}-{}_{\parallel}R_{\parallel}\, R_g \end{bmatrix}''. \tag{11.17}$$

The corresponding residue series are thus given by

$$\begin{bmatrix} \Pi_{\parallel} \\ \Pi_{\perp} \end{bmatrix}' = -\pi k M \sum_{p} \frac{e^{2iC_pkh}}{[\partial\Delta/\partial C]_{C=C_p}}\, H_0^{(2)}(kS_p\rho) \begin{bmatrix} G_{\parallel p} \\ G_{\perp p} \end{bmatrix}' \tag{11.18}$$

with

$$G_{\parallel p}=f_p(z)\,(e^{2iC_pkh}-R_g^h\,{}_{\perp}R_{\perp}) \tag{11.19}$$

and

$$G_{\perp p}=if_p^h(z)\,R_g\,{}_{\parallel}R_{\perp}. \tag{11.20}$$

The summation is over the poles of the integrand $[F(C)]''$. Clearly this corresponds to the roots of the equation, $\Delta=0$, which are designated $C=C_p$. It is understood that all quantities in the summand of eq (18) are to be evaluated at $C=C_p$. The "height-factors" $f_p(z)$ and $f_p^h(z)$ have the usual form, that is

$$2f_p(z)=(R_g)^{-\frac{1}{2}}\,e^{ikC_pz}+(R_g)^{\frac{1}{2}}\,e^{-ikC_pz} \tag{11.21}$$

and

$$2f_p^h(z)=(R_g^h)^{-\frac{1}{2}}\,e^{ikC_pz}+(R_g^h)^{\frac{1}{2}}\,e^{-ikC_pz}. \tag{11.22}$$

The above results reduce to those of Budden when the ground is perfectly conducting.

The modes excited in the waveguide can be logically grouped into two sets. The first has a TM (transverse magnetic) character and the second has a TE (transverse electric) character. To obtain the attenuation and the phase constants of these individual modes, it is adequate to consider the anisotropy as sort of a perturbation to the corresponding TE and TM modes for the isotropic case.[6]

To simplify the discussion the ground is considered to be perfectly conducting. That is, $R_g=1$ and $R_g^h=-1$. The modal equation now becomes

$$(e^{i2kCh}-{}_{\parallel}R_{\parallel})\,(e^{i2kCh}+{}_{\perp}R_{\perp})+{}_{\parallel}R_{\perp}\,{}_{\perp}R_{\parallel}=0. \tag{11.23}$$

When mode coupling is disregarded this breaks into two equations

$${}_{\parallel}R_{\parallel}e^{-i2kCh}=1=e^{-i2\pi n} \tag{11.24a}$$

$${}_{\perp}R_{\perp}e^{-i2kCh}=-1=-e^{-i2\pi m} \tag{11.24b}$$

where n and m take integral values. As mentioned in the previous section the reflection coefficients for highly oblique incidence may be approximated by

$${}_{\parallel}R_{\parallel}\cong -e^{-2\beta_{\parallel}C} \tag{11.25}$$

and

$${}_{\perp}R_{\perp}\cong -e^{-2\beta_{\perp}C} \tag{11.26}$$

where to a first order, $\beta_{\parallel}$ and $\beta_{\perp}$ are independent of C. It thus follows that the first approximation (indicated by a superscript (1)) for the solutions of the modal equation are

$$C_p\cong C_n^{(1)}=\frac{\pi(n-\frac{1}{2})}{kh-i\beta_{\parallel}}\ (n=1, 2, \ldots) \tag{11.27}$$

for the TM modes, and

$$C_p\cong C_m^{(1)}=\frac{\pi m}{kh-i\beta_{\perp}}\ (m=1, 2, 3, \ldots) \tag{11.28}$$

for the TE modes. These have exactly the same form as when the ionosphere is assumed to be isotropic. The difference lies in the value of the coefficients $\beta_{\parallel}$ and $\beta_{\perp}$ which are functions of the earth's magnetic field.

[6] It should be noted that the negative order modes in the case of an anisotropic ionosphere are not the same as the positive order modes. That is, $C_{-1}\neq -C_0$, $C_{-2}\neq -C_1$, etc.

A second approximation to the mode equations is obtained in the following way. The modal equation is rewritten in the two equivalent forms

$$1-{}_{\|}R_{\|}e^{-i2kCh}=2\delta(C) \tag{11.29}$$

$$1+{}_{\perp}R_{\perp}e^{-i2kCh}=2\gamma(C) \tag{11.30}$$

where

$$2\delta(C)=-\frac{{}_{\|}R_{\perp}\;{}_{\perp}R_{\|}e^{-i2kCh}}{e^{i2kCh}+{}_{\perp}R_{\perp}} \tag{11.31}$$

and

$$2\gamma(C)=-\frac{{}_{\|}R_{\perp}\;{}_{\perp}R_{\|}e^{-i2kCh}}{e^{i2kCh}-{}_{\|}R_{\|}}. \tag{11.32}$$

It is to be expected that $\delta(C)$ is a small quantity for the TM type modes and $\gamma(C)$ is a small quantity for the TE type modes. The second approximations then are obtained by replacing $\delta(C)$ by $\delta(C_n^{(1)})$ for the TM set, and replacing $\gamma(C)$ by $\gamma(C_m^{(1)})$ for the TE set. Solving eqs (29) and (30) with these substitutions leads readily to

$$C_p\simeq C_n^{(2)}=\frac{\pi(n-\frac{1}{2})-i\delta(C_n^{(1)})}{kh-i\beta_{\|}} \tag{11.33}$$

for the TM type modes, and

$$C_p\simeq C_m^{(2)}=\frac{\pi m-i\gamma(C_m^{(1)})}{kh-i\beta_{\perp}}. \tag{11.34}$$

for the TE type modes.

These should be adequate solutions since $|\delta(C_n{}^{(1)})|$ and $|\gamma(C_m{}^{(1)})|$ are small compared to unity for the important modes. In fact for most cases of practical interest, the first-order approximations should suffice.

To provide some idea of the character of the TM and the TE type modes excited by a vertical dipole source, the ratio of the tangential magnetic field in the two principal planes is considered. For the pth mode, this ratio is given by

$$\left(\frac{H_\rho}{H_\phi}\right)=\left(\frac{\partial^2\Pi_{\perp p}}{\partial\rho\partial z}\right)\Big/\left(-i\epsilon\omega\frac{\partial\Pi_{||p}}{\partial\rho}\right) \tag{11.35}$$

$$\simeq=\frac{(\partial\Pi_{\perp p}/\partial z)}{ik\Pi_{||p}}\text{ for } k\rho>>1$$

$$\simeq-\frac{C_{p||}R_{\perp}R_g}{[e^{2iCkh}-R_g^h{}_{\perp}R_{\perp}]}\left\{\frac{\frac{1}{kC_p}\frac{\partial}{\partial z}f_p^h(z)}{f_p(z)}\right\} \tag{11.36}$$

where it is understood that the reflection coefficients are to be evaluated at $C=C_p$. The preceding expression reduces to

$$\left(\frac{H_\rho}{H_\phi}\right)\simeq\frac{C_{p]||}R_{\perp}}{e^{2iCkh}+{}_{\perp}R_{\perp}} \tag{11.37}$$

which was given originally by Budden [11]. In general this ratio is small except near the TE resonance wherein the denominator becomes very small. For the important TM modes which are of low order, both C_p and ${}_{||}R_{\perp}$ are small compared to unity and thus the magnetic field H_ρ in the direction of propagation has a relatively small magnitude.

12. Higher Approximations to the Curved Earth Theory

In the previous sections the mode series for a concentric spherical earth-air-ionosphere system was developed. In order to simplify the discussion and lead to results suitable for immediate use, rather crude approximations were introduced. In this section the problem is reformulated in a more rigorous fashion and higher order approximations for the various spherical wave functions are introduced. This analysis is really an extension of the work of Watson, Rydbeck, and Bremmer. The final results indicate the range of validity of the lower order approximations used in the earlier sections. The formulas are in a form which is suitable for numerical computation.

The earth is represented by a homogeneous sphere of radius a and is surrounded by a concentric homogeneous sharply-bounded ionosphere of radius c. The source is a vertical electric dipole of strength $I\,ds$ and is located at $r=b$. The electrical constants of the air space are denoted ϵ and μ and subscripts g and i are added to these when reference is made to the ground and the ionosphere, respectively (see fig. 5).

The fields can be expressed in terms of a Hertz vector which has only a radial component U, and thus, for the region $a<r<a+h$,

$$E_r=\left(k^2+\frac{\partial^2}{\partial r^2}\right)(rU) \qquad H_r=0$$

$$E_\theta=\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(rU) \qquad H_\theta=0$$

$$E_\phi=0 \qquad H_\phi=-i\epsilon\omega\frac{\partial U}{\partial\theta} \tag{12.1}$$

where $k=(\epsilon\mu)^{1/2}\omega$.[7] A subscript g and i are also added to the field quantities when reference is made to the regions $r<a$ and $r>c$, respectively. Furthermore, $k_g=(\epsilon_g\mu_g)^{1/2}\omega$ and $k_i=(\epsilon_i\mu_i)^{1/2}\omega$ are the respective wave numbers for these two regions.

The Hertz functions satisfy the inhomogeneous wave equation

$$(\nabla^2+k^2)\,U=\overline{C}\frac{\delta(r-b)\,\delta(\theta)}{2\pi r^2\sin\theta} \tag{12.2}$$

for $a<r<a+h$, where the δ's are unit impulse functions. The factor $2\pi r^2\sin\theta$ is the Jacobian of the transformation from rectangular to spherical coordinates. The constant $\overline{C}$ is to be chosen so that U has the proper singularity at the dipole, that is

$$-bU\rightarrow\frac{e^{-ikR}}{4\pi i\omega\epsilon R}I\,ds \text{ for } R\rightarrow 0$$

where $R=[r^2+b^2-2br\cos\theta]^{1/2}$, and therefore $\overline{C}=(i/\omega\epsilon)I\,ds$.

The field in the region $a<r<a+h$ is now written as the sum of the two parts U_e+U_s, where U_e has the proper dipole singularity at $R=0$, and U_s is finite at the point. As U_s is a solution of the homogeneous wave equation, it can be written in the form

$$U_s=\frac{ik\overline{C}}{4\pi}\sum_{q=0}^{\infty}(2q+1)\,[A_q h_q^{(2)}(kr)+B_q j_q(kr)]\,P_q(\cos\theta) \tag{12.3}$$

where $j_q(kr)$ and $h_q^{(2)}(kr)$ are spherical Hankel functions of the first and fourth kind, respectively, and $P_q(\cos\theta)$ is the Legendre function. The summation is over positive integral values of q. The corresponding expression for U_e is given by

$$U_e=\frac{ik\overline{C}}{4\pi}\sum_{q=0}^{\infty}(2q+1)\,P_q(\cos\theta)\begin{cases} j_q(kr)\,h_q^{(2)}(kb); & \text{for } r<b \\ h_q^{(2)}(kr)\,j_q(kb); & \text{for } r>b. \end{cases} \tag{12.4}$$

[7] The function $U=-i\eta\psi$ in terms of scalar function ψ used previously for the potential.

Since there are no singularities other than the source dipole, the Hertz functions U_g and U_i are solutions of the homogeneous wave equations

$$(\nabla^2+k_g^2)(rU_g)=0 \qquad \text{for } 0\leqq r\leqq a \tag{12.5}$$

and

$$(\nabla^2+k_i^2)(rU_i)=0 \qquad \text{for } r\geqq c. \tag{12.6}$$

Noting that U_g is to be finite at $r=0$ the solution must be of the form

$$U_g=\frac{ik\overline{C}}{4\pi}\sum_{q=0}^{\infty}(2q+1)P_q(\cos\theta)a_q j_q(k_g r) \tag{12.7}$$

where a_q is a coefficient which is independent of r and θ. Furthermore, since U_i is to give rise to an outgoing wave at $r=\infty$, the solution is of the form

$$U_i=\frac{ik\overline{C}}{4\pi}\sum_{q=0}^{\infty}(2q+1)P_q(\cos\theta)b_q h_q^{(2)}(k_i r) \tag{12.8}$$

where b_q is a coefficient.

The four unknown coefficients A_q, B_q, a_q, and b_q, can be found from the boundary conditions at $r=a$ and c. These require the continuity of the tangential field components. In order to facilitate the solution and to readily permit later generalizations, the (four) boundary conditions as stated above can be replaced by two impedance type boundary conditions. For the qth terms of the series these read

$$E_\theta^{(q)}=-Z_g^{(q)}H_\phi^{(q)} \quad \text{at} \quad r=a \tag{12.9}$$

and

$$E_\theta^{(q)}=Z_i^{(q)}H_\phi^{(q)} \quad \text{at} \quad r=c \tag{12.10}$$

where

$$Z_g^{(q)}=\frac{1}{i\epsilon\omega}\frac{\frac{\partial}{\partial r}[rj_q(k_g r)]}{rj_q(k_g r)}$$

and

$$Z_i^{(q)}=-\frac{1}{i\epsilon\omega}\frac{\frac{\partial}{\partial r}[rh_q^{(2)}(k_i r)]}{rh_q^{(2)}(k_i r)}.$$

Replacing kr by x, eqs (9) and (10) may be rewritten

$$\frac{1}{x}\frac{\partial}{\partial x}(xU)=i(Z_g^{(q)}/\eta)U \qquad \text{for } x=ka \tag{12.11}$$

and

$$\frac{1}{x}\frac{\partial}{\partial x}(xU)=-i(Z_i^{(q)}/\eta)U \qquad \text{for } x=kc. \tag{12.12}$$

Applying these to eq (3) enable A_q and B_q to be obtained explicitly in terms of known quantities. Using these results leads readily to the following exact solution for $a\leqq r\leqq b$.

$$U_s=\frac{ik\overline{C}}{4\pi}\sum_{q=0}^{\infty}(2q+1)h_q^{(2)}(kb)h_q^{(1)}(kr)P_q(\cos\theta)\frac{F_q}{D_q} \tag{12.13}$$

where

$$F_q = \left[1 + R_g^{(q)} \frac{h_q^{(1)}(ka)}{h_q^{(2)}(ka)} \frac{h_q^{(2)}(kr)}{h_q^{(1)}(kr)}\right]\left[1 + R_i^{(q)} \frac{h_q^{(2)}(kc)}{h_q^{(1)}(kc)} \frac{h_q^{(1)}(kb)}{h_q^{(2)}(kb)}\right], \tag{12.14}$$

$$D_q = 1 - R_g^{(q)} R_i^{(q)} \frac{h_q^{(1)}(ka)}{h_q^{(2)}(ka)} \frac{h_q^{(2)}(kc)}{h_q^{(1)}(kc)} \tag{12.15}$$

$$R_g^{(q)} = -\frac{ln'[ka\, h_q^{(1)}(ka)] - iZ_g^{(q)}/\eta}{ln'[ka\, h_q^{(2)}(ka)] - iZ_g^{(q)}/\eta} \tag{12.16}$$

$$R_i^{(q)} = -\frac{ln'[kc\, h_q^{(2)}(kc)] + iZ_i^{(q)}/\eta}{ln'[kc\, h_q^{(1)}(kc)] + iZ_i^{(q)}/\eta}. \tag{12.17}$$

The symbol ln' denotes logarithmic differentiation, for example

$$ln'[ka\, h_q^{(1)}(ka)] = \left.\frac{\frac{\partial}{\partial x}[xh_q^{(1)}(x)]}{xh_q^{(1)}(x)}\right]_{x=ka.} \tag{12.18}$$

The above result, although rigorous, is not of practical value for vlf propagation calculations because of poor convergence of the series solutions. In fact, something of the order of 2 ka terms are required to achieve 5 percent accuracy. At 15 kc, for example, 2 $ka \simeq 2{,}000$ which is a rather large number. An important observation, however, is that terms of order q beyond 2 ka contribute little to the series. Thus the spherical Bessel functions $j_q(k_g a)$ may be replaced by the Debye or second-order approximation since $|k_g a| >> q$ in the important range of q so long as $|k_g| >> k$ (i.e., well conducting ground). Thus

$$ln'[k_g a j_q(k_g a)] \cong i\left[1 - \frac{q^2}{k_g^2}\right]^{\frac{1}{2}}. \tag{12.19}$$

Similarly, for $|k_i| >> k$

$$ln'[k_i a h_q^{(2)}(k_i a)] \cong -i\left[1 - \frac{q^2}{k_i^2}\right]^{\frac{1}{2}}. \tag{12.20}$$

Since the total field is of the form

$$U = \sum_{q=0}^{\infty} (2q+1) f(q) P_q(\cos\theta) \tag{12.21}$$

it can be rewritten as a contour integral over q where the integrand has poles when q takes integrand values. Such a representation is

$$U = i\int_{C_1+C_2} \frac{qdq}{\cos q\pi} f(q-\tfrac{1}{2}) P_{q-\frac{1}{2}}[\cos(\pi-\theta)] \tag{12.22}$$

where the contour C_1+C_2 encloses the real axis as illustrated in figure 9. Noting that the poles of the integrand are located at $q=1/2, 3/2, 5/2$ . . . etc., it can be readily verified by the theorem of residues that this integral is equivalent to eq (21). Now, subject to the validity of the second-order approximations, for the wave functions of order $k_g a$ and $k_i a$ mentioned above, the function $f(q-\frac{1}{2})$ is an even function of q. This means that the part of the contour C_1 just above the positive real axis can be replaced by C_1' which is located just below the negative real axis (see fig. 9). The contour $C_1'+C_2$ is now entirely equivalent to L, a straight line running along just below the real axis. Replacing $q-\frac{1}{2}$ by ν the contour representation for U takes the form

$$U = -i\int_L \frac{(\nu+\frac{1}{2})}{\sin \nu\pi} f(\nu) P_\nu[\cos(\pi-\theta)]\, d\nu. \tag{12.23}$$

It is to be noted that this manipulation of the contours is only strictly justified when $f(q-\frac{1}{2})$ is an even function of q. This is well justified when $|k_g|^2$ and $|k_i|^2$ are both $>>k^2$.

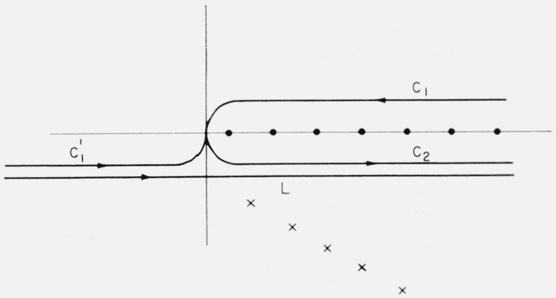

FIGURE 9. *The contours in the complex q plane.*

The next step in the analysis is to close L by an infinite semicircle in the negative half-plane. The contribution from this part of the contour vanishes as the radius of the semicircle approaches infinity because of the exponentially decreasing character of the integrand. The value of the integral for U along the contour L is now equal to $-2\pi i$ times the sum of the residues of the integrand evaluated at the poles of $f(\nu)$ located in the lower half-plane of ν. It then follows that

$$U=-ik\overline{C}\sum_{\nu}\frac{(\nu+\frac{1}{2})}{\sin\nu\pi}h_{\nu}^{(2)}(kb)h_{\nu}^{(1)}(kr)\frac{F_{\nu}}{D_{\nu}'}P_{\nu}[\cos(\pi-\theta)] \tag{12.24}$$

where $D_{\nu}^{(\prime)}=\partial D_{\nu}/\partial\nu$. All quantities in the summand are to be evaluated at the poles of $f(\nu)$ which are the roots of the equation

$$D_{\nu}=0.$$

This equation is precisely the same as the one discussed earlier (*i.e.*, eq (5.5)). At that time the relevant spherical wave functions of order ka and kc were simplified by the use of the Debye or second order approximation. The Hankel or third order approximation will now be employed. It may be written [26]

$$xh_{\nu}^{\binom{1}{2}}(x)\simeq\left(\frac{\pi r}{6}\right)^{\frac{1}{2}}\frac{x}{\nu+\frac{1}{2}}\left[1-\frac{(\nu+\frac{1}{2})^2}{x^2}\right]^{\frac{3}{4}}H_{1/3}^{\binom{1}{2}}(\rho)\exp\left[\pm i\left(\frac{5}{12}\pi-\rho\right)\right]S_{\nu}^{\binom{1}{2}}(x) \tag{12.25}$$

where

$$S_{\nu}^{\binom{1}{2}}(x)=\left[1-\frac{(\nu+\frac{1}{2})^2}{x^2}\right]^{-\frac{1}{4}}\exp\left(\mp i\frac{\pi}{4}\right)\exp\left[\pm i\int_{\nu+\frac{1}{2}}^{x}\left(1-\frac{(\nu+\frac{1}{2})^2}{x^2}\right)^{\frac{1}{2}}dx\right] \tag{12.26}$$

and $H_{1/3}^{\binom{1}{2}}(\rho)$ is the Hankel function of order 1/3 of argument ρ given by

$$\rho=\frac{\nu+\frac{1}{2}}{3}\left[\frac{x^2}{(\nu+\frac{1}{2})^2}-1\right]^{\frac{3}{2}}. \tag{12.27}$$

For Re $(\nu+\frac{1}{2})<<x$ or for $|\rho|>>1$, the above reduces to

$$xh_{\nu}^{\binom{1}{2}}(x)\simeq S_{\nu}^{\binom{1}{2}}(x) \tag{12.28}$$

which is the Debye approximation used in section (5).

The third order representation for the logarithmic derivative is

$$ln'[xh_{\nu}^{\binom{1}{2}}(x)]\simeq e^{\pm i\frac{2\pi}{3}}\left[1-\frac{(\nu+\frac{1}{2})^2}{x^2}\right]^{\frac{1}{2}}\frac{H_{2/3}^{\binom{1}{2}}(\rho)}{H_{1/3}^{\binom{1}{2}}(\rho)} \tag{12.29}$$

while the corresponding second order approximation, valid for $|\rho|>>1$ is simply

$$ln'[xh_{\nu}^{\binom{1}{2}}(x)]\simeq\pm i\left[1-\frac{(\nu+\frac{1}{2})^2}{x^2}\right]^{\frac{1}{2}}. \tag{12.30}$$

For convenience in what follows, it is desirable to introduce two new spherical reflection coefficients r_g and r_i which are connected to R_g and R_i in the following manner:

$$r_g = -\frac{ln'[kah_\nu^{(2)}(ka)]}{ln'[kah_\nu^{(1)}(ka)]} R_g = \frac{1 - \dfrac{i\Delta_g}{ln'[kah_\nu^{(1)}(ka)]}}{1 - \dfrac{i\Delta_g}{ln'[kah_\nu^{(2)}(ka)]}} \tag{12.31}$$

and

$$r_i = -\frac{ln'[kch_\nu^{(1)}(kc)]}{ln'[kch_\nu^{(2)}(kc)]} R_i = \frac{1 + \dfrac{i\Delta_i}{ln'[kch_\nu^{(2)}(kc)]}}{1 + \dfrac{i\Delta_i}{ln'[kch_\nu^{(1)}(kc)]}} \tag{12.32}$$

where

$$\Delta_g \simeq \frac{k}{k_g}\left[1 - \left(\frac{\nu+\frac{1}{2}}{k_g a}\right)^2\right]^{\frac{1}{2}} \tag{12.33}$$

and

$$\Delta_i \simeq \frac{k}{k_i}\left[1 - \left(\frac{\nu+\frac{1}{2}}{k_i a}\right)^2\right]^{\frac{1}{2}}. \tag{12.34}$$

These new reflection coefficients may be expressed to a high order of approximation by using the Hankel or third-order approximation for the spherical Bessel functions of argument ka or kc. Thus

$$r_g \cong \frac{1 - \dfrac{\Delta_g}{\left[1 - \left(\frac{\nu+\frac{1}{2}}{ka}\right)^2\right]^{\frac{1}{2}}} e^{-i\frac{\pi}{6}} \dfrac{H_{1/3}^{(1)}(\rho_a)}{H_{2/3}^{(1)}(\rho_a)}}{1 + \dfrac{\Delta_g}{\left[1 - \left(\frac{\nu+\frac{1}{2}}{ka}\right)^2\right]^{\frac{1}{2}}} e^{i\frac{\pi}{6}} \dfrac{H_{1/3}^{(2)}(\rho_a)}{H_{2/3}^{(2)}(\rho_a)}} \tag{12.35}$$

and

$$r_i \cong \frac{1 - \dfrac{\Delta_i}{\left[1 - \left(\frac{\nu+\frac{1}{2}}{kc}\right)^2\right]^{\frac{1}{2}}} e^{i\frac{\pi}{6}} \dfrac{H_{1/3}^{(2)}(\rho_c)}{H_{2/3}^{(2)}(\rho_c)}}{1 + \dfrac{\Delta_i}{\left[1 - \left(\frac{\nu+\frac{1}{2}}{kc}\right)^2\right]^{\frac{1}{2}}} e^{-i\frac{\pi}{6}} \dfrac{H_{1/3}^{(1)}(\rho_c)}{H_{2/3}^{(1)}(\rho_c)}} \tag{12.36}$$

where

$$\rho_a = \frac{\nu+\frac{1}{2}}{3}\left[\frac{(ka)^2}{(\nu+\frac{1}{2})^2} - 1\right]^{3/2} \tag{12.37}$$

and

$$\rho_c = \frac{\nu+\frac{1}{2}}{3}\left[\frac{(kc)^2}{(\nu+\frac{1}{2})^2} - 1\right]^{3/2}. \tag{12.38}$$

To this same approximation

$$r_g \cong e^{-i4\pi/3} \frac{H_{2/3}^{(2)}(\rho_a)}{H_{1/3}^{(2)}(\rho_a)} \frac{H_{1/3}^{(1)}(\rho_a)}{H_{2/3}^{(1)}(\rho_a)} R_g \tag{12.39}$$

and

$$r_i \cong e^{i4\pi/3} \frac{H_{2/3}^{(1)}(\rho_c)}{H_{1/3}^{(1)}(\rho_c)} \frac{H_{1/3}^{(2)}(\rho_c)}{H_{2/3}^{(2)}(\rho_c)} R_i. \tag{12.40}$$

When $|\rho_a|$ and $|\rho_c| >> 1$

$$r_g \cong R_g \cong \frac{\left[1 - \left(\frac{\nu+\frac{1}{2}}{ka}\right)^2\right]^{\frac{1}{2}} - \Delta_g}{\left[1 - \left(\frac{\nu+\frac{1}{2}}{ka}\right)^2\right]^{\frac{1}{2}} + \Delta_g} \tag{12.41}$$

and

$$r_i \cong R_i \cong \frac{\left[1-\left(\frac{\nu+\frac{1}{2}}{kc}\right)^2\right]^{\frac{1}{2}}-\Delta_i}{\left[1-\left(\frac{\nu+\frac{1}{2}}{kc}\right)^2\right]^{\frac{1}{2}}+\Delta_i}. \tag{12.42}$$

On writing $\nu+\frac{1}{2}=kaS=kcS'$, these latter forms are readily identified as the Fresnel reflection coefficients

$$r_g \simeq R_g \simeq \frac{N_g^2(1-S^2)^{\frac{1}{2}}-(N_g^2-S^2)^{\frac{1}{2}}}{N_g^2(1-S^2)^{\frac{1}{2}}+(N_g^2-S^2)^{\frac{1}{2}}} \tag{12.43}$$

and

$$r_i \simeq R_i \simeq \frac{N_i^2[1-(S')^2]^{\frac{1}{2}}-[N_i^2-(S')^2]^{\frac{1}{2}}}{N_i^2[1-(S')^2]^{\frac{1}{2}}+[N_i^2-(S')^2]^{\frac{1}{2}}}. \tag{12.44}$$

Attention is turned specifically to the determination of the roots of the equation

$$D_\nu=0. \tag{12.45}$$

This may be written

$$R_g R_i \frac{h_\nu^{(1)}(ka)h_\nu^{(2)}(kc)}{h_\nu^{(2)}(ka)h_\nu^{(1)}(kc)}=e^{-2\pi i n} \tag{12.46}$$

where R_g and R_i are defined by eq (31) and (32) and n may take integral values. Emplying the third-order approximations, this may be rewritten, for Re $(\nu+\frac{1}{2})<ka$

$$r_g r_i \frac{H_{2/3}^{(2)}(\rho_c)H_{2/3}^{(2)}(\rho_a)}{H_{2/3}^{(1)}(\rho_c)H_{2/3}^{(2)}(\rho_a)} e^{-i2(\gamma_c-\gamma_a)}e^{i2(\rho_c-\rho_a)}=e^{-i2\pi n} \tag{12.47}$$

where

$$\gamma_a=\int_{\nu+\frac{1}{2}}^{ka}\left[1-\frac{(\nu+\frac{1}{2})^2}{x^2}\right]^{\frac{1}{2}}dx \tag{12.48}$$

and

$$\gamma_c=\int_{\nu+\frac{1}{2}}^{kc}\left[1-\frac{(\nu+\frac{1}{2})^2}{x^2}\right]^{\frac{1}{2}}dx \tag{12.49}$$

while, for $kc>\text{Re}\ (\nu+\frac{1}{2})>ka$,

$$r_g r_i \frac{H_{2/3}^{(2)}(\rho_c)H_{2/3}^{(2)}(\rho_a)}{H_{2/3}^{(1)}(\rho_c)H_{2/3}^{(1)}(\rho_a)} e^{-i2\gamma_c}e^{i2\rho_c}=e^{-i2\pi n}. \tag{12.50}$$

In the above formulas the (spherical) reflection coefficients r_g and r_i are defined by eqs (35) and (36). When $|\rho_a|$ and $|\rho_c| >>1$ or if Re $(\nu+\frac{1}{2})<<ka$, the relevant equation for the modes is simply

$$r_g r_i e^{-i2(\gamma_c-\gamma_a)}=e^{-i2\pi n} \tag{12.51}$$

where r_g and r_i are defined by eq (41) and (42) which are the Fresnel form of the reflection coefficients. Equation (51) is identical to eq (5.10) which was discussed previously.

13. Influence of Stratification at the Lower Edge of the Ionosphere

Attention in previous sections has been largely confined to a sharply bounded homogeneous ionosphere. In view of the general uncertainty about the electrical properties of the lower edge of the ionosphere, a more elaborate model might hardly seem worthwhile. Furthermore, despite the geometrical simplicity of the above models, the computation of the modes is very involved in the general case. Despite these disparaging remarks the inhomogeneity of the lower ionosphere may be considered in some cases without greatly increasing the complexity. Some of these generalizations are discussed here for what they are worth.

The theoretical treatment given in section (3) for a vertical electric dipole located in the air space between a flat ground and a plane interface of a homogeneous ionosphere may be easily generalized to a stratified ionosphere. The essential modification is to replace the ionosphere reflection coefficient $R_i(C)$ by a more elaborate form which is denoted $\overline{R}_i(C)$. For example, a two layer ionosphere is chosen. The lower edge is at $z=h$ and from there to $z=h+s$, the refractive index (assumed constant and isotropic) is N_1; at this point the refractive index (also assumed constant and isotropic) is N_2 and remains at this value thereafter. It is not at all difficult to show that, for vertical polarization [31],

$$\overline{R}_i(C)=\frac{N_1^2C-(N_1^2-S^2)^{\frac{1}{2}}Q}{N_1^2C+(N_1^2-S^2)^{\frac{1}{2}}Q} \tag{13.1}$$

where

$$Q=\frac{N_1^2(N_2^2-S^2)^{\frac{1}{2}}+N_2^2(N_1^2-S^2)^{\frac{1}{2}}\tanh\,[iks(N_1^2-S^2)^{\frac{1}{2}}]}{N_2^2(N_1^2-S^2)^{\frac{1}{2}}+N_1^2(N_2^2-S^2)^{\frac{1}{2}}\tanh\,[iks(N_1^2-S^2)^{\frac{1}{2}}]}. \tag{13.2}$$

Here it may be observed that if $|kN_1s|<<1$ the reflection coefficient becomes

$$\overline{R}_i(C)\cong\frac{N_1^2C-(N_2^2-S^2)^{\frac{1}{2}}}{N_2^2C+(N_2^2-S^2)^{\frac{1}{2}}}. \tag{13.3}$$

whereas if $|kN_1s|>>1$, it becomes

$$\overline{R}_i(C)\cong\frac{N_1^2C-(N_1^2-S^2)^{\frac{1}{2}}}{N_1^2C+(N_1^2-S^2)^{\frac{1}{2}}}. \tag{13.4}$$

These two limiting cases correspond respectively, to the conditions of an electrically thin and an electrically thick stratum. In the former case the effective reflection level is at $z=h+s$ and in the latter case it is at $z=h$.

The formula for $\overline{R}_i(C)$ may easily be generalized to any number of layers. For example, in the case of discrete layers or strata, $0<z<h$ corresponds to the air; $h<z<h+s_1$ corresponds to a stratum with index N_1, $h+s_1<z<h+s_1+s_2$ corresponds to a stratum with index N_2, and so on. (See fig. 10.) With this generalization, Q, in eq (1), is to be replaced by

$$Q_1=\frac{N_1^2(N_2^2-S^2)^{\frac{1}{2}}Q_2+N_2^2(N_1^2-S^2)^{\frac{1}{2}}\tanh\,[iks_1(N_1^2-S^2)^{\frac{1}{2}}]}{N_2^2(N_1^2-S^2)^{\frac{1}{2}}+N_1^2(N_2^2-S^2)^{\frac{1}{2}}Q_2\tanh\,[iks_1(N_1^2-S^2)^{\frac{1}{2}}]} \tag{13.5}$$

$$Q_2=\frac{N_2^2(N_3^2-S^2)Q_3+N_3^2(N_2^2-S^2)\tanh\,[iks_2(N_2^2-S^2)^{\frac{1}{2}}]}{N_3^2(N_2^2-S^2)+N_2^2(N_3^2-S^2)Q_3\tanh\,[iks_2(N_2^2-S^2)^{\frac{1}{2}}]} \tag{13.6}$$

and so on. Q_3, Q_4, Q_5 . . . are obtained by cyclic permutation of indices. It should be noted, however, for M discrete strata that $Q_M=1$ since effectively $s_M=\infty$. The resultant Hertz vector for the air space $0<z<h$ is then formally given by eq (3.13) with the more general meaning now attached to the ionosphere reflection coefficient. In the general case, the rigorous evaluation of the integrals would be extremely involved. However, using arguments similar to those for the homogeneous ionosphere, the field may be approximated as a sum of residues evaluated at the poles of the integrand. Thus the contributions from the branch points are again neglected since for finitely conducting layers they correspond to heavily damped waves. Therefore, the residue series formula given by eq (3.15) is also applicable if the reflection co-

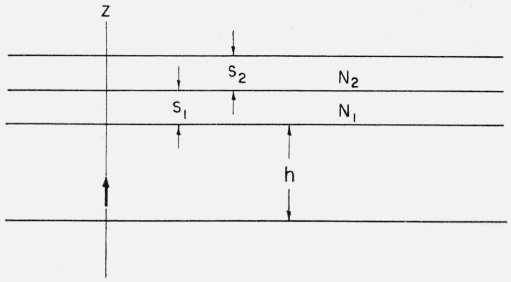

FIGURE 10. *Stratified model for ionosphere.*

efficient $R_i(C_n)$ is replaced by $\overline{R}_i(C_n)$. The modal equation now reads

$$\overline{R}_i(C_n)R_g(C_n)\exp(-2ikhC_n)=\exp(-i2\pi n) \tag{13.7}$$

where n takes integral values.

A numerical treatment and an application for the special case of a two-layer ionosphere has been carried out and reported in the literature [21]. There is no intrinsic difficulty in extending such calculations to an indefinitely large number of such layers each with infinitesimal thickness. For finitely conducting strata such a process converges and leads to an adequate representation for a continuous refractive index profile.

A great simplification to the formulas for a stratified ionosphere is effected if the refractive indices for all layers are large. For example, if $|N_1|, |N_2| \dots |N_M| >> 1$, then

$$Q_1 \simeq \frac{N_1Q_2+N_2\tanh(iks_1N_1)}{N_2+N_1Q_2\tanh(iks_2N_1)} \tag{13.8}$$

$$Q_2 \simeq \frac{N_2Q_3+N_3\tanh(iks_2N_2)}{N_3+N_2Q_3\tanh(iks_2N_2)} \tag{13.9}$$

and so on. Thus to this approximation, Q_1 does not depend on the angle of incidence or the factor C. In this case, the modal equation simplifies to

$$khC_n \simeq \pi n + i\Delta/C_n \tag{13.10}$$

where

$$\Delta = \frac{1}{N_g} + \frac{Q_1}{N_1}. \tag{13.11}$$

Regarding Δ/C_n as a small quantity, this can be solved to give

$$S_n = \left[1-\left(\frac{\pi n}{kh}\right)^2\right]^{\frac{1}{2}} - i\frac{\epsilon_n}{2kh}\Delta\left[1-\left(\frac{\pi n}{kh}\right)^2\right]^{-\frac{1}{2}} \tag{13.12}$$

where $\epsilon_0=1$, $\epsilon_n=2(n\neq 0)$. This is valid if $|\Delta kh| << 1$ and $|\Delta| << kh\,[1-(\pi n/kh)^2]$. Thus, at extremely low frequencies and for highly conducting layers, the propagation factor S_n is expressible in a relatively simple form.

The special case of a two layer ionosphere was considered in some detail in a previous paper [21]. Such a model was sufficient to explain the variation of the observed attenuation rate for frequencies of the order of 500 cps to 15 kc.

Exponential Profiles

At the extremely low frequencies it was seen that for a stratified model of the ionosphere. the factor Q does not depend on C or S. In fact, it is not difficult to show that the surface impedance Z at $z=h$ looking outwards is given by

$$Z=\eta_0 Q_1/N_1 \qquad \text{where } \eta_0 \simeq 120\pi.$$

From this viewpoint is becomes quite easy to write down expressions for Q_1 for other profiles. For example, if the refractive index varies in an exponential fashion relatively simple formulas are obtained. Two cases which could be considered are

$$N(z)=1.0 \text{ for } 0<z<h,$$

$$N(z)=\overline{N} \exp\left[\mp(z-h)/l\right] \text{ for } z>h,$$

where the $(-)$ sign corresponds to a refractive index decreasing with height and the $(+)$ sign corresponds to a refractive index increasing with height. In the above, l is a scale factor; for example, at within a distance l above the lower edge of the ionosphere, the refractive index has changed from $\overline{N}$ to $\overline{N}/e$ or $\overline{N}e$ for the two respective cases.

Within the layers of the ionosphere propagation is vertically upwards and thus a component E of the electric field satisfies

$$\frac{\partial^2 E}{\partial z^2}+k^2N^2(z)E=0 \qquad \text{for } z>h.$$

Solutions of this equation for the exponential form of $N^2(z)$ are

$$E=\begin{matrix}\text{const}\times I_0\ (ik\overline{N}le^{-(z-h)/l}) \\ \text{const}\times K_0(ik\overline{N}le^{+(z-h)/l})\end{matrix} \tag{13.13}$$

for the two respective cases, where I_0 and K_0 are modified Bessel functions. The transverse component of the magnetic field is then found from Maxwell's equations, e.g., $i\mu\omega H=\partial E/\partial z$. The surface impedance Z at the lower edge of this model of the ionosphere is then defined by

$$Z=E/H]_{z=h}.$$

For the case when the refractive index *decreases* with height

$$Z=\frac{\eta_0}{N_1}Q, \text{ with } Q=\frac{I_0(ik\overline{N}l)}{I_1(ik\overline{N}l)} \tag{13.14}$$

and when the refractive index increases with height,

$$Z=\frac{\eta_0}{N_1}Q, \text{ with } Q=\frac{K_0(ik\overline{N}l)}{K_1(ik\overline{N}l)}. \tag{13.15}$$

For low frequencies, $ik\overline{N}l\cong\sqrt{i}\,x$ with $x=|\overline{N}|\,kl$. The arguments of modified Bessel functions are thus proportional to $\sqrt{i}$. Numerical values are shown in the table for certain real values of x. In both these cases, it may be observed that as the scale length l approaches infinity, Z becomes $\eta_0/\overline{N}$ as it should.

The simplicity of the above formulas for the exponential profiles is due to the inherent assumption that the refractive index $\overline{N}(z)$ is large compared to unity for $z>h$. When this condition is violated, such as it would be at frequencies above several kilocycles per second, the solution is not expressible in closed form, except for horizontal polarization [32] which is really only of academic interest at vlf. Nevertheless for calculation of attenuation rates at extremely-low-frequencies (less than 1 kc, say), the exponential models find direct application. For example, if $Q=1$ the homogeneous ionosphere is regained and the attenuation rate $(-k \text{ Im } S_0)$ is proportional to $\omega^{\frac{1}{2}}$; on the other hand for an exponential profile with $N(z)$ increasing upwards, the attenuation rate increases with frequency more rapidly as suggested by experimental data. In fact, it is quite easy to see, for both decreasing and increasing exponential profiles, that

$$\frac{\text{Attenuation rate for exponential model}}{\text{Attenuation rate for homogeneous model}}=\sqrt{2}\,|Q|\sin\left(\frac{\pi}{4}-\arg Q\right).$$

TABLE 1. *Selected numerical values of the factor* Q

	$Q=\frac{I_0(\sqrt{i}\,x)}{I_1(\sqrt{i}\,x)}$		$Q=\frac{K_0(\sqrt{i}\,x)}{K_1(\sqrt{i}\,x)}$	
x	$\|Q\|$	arg Q	$\|Q\|$	arg Q
0.2	10.000	−44°43′	0.3854	23°58′
0.5	4.003	−43°13′	0.5880	17°16′
1.0	2.026	−37°55′	0.7344	11°59′
1.5	1.417	−29°52′	0.8047	9°15′
2.0	1.180	−20°59′	0.8459	7°28′
2.5	1.100	−13°44′	0.8728	6°20′
3.0	1.083	−9°09′	0.8919	5°28′
4.0	1.080	−5°42′	0.9169	4°18′
5.0	1.073	−4°37′	0.9325	3°33′
6.0	1.060	−3°49′	0.9433	3°02′
7.0	1.051	−3°19′	0.9511	2°37′
8.0	1.045	−2°47′	0.9570	2°16′
9.0	1.040	−2°26′	0.9618	2°06′
10.0	1.035	−2°11′	0.9654	1°54′
0.1			0.25518	27°22′
0.3			0.4719	21°08′
0.7			0.6622	14°38′

The right-hand side of the above equation is a function of the quantity $x=|\overline{N}|l\simeq(\overline{\omega}_r/\omega)^{\frac{1}{2}}l$ where $\overline{\omega}=\overline{\sigma}/\epsilon_0$ is the conductivity at the lower edge of the ionosphere and l is the vertical distance in which $\overline{N}$ changes by a factor 2.718.

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14. References

[1] G. N. Watson. The transmission of electric waves round the earth, Proc. Roy. Soc. **95**, 546 (1919).

[2] O. E. H. Rydbeck. On the propagation of radio waves, Trans. Chalmers Univ., **34** (1944).

[3] H. Bremmer. Terrestrial radio waves (Elsevier Publishing Co., Amsterdam, Netherlands, 1949).

[4] W. O. Schumann. Uber die ausbreitung sehr langer elektrischer wellen um die erde und die signale des blitze, Il Nuovo cimento **IX** (1952).

[5] W. O. Schumann. Uber die oberfelder bei der Ausbreitung langer, elektrischer wellen im system erde-luft-ionosphare und 2 anwendungen (horizontaler und senkrechter dipol) Z. angew. Phys. **6,** 34, (1954).

[6] W. O. Schumann. Uber die strahlung langer wellen des horizontalen dipols in dem lufthohlraum zwischen erde und ionosphare, Z. angew. Phys. **6,** 225, (1954).

[7] H. Kaden. Die reflexions und schirmwirkung metallischer hullen in einer ebenen elektromagnetischen welle, Sonderdruck aus Archiv der Elektrischen Uberstragung, 403 (1957).

[8] W. L. Taylor and L. J. Lange. Some characteristics of vlf propagation using atmospheric waveforms (Proc. 2d Conf. on Recent advances in atmospheric electricity, Pergamon Press, New York, N.Y., 609, 1959).

[9] F. W. Chapman and R. C. V. Macario. Propagation of audio frequency radio waves to great distances, Nature, **177,** 930 (1956).

[10] S. Namba. General theory of the propagation in the upper atmosphere, Proc. IRE **21,** 238 (1933).

[11] K. G. Budden. The propagation of a radio atmospheric II, Phil. Mag. **43,** 1179 (1952).

[12] K. G. Budden. The propagation of very low frequency radio waves to great distances, Phil. Mag., **44,** 504 (1953).

[13] K. G. Budden. The "wave-guide mode" theory of the propagation of very-low-frequency radio waves, Proc. IRE, **45,** 772 (1957).

[14] IA. L. Al'pert. Low frequency electromagnetic wave propagation over the surface of the earth (Monograph published by Academy of Science, U.S.S.R. 1955).

[15] IA. L. Al'pert. The field of long and very long radio waves over the earth under actual conditions, Radiotekh. i Elektron., **1,** 281 (1956).

[16] L. Lieberman. Anomalous propagation below 500 cps, Proc. Symp. on Propagation of vlf Radio Waves **3,** paper 25, Boulder, Colo., (1957).

[17] J. R. Wait and L. B. Perry. Calculations of ionospheric reflection coefficients at very low radio frequencies J. Geophys. Research **62,**, 43 (1957).

[18] J. R. Wait. On the mode theory of vlf ionospheric propagation, Rev. Geofis. pura e appl. **37,** 103 (1957).

[19] J. R. Wait and H. H. Howe. The waveguide mode theory of vlf ionospheric propagation, Proc. IRE **45,** (1957).

[20] J. R. Wait. The mode theory of vlf ionospheric propagation for finite ground conductivity, Proc. IRE **45,** 760 (1957).

[21] J. R. Wait. An extension to the mode theory of vlf ionospheric region, J. Geophys. Research **63,** 125 (1958).

[22] J. R. Wait. Propagation of very-low-frequency pulses to great distances, J. Research, NBS **61,** 187 (1958).

[23] H. H. Howe and J. R. Wait. Mode calculations for vlf ionospheric propagation (Vlf Symp. paper 36, Boulder, Colo., 1957).

[24] B. Friedman. Low frequency propagation in the ionosphere, (Proc. Symp. on the propagation of radio waves, Liege, Belgium, 1958).

[25] A. N. Sommerfeld. Uber die ausbreitung der wellen in der drachtlosen telegraphie, Ann. Physik **81,** 1135 (1926).

[26] A. N. Sommerfeld. Partial differential equations (Academic Press, New York, N.Y., 1949).

[27] F. Magnus and F. Oberhettinger. Special functions of mathematical physics (Chelsea Pub. Co., New York, N.Y., 71, 1949).

[28] W. O. Schumann. Uber elektrische eigenschwingungen des hohlraumes erde-luft-ionosphere are arregt durch blitzentladungen, Z. angew. Phys. **9,** 373 (1957).

[29] J. R. Wait. Low frequency radiation from a horizontal antenna over a spherical earth, Can. J. Phys. **34,** 586 (1956).

[30] K. G. Budden. The reflection of very low frequency radio waves at the surface of a sharply bounded ionosphere with superimposed magnetic field, Phil. Mag. **44,** 504 (1951).

[31] J. R. Wait. Transmission and reflection of electromagnetic waves in the presence of stratified media, J. Research NBS **61,** 205 (1958).

[32] J. R. Wait. Oblique reflection of radio waves from an inhomogeneous region, J. Appl. Phys. **23,** 1403 (1952).

15. Additional References

Al'pert, IA. L., On the speed of propagation of electromagnetic waves at audio frequencies, J. Exp. Theoret. Phys. (U.S.S.R.) **33,** 1305 (1957).

Al'pert, IA. L., Investigation of the propagation of long and very long radio waves by the method of analyzing the shape of atmospherics, Radiotekh. i Elektron., **1,** 293 (1956).

Al'pert, IA. L., and Bordina, S. V., On the velocity of propagation of audio frequency electromagentic waves, Radiotekh. i Elektron. **4,** 195 (1959).

Anderson, C. N., Correlation of long wave transatlantic radio with other factors affected by solar activity, Proc. IRE **16,** 207 (1928).

Anderson, C. N., Notes on radio transmission, Proc. IRE **19,** 1150 (1931).

Austin, L. W., A series of reports on long-distance receiving measurements at Washington, Proc. IRE **12–19** (1931).

Austin, L. W., Preliminary note on proposed changes in the constants of the Austin-Cohen transmission formula, Proc. IRE **14,** 377 (1926).

Bracewell, R. N., Budden, K. G., Ratcliffe, J. A., Straker, T. W., and Weekes, K., The ionospheric propagation of low and very-low frequency radio waves over distances less than 1,000 km, Proc. Inst. Elec. Engrs. **98,** 221 (1951).

Bremmer, H., Propagation of electromagnetic waves, Handbuch der Physik **16,** 423 (1958).

Bremmer, H., Mode expansion in the low frequency range for propagation through a curved stratified atmosphere, J. Research NBS **63D,** 75 (1959).

Brown, J. N., Round-the-world signals at very low frequency, J. Geophys. Research **54,** 367 (1949).

Budden, K. G., The numerical solution of the differential equations governing the reflexion of long radio waves from the ionosphere. II, Phil. Trans. Roy. Soc. London, Sec. A. **248,** 45 (1955).

Budden, K. G., The numerical solution of differential equations governing reflexion of long radio waves from the ionosphere. I, Proc. Roy. Soc. **227,** 516 (1955).

Budden, K. G., A reciprocity theorem on the propagation of radio waves via the ionosphere, Phil. Mag., **44,** 604 (1953, 1954).

Crombie, D. D., Differences between east-west and west-east propagation of vlf signals over long distances, J. Atmospheric and Terrest. Phys. **12,** 110 (1958).

Eckersley, T. L., Studies in radio transmission, J. Inst. Elec. Engrs., **71,** 405 (1932).

Friedman, B. F., Propagation in a nonhomogeneous atmosphere, (Theory of Electromagnetic Waves, Academic Press, New York, N.Y., 1951).

Helliwell, R. A., and Gehrels, E., Observations of magneto-ionic duct propagation using man-made signals of very low frequency, Proc. IRE **46,** 785, (1958).

Heritage, J. L., Weisbrod, S., and Bickel, J. E., A study of signal vs. distance data at vlf, Vlf Symp. Record, **4,** 77 (1957).
Hollingworth, J., The propagation of radio waves, J. Inst. Elec. Engrs., **64,** 579 (1926).
Jean, A. G., Lange, L. J., and Wait, J. R., Ionospheric reflection coefficients at vlf from sferics measurements, Geofis. pura e appl. **38,** 147 (1957).
Karbowiak, A. E., Guided wave propagation in submillimetric region, Proc. IRE **46,** 1706 (1958).
Kendrick, G. W., Radio transmission formulae, Phys. Rev. **31,** 1040 (1928).
Kitchen, F. A., Pressey, B. G., and Tremellen, K. W., A review of present knowledge on the ionospheric propagation of very-low, low- and medium-frequency waves, Proc. Inst. Elec. Engrs. **100,** 100 (1953).
Norton, K. A., Low and medium frequency radio propagation, (Proc. Symp. on propagation of radio waves, Liege, Belgium, 1958).
Norton, K. A., The calculation of the ground wave field intensity, Proc. IRE **29,** 723 (1941).
Pierce, J. A., Sky wave field intensity 1: Low- and very-low radio frequencies, Cruft Lab. Tech. Rept. 158, Harvard Univ. (1952).
Rosner, O., Die ausbreitung elektromagnetischer TE-(H)-wellen, erzeugt durch eine horizontal strom durchflossene leiterscheife, Z. angew. Phys. **9,** 448 (1957).
Round, H. J., Eckersley, T. L., Tremellen, K., and Lunnon F. C., Report on measurements made on signal strength at great distances during 1922 and 1923 by an expedition sent to Australia, J. Inst. Elec. Engrs. **63,** 933 (1925).
Schmelovsky, K. H., Problem der ausbreitung in tropospherischen und ionosphischen wellenleiter, Abhanglungen des Meteorologischen und Hydrologischen Dienstes der Deutschen Demokratischen Republik, **7** (1958).
Schumann, W. O., Uber wellenausbrietung im plasma zwischen zwei unendlich gut leitenden ebenen in richtung eines aufgepragten auberen magnetfeldes, Z. angew. Phys. **8,** 482 (1956).
Schumann, W. O., Uber die zeitliche form und das spehtrum ausqesendeter dipolsignale in einer dielectrischen hohlkugel mit leitenden wanden (Bavarian Academy of Sciences, Munich, Germany, 1956).
Storey, L. R. O., An investigation of whistling atmospherics, Phil. Trans. Roy. Soc., **246,** 113 (1953).
Tremellen, K., A study of ionospheric ray field intensities in the 10–30 and 500–1,100 kc bands, Marconi Rev. **13,** 153 (1950).
Wait, J. R., The attenuation versus frequency characteristics of vlf radio waves, Proc. IRE **45,** 768 (1957).
Wait, J. R., Transmission loss curves for propagation at very low radio frequencies, IRE Trans. on Commun. Systems, **CS–6,** 58 (1958).
Wait, J. R., and Howe, H. H., Amplitude and phase curves for ground-wave propagation in the band 200 cps to 500 kc NBS Circ. **574** (1956).
Wait, J. R., Diffraction corrections to the geometrical optics of low frequency propagation (Proc. Symp. on propagation of radio waves, Liege, Belgium, 1958).
Wait, J. R., Radiation from a vertical antenna over a curved stratified ground, J. Research NBS **56,** 237 (1956).
Wait, J. R., Note on the theory of radio propagation over an ice covered sea, Def. Research Tele. Est., Radio Physics Lab., Project Rept. 18–0–7 (1954). (Gives solution for dielectric slab over a conductive base.)
Wait, J. R., and Murphy, A., Multiple reflections between the earth and the ionosphere in vlf propagation, Geofis. pura e appl. **35,** 61, (1956).
Wait, J. R., Reflection of electromagnetic waves obliquely from an inhomogeneous medium, J. Appl. Phys. **23,** 1403 (1952).
Wait, J. R., and Perry, L. B., Calculations of ionospheric reflection coefficients at very low radio frequencies, J. Geophys. Research **62,** 43 (1957).
Wait, J. R., Mixed path ground wave propagation: 1. Short distances, J. Research NBS **57** (1956).
Wait, J. R., Amplitude and phase of the low-frequency ground wave near a coastline, J. Research NBS **58,** 237 (1957).
Wait, J. R., and Householder, James, Mixed-path ground-wave propagation: 2. Larger distances, J. Research NBS **59,** 19 (1957).
Watt, A. D., and Maxwell, E. L., Characteristics of atmospheric noise from 1 to 100 kc, Proc. IRE **45,** 787 (1957).
Weidner, Julius, Uber eine verfeinerung der theorie der ausbreitung langer elektrischer wellen um die erde z. angew. Phys. **7,** 77 (1955).
Wilkes, M. V., The theory of reflection of very long wireless waves from the ionosphere, Proc. Roy. Soc. **A175,** 143 (1940).
Wilkes, M. V., The oblique reflection of very long wireless waves from the ionosphere, Proc. Roy. Soc. **A189,** 130 (1947).
Yokoyama, E., and Nakai, T., East-west and north-south attenuations of long radio waves in the Pacific, Proc. IRE **17,** 1240 (1929).
Yokoyama, E., and Nakai, T., Effect of sunspots and magnetism on long distance reception on vlf waves Proc. IRE **19,** 882 (1931).
Yokoyama, E., and Tanimua, I., Some long distance transmission phenomena of low frequency waves, Proc. IRE, **21** 263 (1933).

The following two papers have just appeared:
Barron, D. W., The "waveguide mode" theory of radio wave propagation when the ionosphere is not sharply bounded, Phil. Mag. **50,** 1068 (1959). (This is a continuation of early unpublished work of Budden.)
Poverlein, H., Lang-und Langstwellausbreitung, Fortschr. der Hochfrequenztech. **4,** 47 (1959). (Many additional references are found in this excellent review article.)

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